Poisson Brackets

23 May 2005

In the <u>Schrödinger picture</u>, observables are fixed while states evolve in time; in the Heisenberg picture states are fixed while observables evolve in time:

Schrödinger picture	Heisenberg Picture
$x \in X$ is a state	$F \in C^\infty(X)$ is an observable
$x \longmapsto x(t)$	$F \longmapsto F_t$
where	where
$\frac{d}{dt} x(t) = v_H(x(t))$	$\frac{d}{dt} F_t = v_H F_t$

and

$$F(x(t)) = F_t(x)$$

is the result of measuring F in state x after waiting a time $t \in \mathbb{R}$.

We define the Poisson bracket of $F, G \in C^\infty(X)$ by

$$\{F, G\} = v_F G \in C^\infty(X)$$

In this language, Hamilton's equations become

$$\frac{d}{dt} F_t = \{H, F_t\}$$

Example: If $X = T^*Q$ then

$$\omega = dq^i \wedge dp_i$$

so

$$\begin{aligned} \theta : v &\longmapsto \omega(v,-) \\ \frac{\partial}{\partial q^i} &\longmapsto dp_i \\ \frac{\partial}{\partial p_i} &\longmapsto -dq^i \end{aligned}$$

So

$$\begin{aligned} v_F &= \theta^{-1}(dF) \\ &= \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i} \end{aligned}$$

so

$$\begin{aligned} \{F,G\} &= v_F G \\ &= \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} \, . \end{aligned}$$

Noether's Thm (Hamiltonian version) — Suppose (X,ω) is a symplectic manifold, $H \in C^\infty(X)$ has v_H integrable. Then

$$\{H,F\} = 0 \iff F \text{ is a conserved quantity}$$

where we say F is conserved (for time evolution generated by H) if $F_t = F \;\; \forall t \in \mathbb{R}$. Suppose also that v_F is integrable and let

$$x \longmapsto x(s)$$

be the flow generated by v_F:

$$\frac{d}{ds} x(s) = v_F(x(s))$$

We say F generates symmetries of H if

$$H(x) = H(x(s)) \qquad \forall s \in \mathbb{R}$$

Then:

$$F \text{ is conserved} \iff F \text{ generates symmetries of } H.$$

<u>Proof:</u>

$$\begin{aligned} F \text{ is a conserved quantity} &\iff F = F_t \ \forall t \in \mathbb{R} \\ &\iff \frac{d}{dt} F_t = 0 \quad \forall t \\ &\iff \{H, F_t\} = 0 \quad \forall t \\ &\implies \{H, F\} = 0 \end{aligned}$$

& $(\Leftarrow)$ is true too: F_t & F solve $\frac{d}{dt} - = \{H, -\}$ so by uniqueness of solutions of 1st order ODE we get $F = F_t$.

Next:
$$\begin{aligned} F \text{ generates symmetries of } H &\iff H(x) = H(x(s)) \quad \forall s \in \mathbb{R} \\ &\iff \frac{d}{ds} H(x(s)) = 0 \\ &\iff v_F(H) = 0 \\ &\iff \{F, H\} = 0 \end{aligned}$$

by exact same argument. So we just need

$$\{F, G\} = -\{G, F\}$$

which is true in the previous example, but also in general.

$$
\begin{aligned}
v_F &= \theta^{-1}(dF) \\
\theta(v_F) &= dF \\
\omega(v_F,-) &= dF(-) \\
\omega(v_F,v_G) &= dF(v_G) \\
&= v_G(F) \\
&= \{G,F\}
\end{aligned}
$$

So

$$
\begin{aligned}
\{F,G\} = v_F G &= \omega(v_G,v_F) \\
&= -\omega(v_F,v_G) \\
&= -\{G,F\} \qquad \blacksquare
\end{aligned}
$$

In short: antisymmetry of ω says:

F generates symmetries of $H \iff H$ generates symmetries of F.

<u>Example</u>: Taking $F = H$, we see that energy is conserved!

$$\frac{d}{dt} H_t = 0 \quad \text{since} \quad \{H,H\} = 0$$

by antisymmetry of $\{-,-\}$.

<u>Example</u>: If F & G are conserved quantities, then

$$\alpha F + \beta G \qquad \alpha,\beta \in \mathbb{R}$$

$$FG$$

$$\& \quad \{F,G\}$$

are also conserved quantities.

$\alpha F + \beta G$ is conserved quantity, since:

$$\begin{aligned}\{H, \alpha F + \beta G\} &= v_H(\alpha F + \beta G) \\ &= \alpha v_H F + \beta v_H G \\ &= \alpha\{H,F\} + \beta\{H,G\}\end{aligned}$$

FG is a conserved quantity, since:

$$\begin{aligned}\{H, FG\} &= v_H(FG) \\ &= (v_H F)G + F v_H G \\ &= \{H,F\}G + F\{H,G\}\end{aligned}$$

Finally, $\{F,G\}$ is conserved since

$$\{H,\{F,G\}\} = \{\{H,F\},G\} + \{F,\{H,G\}\}$$

— but this is trickier! This uses the fact that ω is closed — and is <u>equivalent</u> to this fact.

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25 May 2005

To prove the Jacobi identity, we use that ω is closed & this formula:

$$d : \Omega^p(Q) \longrightarrow \Omega^{p+1}(Q)$$

is given by

$$d\omega(v_0, \dots, v_p) = \sum_{0 \le i \le p} (-1)^i \, v_i \, \omega(v_0, \dots, \hat{v}_i, \dots, v_p)$$

$v_i \in \mathrm{Vect}(Q)$

$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([v_i, v_j], v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_p).$$

note: this part vanishes for coordinate v. flds.

These imply :

$$0 = d\omega(v_F, v_G, v_H) = v_F\, \omega(v_G, v_H) - v_G\,(v_F, v_H) + v_H\, \omega(v_F, v_G)$$
$$- \omega([v_F, v_G], v_H) + \omega([v_F, v_H], v_G) - \omega([v_G, v_H], v_F)$$

& note that

$$\omega(v_G, -) = dG(-)$$
$$\omega(v_G, v_H) = dG(v_H) = v_H G = \{H, G\}$$

$$v_F\, \omega(v_G, v_H) = v_F \{H, G\} = \{F, \{H, G\}\}$$

and

$$\omega([v_F, v_G], v_H) = -\omega(v_H, [v_F, v_G]) = -dH([v_F, v_G]) = -[v_F, v_G](H) = -v_F v_G H + v_G v_F H = -\{F, \{G, H\}\} + \{G, \{F, H\}\}$$

Using the symmetry of $\{\cdot,\cdot\}$, the 9 terms we get reduce to 3 and we obtain the Jacobi identity :

$$\{H, \{F, G\}\} = \{\{H, F\}, G\} + \{F, \{H, G\}\}.$$

<u>Thm</u>: If (Q, ω) is a symplectic manifold then $(Q, \{\cdot,\cdot\})$ is a <u>Poisson manifold</u>, i.e. $(C^\infty(Q), \{\cdot,\cdot\})$ is a <u>Poisson algebra</u>:

1) $C^\infty(Q)$ is a commutative associative algebra (with usual $+, \cdot,$ etc.)

2) $(C^\infty(Q), \{\cdot,\cdot\})$ is a <u>Lie algebra</u>:

   1) $\{F, \alpha G + \beta H\} = \alpha \{F, G\} + \beta \{F, H\}$
   2) $\{F, G\} = - \{G, H\}$
   3) $\{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}$ (Jacobi)

3) For any $F \in C^\infty(Q)$, $\{F, \cdot\}$ is a <u>derivation</u> of $C^\infty(Q)$: it's linear and

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$

If $(Q, \{\cdot,\cdot\})$ is a Poisson manifold and $F \in C^\infty(Q)$, $\{F, \cdot\}$ is a derivation, so (by a nice theorem) there's a unique vector field $v_F \in \mathrm{Vect}(Q)$ s.t.

$$\{F, G\} = v_F G.$$

<u>Thm</u>: If $(Q, \{\cdot,\cdot\})$ is a Poisson manifold:

$$\begin{aligned} v: C^\infty(Q) &\longrightarrow \mathrm{Vect}(Q) \\ F &\longmapsto v_F \end{aligned}$$

is a <u>Lie algebra homomorphism</u>: it's linear &

$$v_{\{F,G\}} = [v_F, v_G]$$

Proof: $\{F, -\}$ & thus v_F is clearly linear in F. Also:

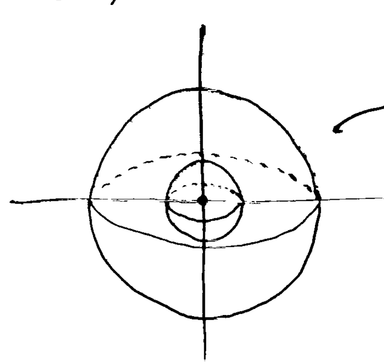
$$\begin{aligned} v_{\{F,G\}} H &= \{\{F,G\}, H\} \\ &= \{F, \{G,H\}\} + \{\{F,H\}, G\} \quad \text{) Jacobi identity.} \\ &= \{F, \{G,H\}\} - \{G, \{F,H\}\} \\ &= v_F v_G H - v_G v_F H \\ &= [v_F, v_G] H. \end{aligned}$$

Here's an example of a Poisson manifold that's not a symplectic manifold: $\mathbb{R}^3$ is odd-dimensional, hence not symplectic, but we can define $\{\cdot, \cdot\}$ by

$$\{x, y\} = z$$
$$\{y, z\} = x$$
$$\{z, x\} = y$$

x, y, z the coordinate functions $\mathbb{R}^3 \to \mathbb{R}$.

— cross product in disguise. Using the fact that $\{F, -\}$ is a derivation, you can calculate $\{F, G\}$ for all polynomials. Then by some approximation argument you can define $\{F, G\}$ for all $F, G \in C^\infty(Q)$



it has spheres as "symplectic leaves"

(In fact, any Poisson manifold has a foliation of symplectic manifolds)

Note: somehow we got into calling our symplectic manifolds Q instead of X.

27 May 2005

Liouville's Theorem

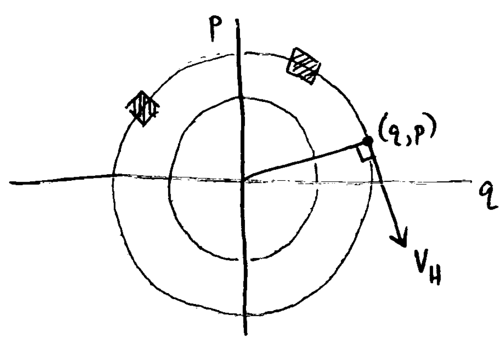
Let our configuration space be $Q = \mathbb{R}$, our phase space be $X = T^*Q \cong \mathbb{R}^2 \ni (q,p)$ and let

$$H = \frac{1}{2}(q^2 + p^2)$$

be the harmonic oscillator Hamiltonian. Then Hamilton's eqns say

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -q$$



so the time evolution $x \longmapsto x(t)$ is rotation by t clockwise.

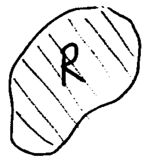
As a region $R \subseteq X$ evolves in time its <u>area</u> is preserved — or conserved! In fact this is completely general: For any symplectic manifold & any (integrable) Hamiltonian, the flow $x \longmapsto x(t)$ given by Hamilton's eqns:

$$\frac{d}{dt} x(t) = v_H(x(t))$$

preserves the symplectic structure ω. In our example,

$$\omega = dq \wedge dp$$

is the "area element":



$$\text{Area}(R) = \int_R \omega$$

For a $2n$-dimensional symplectic manifold,

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp^i$$

is symplectic coordinates, and

note: [illegible] to [illegible] since we're just moving 2-forms around $dq_i \wedge dp^i$.

$$\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{n} = n!\, dq_1 \wedge dp^1 \wedge dq_2 \wedge dp^2 \wedge \cdots \wedge dq_n \wedge dp^n$$

is a $2n$-form which measures volume, and $\frac{\omega^n}{n!}$ is called the Liouville form.

Liouville's Thm: Let (X, ω) be any symplectic manifold and let $H \in C^\infty(X)$ be such that v_H is integrable, so that Hamilton's eqns. determine time evolution:

$$\begin{aligned} \phi_t : X &\longrightarrow X \\ x &\longmapsto x(t) \end{aligned}$$

Then ω is preserved by ϕ_t:

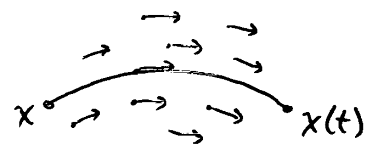
$$\phi_t^* \omega = \omega$$

& thus

$$\phi_t^* \frac{\omega^n}{n!} = \frac{\omega^n}{n!}.$$

Proof: In general if v is any integrable vector field on any manifold X, we get a flow

$$\begin{aligned}\phi_t : X &\longrightarrow X \\ x &\longmapsto x(t)\end{aligned}$$



with

$$\frac{d}{dt} x(t) = v(x(t))$$

and for any differential form $\mu \in \Omega^p(X)$ we have

$$\frac{d}{dt} \phi_t^* \mu \Big|_{t=0} = L_v \mu$$

where

$$L_v : \Omega^p(X) \longrightarrow \Omega^p(X)$$

is called the Lie derivative, which can be computed by Weil's formula

$$L_v = i_v d + d i_v$$

where

$$d : \Omega^p(X) \longrightarrow \Omega^{p+1}(X)$$

is the exterior derivative, and

$$i_v : \Omega^p(X) \longrightarrow \Omega^{p-1}(X)$$

is the interior product, given by

$$(i_v \mu)(v_1, \dots, v_{p-1}) := \mu(v, v_1, \dots, v_{p-1})$$

(Weil's formula isn't hard to prove, but we won't do it now)

In our situation,

$$
\begin{aligned}
\frac{d}{dt}\phi_t^*\omega\Big|_{t=0} &= L_{v_H}\omega \\
&= \underbrace{i_{v_H} d\omega}_{\substack{\| \\ 0 \\ (\omega \text{ closed})}} + d i_{v_H}\omega \\
&= d i_{v_H}\omega \\
&= ddH \\
&= 0 .
\end{aligned}
\qquad
\begin{aligned}
dH(-) &= \omega(v_H, -) \\
&= i_{v_H}\omega(-)
\end{aligned}
$$

One can similarly show $\frac{d}{dt}\phi_t^*\omega = 0$ for <u>any</u> time t so

$$\phi_t^*\omega = \omega. \qquad \blacksquare$$

Why is Weil's formula true? In fact

$$
\begin{aligned}
d &: \Omega^p \longrightarrow \Omega^{p+1} && \text{grade } 1 \\
L_v &: \Omega^p \longrightarrow \Omega^p && \text{grade } 0 \\
i_v &: \Omega^p \longrightarrow \Omega^{p-1} && \text{grade } -1
\end{aligned}
$$

are all <u>superderivations</u> or <u>graded derivations</u>.
A superderivation of grade k is a linear operator $D: \Omega^p \longrightarrow \Omega^{p+k}$ such that:

$$D(\underbrace{\mu}_{\in\Omega^p} \wedge \underbrace{\nu}_{\in\Omega^q}) = D\mu \wedge \nu + (-1)^{pk}\mu \wedge D\nu$$

If D, D' are superderivations of grade k, k', then supercommutator is again a derivation:

$$[D, D'] = DD' - (-1)^{kk'} D'D$$

In particular, Weil's formula says

$$L_v = [d, i_v] = di_v + i_v d,$$

and we can reduce proving it to proving L_v & $[d, i_v]$ agree on 1-forms — by product rule.