Classical Mechanics Homework

January 29, 2∞8

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Conservation of Energy in the n-body Problem

Suppose we have n particles $q_i\colon \mathbb{R} \to \mathbb{R}^3$ with the force on q_i due to q_j written as F_{ij}. Newton's 3rd law says:

$$F_{ij} = -F_{ji}.$$

Suppose further our forces are central, that is, only dependent on the distance from q_i to q_j. Then F_{ij} $(i \neq j)$ can be written as

$$F_{ij}(t) = f_{ij}(||q_i(t) - q_j(t)||)\frac{q_i(t) - q_j(t)}{||q_i(t) - q_j(t)||}$$

for some $f_{ij}\colon (0, \infty) \to \mathbb{R}$. For convenience, set $F_{ii} = 0$. We can define potentials $V_{ij}\colon (0, \infty) \to \mathbb{R}$ such that

$$\frac{d}{dx}V_{ij}(x) = -f_{ij}(x)$$

and then the total potential for q_i is defined as

$$V_i(t) = \sum_{j=1}^{n} V_{ij}(||q_i(t) - q_j(t)||).$$

Again for convenience, set $V_{ii} = 0$. Since $V_{ij} = V_{ji}$, we are lead to the definition of the total potential as

$$V(t) = \frac{1}{2}\sum_{j=1}^{n} V_j(t).$$

Similarly, the total kinetic is defined as

$$T(t) = \frac{1}{2}\sum_{j=1}^{n} m_i \dot{q}_i(t)^2$$

and thus, the total energy is defined as

$$E(t) = V(t) + T(t).$$

Note the total force on the i^{th} particle is:

$$F_i(t) = \sum_{j=1}^{n} F_{ij}(t).$$

Show that if Newton's 2nd law ($F_i(t) = m_i \ddot{q}_i(t)$) holds, then energy is conserved:

$$\frac{d}{dt}E = 0.$$

Solution: First note from the chain rule and our definition of V_{ij}, we have

$$\begin{aligned}\frac{d}{dt}V_{ij}(||x(t)||) &= \frac{d}{d||x(t)||}V_{ij}(||x(t)||)\frac{d||x(t)||}{dt} \\ &= -f_{ij}(||x(t)||)\frac{x(t)}{||x(t)||}\dot{x}(t).\end{aligned}$$

Therefore, from the above and Newton's 3$^{\text{rd}}$ law, we have

$$\begin{aligned}\frac{d}{dt}V_i(t) &= \frac{1}{2}\sum_{j=1}^{n}\frac{d}{dt}V_{ij}(||q_i(t)-q_j(t)||) \\ &= \frac{1}{2}\sum_{j=1}^{n}-f_{ij}(||q_i(t)-q_j(t)||)\frac{q_i(t)-q_j(t)}{||q_i(t)-q_j(t)||}(\dot{q}_i(t)-\dot{q}_j(t)) \\ &= \frac{1}{2}\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)(\dot{q}_i(t)-\dot{q}_j(t)) \\ &= \frac{1}{2}\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)\dot{q}_i(t)+\frac{1}{2}\sum_{j=1}^{n}F_{ij}(||q_i(t)-q_j(t)||)\dot{q}_j(t) \\ &= \frac{1}{2}\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)\dot{q}_i(t)+\frac{1}{2}\sum_{j=1}^{n}-F_{ji}(||q_i(t)-q_j(t)||)(\dot{q}_j(t)\end{aligned}$$

and so

$$\begin{aligned}\frac{d}{dt}V(t) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)\dot{q}_i(t)+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}-F_{ji}(||q_i(t)-q_j(t)||)\dot{q}_j(t) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)\dot{q}_i(t)+\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}-F_{ji}(||q_i(t)-q_j(t)||)\dot{q}_j(t) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}-F_{ij}(||q_i(t)-q_j(t)||)\right)\dot{q}_i(t)+\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}-F_{ji}(||q_i(t)-q_j(t)||)\right)\dot{q}_j(t) \\ &= \frac{1}{2}\sum_{i=1}^{n}-F_i(t)\dot{q}_i(t)+\frac{1}{2}\sum_{j=1}^{n}-F_j(t)\dot{q}_j(t) \\ &= \sum_{i=1}^{n}-F_i(t)\dot{q}_i(t).\end{aligned}$$

Therefore, by Newton's 2$^{\text{nd}}$ law

$$\begin{aligned}\frac{d}{dt}V(t) &= \sum_{i=1}^{n}-m_i\dot{q}_i(t)\ddot{q}_i(t) \\ &= -\frac{d}{dt}T(t),\end{aligned}$$

and thus

$$\frac{d}{dt}E=0.$$

Conservation of Angular Momentum in the n-body Problem

Let $p_i(t) = m_i\dot{q}(t)$ be the momentum of the i^{th} particle. We've shown that Newton's 2^{nd} law gives us conservation of momentum. The angular momentum of the i^{th} particle is

$$J_i(t) = q_i(t) \times p_i(t).$$

The total angular momentum is:

$$J(t) = \sum_{i=1}^{n} J_i(t).$$

Show that Newton's 2^{nd} gives us conservation of angular momentum:

$$\frac{d}{dt}J(t) = 0.$$

Solution: We have

$$\begin{aligned}
\frac{d}{dt}J(t) &= \sum_{i=1}^{n} \frac{d}{dt} q_i(t) \times p_i(t) \\
&= \sum_{i=1}^{n} \dot{q}_i(t) \times p_i(t) + q_i(t) \times \dot{p}_i(t) \\
&= \sum_{i=1}^{n} q_i(t) \times F_i(t) \\
&= \sum_{i=1}^{n} q_i(t) \times \left(\sum_{j=1}^{n} F_{ij}(||q_i(t) - q_j(t)||) \right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} q_i(t) \times F_{ij}(||q_i(t) - q_j(t)||) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(q_i(t) \times \frac{f_{ij}(||q_i(t) - q_j(t)||)}{||q_i(t) - q_j(t)||} q_i(t) - q_i(t) \times \frac{f_{ij}(||q_i(t) - q_j(t)||)}{||q_i(t) - q_j(t)||} q_j(t) \right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} -q_i(t) \times \frac{f_{ij}(||q_i(t) - q_j(t)||)}{||q_i(t) - q_j(t)||} q_j(t) \\
&= 0
\end{aligned}$$

where the last equality follows from the identity:

$$-q_i(t) \times \frac{f_{ij}(||q_i(t) - q_j(t)||)}{||q_i(t) - q_j(t)||} q_j(t) = q_j(t) \times \frac{f_{ji}(||q_i(t) - q_j(t)||)}{||q_i(t) - q_j(t)||} q_i(t).$$