Classical Mechanics Homework
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John Baez
homework by: Scot Childress

Conservation Laws for the n-Body Problem

1. Show that in an n-body system where the force of the j-th body on the i-th body is given by

$$F_{ij}(t) = f_{ij}(|q_i(t) - q_j(t)|)\lambda_{ij}(t)$$

(where λ_{ij} denotes the unit vector in the direction $q_i - q_j$ at the time of interest and $f_{ij} = f_{ji}$; i.e. – Newton's 3rd Law) the energy is conserved.

Note that $E(t) = T(t) + V(t)$. Let's work on T first. We're going to play fast and loose with notation; hopefully all computations will be clear. Recall that T is given as $T = \sum_i (1/2)m_i\dot{q}^2$ so that

$$\begin{aligned}
\frac{dT}{dt} &= \sum_i m_i \dot{q}_i \cdot \ddot{q}_i \\
&= \sum_i \dot{q}_i \cdot F_i && \text{(Newton's 2nd Law)} \\
&= \sum_i \sum_{i \neq j} \dot{q}_i \cdot F_{ij} \\
&= \sum_i \sum_{i \neq j} f_{ij}(\dot{q}_i \cdot \lambda_{ij}) \\
&= \sum_i \sum_{i<j} f_{ij}[\dot{q}_i \cdot \lambda_{ij} + \dot{q}_j \cdot \lambda_{ji}] && \text{(Newton's 3rd Law)} \\
&= \sum_i \sum_{i<j} f_{ij}(\dot{q}_i - \dot{q}_j) \cdot \lambda_{ij}.
\end{aligned}$$

Now working with V we have that:

$$\begin{aligned}
\frac{dV}{dt} &= \sum_i \frac{dV_i}{dt} \\
&= \sum_i \sum_{i<j} \frac{dV_{ij}}{dt} \\
&= -\sum_i \sum_{i<j} f_{ij} \cdot (\dot{q}_i - \dot{q}_j) \cdot \lambda_{ij}
\end{aligned}$$

since $V'_{ij} = -f_{ij}$. Summing the results of these two computations yields $E'(t) \equiv 0$, and hence energy is conserved.

2. Show that angular momentum is conserved in the n-body problem.

$$\begin{aligned}
\frac{dJ}{dt} &= \sum_i \dot{J}_i(t) \\
&= \sum_i \dot{q}_i \times p_i + q_i \times \dot{p} \\
&= \sum_i m_i(\dot{q}_i \times \dot{q}_i) + q_i \times F_i \\
&= \sum_i \sum_{i \neq j} f_{ij}(q_i \times \lambda_{ij}) \\
&= \sum_i \sum_{i<j} f_{ij}(q_i \times \lambda_{ij} + q_j \times \lambda_{ji}) \\
&= \sum_i \sum_{i<j} \frac{f_{ij}}{|q_i - q_j|}(q_i \times q_j + q_j \times q_i)
\end{aligned}$$

and the last term vanishes since $q_i \times q_j = -q_j \times q_i$. (The second to last equality follows from Newton's 3rd Law.)