Classical Mechanics, Lecture 4

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Conservation of Energy in the n-Body Problem

Show that if Newton's 2^{nd} law ($F_i(t) = m_i\,\ddot{q}_i(t)$) holds then energy is conserved:

$$\frac{d}{dt}E(t) \;=\; 0.$$

Solution:

Below are the three equations for the total potential energy, the total kinetic energy, and the total energy of the system respectively:

$$\begin{aligned}
V(t) \;&=\; \sum_{i=1}^{n} V_i(t) \;=\; \sum_{i\,=\,1}^{n}\sum_{j\,\neq\,i} V_{ij}(\mid q_i(t) - q_j(t)\mid) \\
T(t) \;&=\; \sum_{i\,=\,1}^{n} \frac{1}{2}\, m_i\,\dot{q}_i(t)^2. \\
E(t) \;&=\; T(t) \;+\; V(t).
\end{aligned}$$

We can differentiate $E(t)$ with respect to time explicitly.

$$\begin{aligned}
\frac{dE(t)}{dt} \;&=\; \frac{d}{dt}\left[T(t) \;+\; V(t)\right] \\
\frac{dE(t)}{dt} \;&=\; \frac{d}{dt}\left[\sum_{i\,=\,1}^{n} \frac{1}{2}\, m_i\,\dot{q}_i(t)^2 + \sum_{i\,=\,1}^{n} V_i(t)\right] \\
&=\; \sum_{i\,=\,1}^{n} \frac{1}{2}\, m_i\,\frac{d}{dt}\dot{q}_i(t)^2 + \sum_{i\,=\,1}^{n} \frac{dV_i(t)}{dt} \\
&=\; \sum_{i\,=\,1}^{n} m_i\,\ddot{q}_i(t)\,\dot{q}_i(t) + \sum_{i\,=\,1}^{n} \frac{\partial V_i(\mid q_i(t) - q_j(t)\mid)}{\partial q_i}\,\dot{q}_i(t) \\
&=\; \sum_{i\,=\,1}^{n} \dot{q}_i(t)\left[m_i\,\ddot{q}_i(t) + \frac{\partial V_i(\mid q_i(t) - q_j(t)\mid)}{\partial q_i}\right] \qquad (1)
\end{aligned}$$

Because the potential $V_i(\mid q_i(t) - q_j(t)\mid)$ depends only on $\mid q_i(t) - q_j(t)\mid$ we get:

$$(2)\qquad \frac{\partial V_i(\mid q_i(t) - q_j(t)\mid)}{\partial q_i} \;=\; \sum_{j\,\neq\,i} \frac{\partial V_{ij}(\mid q_i(t) - q_j(t)\mid)}{\partial q_i} \;=\; \sum_{j\,\neq\,i} V'_{ij}\,\frac{q_i(t) - q_j(t)}{\mid q_i(t) - q_j(t)\mid}.$$

Newton's 2^{nd} law here reads:

$$(3)\qquad F_i(t) \;=\; m_i\,\ddot{q}_i(t) \;=\; \sum_{j\,\neq\,i} F_{ij}(t) \;=\; \sum_{j\,\neq\,i} f_{ij}(\mid q_i(t) - q_j(t)\mid)\,\frac{q_i(t) - q_j(t)}{\mid\mid q_i(t) - q_j(t)\mid}.$$

Since within the classical regime Newton's 2^{nd} law always holds, when we say "Show that if Newton's 2^{nd} law holds...", we mean given that the symmetric inter-particle interaction potential is derivable from a conservative force, i.e.

$$(4)\qquad V'_{ij} \;=\; -\,f_{ij},$$

implies that indeed the total energy of the system is conserved. Hence substituting equations (2) and (3) into (1) gives:

$$(5) \qquad \frac{dE(t)}{dt} = \sum_{i=1}^{n} \dot{q}_i(t) \left[\sum_{j \neq i} f_{ij}(\mid q_i(t) - q_j(t) \mid) \frac{q_i(t) - q_j(t)}{(\mid q_i(t) - q_j(t) \mid)} + \sum_{j \neq i} V'_{ij} \frac{q_i(t) - q_j(t)}{\mid q_i(t) - q_j(t) \mid} \right].$$

Now substituting (4) into (5), we find the desired result:

$$\boxed{\frac{dE(t)}{dt} \;=\; \sum_{i=1}^{n} \dot{q}_i(t) \sum_{j \neq i} \Big[f_{ij}(\mid q_i(t) - q_j(t) \mid) - f_{ij}(\mid q_i(t) - q_j(t) \mid) \Big] \frac{q_i(t) - q_j(t)}{(\mid q_i(t) - q_j(t) \mid)} \;=\; 0}$$

We could have just as well arrived at the same valid conclusion by omitting (4) and noting that $\frac{\partial V_i j}{\partial q_i} \;=\; -\frac{\partial V_i j}{\partial q_j}$ because the quantity $q_i(t) - q_j(t)$ is anti-symmetric on interchange of i and j. This is why $f_{ij} \;=\; f_{ji}$ but $F_{ij} \;=\; -F_{ji}$. This is of course what makes (4) true when the assumption of the existence of a conservative force interaction connected to a potential with the precise argument dependence of $\mid q_i(t) - q_j(t) \mid$ is made.

Conservation of Angular Momentum in the n-Body Problem:

Show that $\frac{d}{dt} J(t) = 0$ using Newton's 2^{nd} law and

$$F_{ij}(t) \;=\; f_{ij}(\mid q_i(t) - q_j(t) \mid) \frac{q_i(t) - q_j(t)}{\mid q_i(t) - q_j(t) \mid}$$

where $f_{ij} \;=\; f_{ji}$.

Solution:

We start with the following:

$$J(t) \;=\; \sum_{i=1}^{n} J_i(t) \;=\; \sum_{i=1}^{n} m_i q_i(t) \times \dot{q}_i(t)$$

Differentiating the above equation gives:

$$\begin{aligned}
\frac{dJ(t)}{dt} &= \frac{d}{dt} \left[\sum_{i=1}^{n} m_i q_i(t) \times \dot{q}_i(t) \right] \\
&= \sum_{i=1}^{n} m_i \frac{d}{dt} \left[q_i(t) \times \dot{q}_i(t) \right] \\
&= \sum_{i=1}^{n} \left[m_i \dot{q}_i(t) \times \dot{q}_i(t) + q_i(t) \times m_i \ddot{q}_i(t) \right] \\
&= \sum_{i=1}^{n} \left[q_i(t) \times F_i(t) \right] \quad (\text{since:} \quad a \times a \equiv 0) \\
&= \sum_{i=1}^{n} \left[q_i(t) \times \sum_{j \neq i} F_{ij}(t) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\left[q_i(t)\ \times \sum_{j\,\neq\, i} f_{ij}(\mid q_i(t)-q_j(t)\mid)\frac{q_i(t)-q_j(t)}{\mid q_i(t)-q_j(t)\mid}\right] \\
&= \sum_{i=1}^{n}\left[\sum_{j\,\neq\, i} f_{ij}(\mid q_i(t)-q_j(t)\mid)\frac{q_i(t)\ \times\ q_i(t)-q_i(t)\ \times\ q_j(t)}{\mid q_i(t)-q_j(t)\mid}\right] \\
&= -\sum_{i=1}^{n}\sum_{j\,\neq\, i}\left[f_{ij}(\mid q_i(t)-q_j(t)\mid)\frac{q_i(t)\ \times\ q_j(t)}{\mid q_i(t)-q_j(t)\mid}\right]. \qquad (6)
\end{aligned}
$$

But because $f_{ij} = f_{ji}$ and the quantity $[q_i(t) \times q_j(t)]$ is anti-symmetric, i.e. $[q_i(t) \times q_j(t)] = -[q_j(t) \times q_i(t)]$, the sum in (6) vanishes identically. Thus we have shown that,

$$
\boxed{\frac{dJ(t)}{dt} \;=\; -\sum_{i=1}^{n}\sum_{j\,\neq\, i}\left[f_{ij}(\mid q_i(t)-q_j(t)\mid)\frac{q_i(t)\ \times\ q_j(t)}{\mid q_i(t)-q_j(t)\mid}\right] \;=\; 0,}
$$

as was desired.