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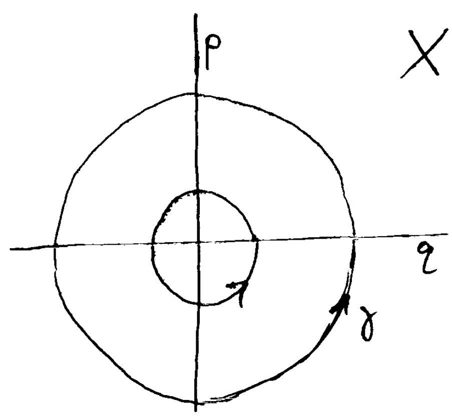
A Taste of Geometric Quantization

In Schrödinger's approach to QM, you take the config. space Q of a classical system, give it a Riemannian metric, & then form a Hilbert space $L^2(Q, \mathrm{vol})$ whose unit vectors are states in the corresponding quantum system, where vol is the measure coming from the Riemannian metric. In <u>geometric quantization</u> we instead construct the Hilbert space from the phase space X, which is a symplectic manifold.

De Broglie & then Bohr & Sommerfeld noticed a "quantization condition" which picked out the allowed "quantum orbits" among the classical ones in say, the harmonic oscillator or hydrogen atom:



If the orbit is some loop $\gamma : S^1 \to X$, it's an allowed orbit if the phase

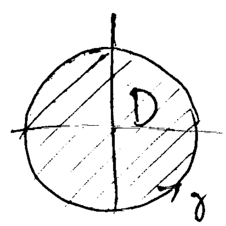
$$e^{i\int_\gamma \alpha/\hbar} = 1$$

where α is the canonical 1-form on X (if $X = T^*Q$) and $\hbar$ is Planck's constant ($\frac{h}{2\pi} = \hbar$). Here $\int_\gamma \alpha$ is a term in the action for the path $\tilde{\gamma}: [0,1] \to X \times \mathbb{R}$ in the extended phase space, & e^{iS} is what appears in the wave approach to CM — the "eikonal approximation" relating particles to waves. So we're demanding that we get complete constructive interference — that our particle's "wavefunction" comes back in phase when the particle goes in γ.

When $X = T^*Q$ we have

$$\int_\gamma \alpha = \int_D d\alpha = \int_D \omega$$

where D is any disk with $\partial D = \gamma$.



If $Q = \mathbb{R}$,
$\omega = dp \wedge dq$
so $\int_D \omega =$ area of D.

The Bohr-Sommerfeld quantization condition says

$$e^{i\int_D \omega/\hbar} = 1$$

or

$$\int_D \omega = 2\pi n\hbar = nh$$

which almost agrees with the modern formula $(n+\frac{1}{2})h$.

In modern terms, $\overset{A:=}{\frac{i\alpha}{\hbar}}$ is a connection on some $U(1)$ bundle over T^*Q,

$$e^{i\int_\gamma \alpha/\hbar} \in U(1)$$

is the holonomy of this connection around the loop γ, and

$$F = \frac{i\omega}{\hbar}$$

is the curvature of A.

So: in geometric quantization, we begin to quantize a system with phase space (X, ω) by finding a $U(1)$ bundle $P \to X$ with a connection A s.t. the curvature F of A is $\frac{i\omega}{\hbar}$. We can do this if and only if $\frac{\omega}{\hbar}$ is an integral closed 2-form, i.e. $[\frac{\omega}{\hbar}] \in H^2_{\text{DeRham}}(X)$ is the image of:

$$H^2(X,\mathbb{Z}) \xrightarrow{i_*} H^2(X,\mathbb{R}) \cong H^2_{DR}(X)$$

Given any $U(1)$ bundle P, its first Chern Class $c_1(P) \in H^2(X,\mathbb{Z})$ is a complete invariant of $U(1)$ bundles, and

$$i_* c_1(P) \in H^2_{DR}(X)$$

equals

$$\left[\frac{F}{2\pi i}\right]$$

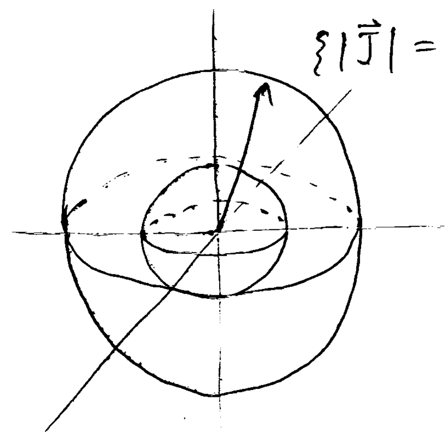
where F is the curvature of any connection on P.

Example: $\mathbb{R}^3$ is a Poisson manifold w.

$$\{x,y\} = z \qquad \{y,z\} = x \qquad \{z,x\} = y.$$

This is the phase space of a spinning point particle — i.e. $\vec{J} \in \mathbb{R}^3$ represents angular momentum. This is not symplectic, but it's foliated by "symplectic leaves" — spheres centered at the origin

$$\{|\vec{J}| = j\} = X_j$$



The sphere of radius j, X_j, is the phase space for a particle of total angular momentum j. In fact X_j has an integral symplectic structure when

$$j = 0, \tfrac{1}{2}, 1, \ldots$$

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The phase space for a spinning point particle of total angular momentum $j \in [0, \infty)$ is

$$X_j = \{ \vec{J} \in \mathbb{R}^3 : \|\vec{J}\| = j \},$$

a sphere with Poisson structure s.t.

$$\{x, y\} = z \qquad \{y, z\} = x \qquad \{z, x\} = y$$

One can check that this comes from a symplectic structure ω on X_j :

$$\{f, g\} = \omega(dg, df)$$

ω is a multiple of the area 2-form on the sphere X_j, normalized so that

$$\int_{X_j} \omega = 4\pi j \qquad \text{(note: } \underline{\text{not}}\ 4\pi j^2 \text{ !)}$$

When is ω integral? There's a theorem that says a closed 2-form ω on some manifold X is integral if for any 2d surface S mapped into X,

S

$$\int_S \omega \in \mathbb{Z}$$

Here we really want

$$\int_{X_j} \frac{\omega}{2\pi} \in \mathbb{Z} \qquad (\hbar = 1)$$

This happens when

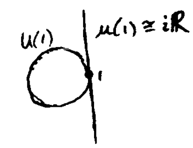
$$\int_{X_j} \frac{\omega}{2\pi} = \frac{1}{2\pi} 4\pi j = 2j \in \mathbb{Z}$$

i.e.

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

— the usual quantization condition for angular momentum! When this condition holds, there exists a $U(1)$ bundle $P \to X_j$ with a connection F whose curvature $F \in \Omega^2(X_j, \mathrm{Ad}\, P) = \Omega^2(X_j, \mathfrak{u}(1)) = \Omega^2(X_j, i\mathbb{R})$ is

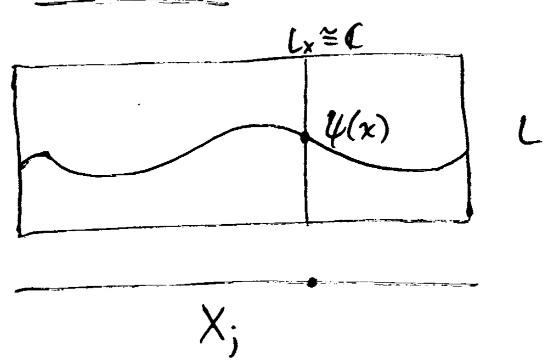
$$F = i\omega$$



We then build the Hilbert space H_j of states of a quantum spin-j particle using P as follows. We form an associated vector bundle

$$L = P \times_{U(1)} \mathbb{C}$$

using the God-given action of $U(1)$ on $\mathbb{C}$. This sort of bundle, with $\mathbb{C}$ as fibers, is called a complex line bundle, or usually a line bundle.



A section ψ of L looks locally like a complex function on the phase space. The Hilbert space H_j will consist of certain sections ψ of L. We shouldn't use all L^2 sections, since we want H_j to be finite dimensional! To get the right answer, we think of X_j as the Riemann sphere $\mathbb{C}P^1$ $= \mathbb{C} \cup \{\infty\}$. This lets us do complex analysis on X_j, and define holomorphic sections of L. Then we let H_j be the space of these holomorphic sections. Then:

$$\dim H_j = (2j+1)$$

as desired. We make H_j into a Hilbert space as follows: $U(1)$ acts on $\mathbb{C}$ in a way that preserves the inner product on $\mathbb{C}$, so the fibers L_x of L get an inner product. This lets us define

$$\langle \psi, \varphi \rangle = \int_{X_j} \langle \psi(x), \varphi(x) \rangle \, \omega$$

What's going on in general? Our phase space X starts out as an integral symplectic manifold, which then acquires a line bundle $L \to X$. To define holomorphic sections of L we need to equip X with extra structure — namely a Kähler structure:

Def: An almost Kähler structure on a manifold X is

1) a linear operator

$$J_x : T_x X \longrightarrow T_x X$$

depending smoothly on x, s.t.

$$J_x^2 = -1$$

This makes $T_x X$ into a complex vector space.

2) a complex inner product

$$\langle \cdot , \cdot \rangle_x : T_x X \times T_x X \longrightarrow \mathbb{C}$$

depending smoothly on X. This makes $T_x X$ into a (fin-dim) Hilbert space.

Given this, we have

$$\begin{aligned} \langle u, v \rangle_x &= \mathrm{Re}\langle u, v \rangle_x + i\,\mathrm{Im}\langle u, v\rangle_x \\ &= g(u,v) + i\,\omega(u,v) \end{aligned}$$

where

g is a Riemannian metric

& ω is a nondegenerate 2-form

Def: An almost Kähler manifold is Kähler if ω is closed, i.e. a symplectic structure.

Then you can do geometric quantization.