Classical Mechanics Homework

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Homework 1

Solve Newton's second law $F = ma$ for $q(t) \in \mathbb{R}^3$ for

$$F(t) = (0, 0, -mg),$$

the force felt by a particle near the Earth's surface only under the influence of gravity. Find $q(t)$ in terms of the initial position $q(0)$ and $\dot{q}(0)$.

Solution

In this case, Newton's second law $F = ma = m\ddot{q}$ just says that

$$\ddot{q} = \frac{F}{m} = (0, 0, -g).$$

In other words, acceleration $\ddot{q}$ is a constant. Since there's no increase in difficulty, we'll just solve all constant acceleration problems in any dimension, i.e. differential equations of the form

$$\ddot{q} = a$$

where $a \in \mathbb{R}^n$ is a constant. Then we'll set $n = 3$ and $a = (0, 0, -g)$ to solve this special case.

It's straightforward to integrate

$$\ddot{q} = a$$

once and get

$$\dot{q} = v + ta$$

where our constant of integration $v \in \mathbb{R}^n$ is seen, upon setting $t = 0$, to be the initial velocity

$$\dot{q}(0) = v$$

so we have

$$\dot{q} = \dot{q}(0) + ta$$

thus far, and we integrate again to get

$$q = x + t\dot{q}(0) + \frac{1}{2}t^2 a$$

where, just as before, our constant of integration $x \in \mathbb{R}^n$ is seen, upon setting $t = 0$, to be the initial positon

$$q(0) = x$$

So our full solution is thus

$$q = q(0) + t\dot{q}(0) + \frac{1}{2}t^2 a$$

Now back to our original problem, where $n = 3$ and $a = (0, 0, -g)$. Taking the initial position and velocity to be

$$q(0) = (x_0, y_0, z_0)$$

and

$$\dot{q}(0) = (v_x, v_y, v_z)$$

we get that our position at time t is

$$q(t) = (x_o + v_x t, y_o + v_y t, z_o + v_z t - \frac{1}{2}gt^2)$$

in terms of its components. In particular, if we take the z-axis to point in the vertical direction, we have that the particle's height at time t is

$$q_z(t) = z_o + v_z t - \frac{1}{2}gt^2,$$

the familiar formula from any freshman physics text.

Homework 2

Solve Newton's second law for $q(t) \in \mathbb{R}$ when the force is given by Hooke's law

$$F(t) = -kq(t)$$

in terms of m, k, $q(0)$, and $\dot{q}(0)$, and then find the period P of the oscillation and the frequency ω, where $\omega = \dfrac{2\pi}{P}$

Solution

Plugging this force into Newton's second law, we get

$$\ddot{q} = -\frac{k}{m}q$$

or

$$\ddot{q} + \frac{k}{m}q = 0$$

a slightly more difficult differential equation than the one in the first problem.

To solve it, we turn to the arcane techniques of the theory of ordinary differential equations: we find the **characteristic polynomial** of this equation,

$$\lambda^2 + \frac{k}{m},$$

which is the polynomial that looks like the differential equation, if we replace each nth derivative of q with the nth power of λ. Now we find the roots of this polynomial,

$$\lambda = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

where we've defined $\omega = \sqrt{\frac{k}{m}}$. Since the characteristic polynomial has no repeated roots, the most general complex-valued solution to our differential equation is of the form

$$\tilde{q}(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

where A and B are complex constants to be determined by initial conditions. But we don't want a complex-valued solution; we want a real solution. This is easy to obtain, because the real part of $\tilde{q}$,

$$q(t) = \mathrm{Re}(\tilde{q}(t) = A' \cos(\omega t) + B' \sin(\omega t)$$

where A' and B' are real constants, satisfies the same differential equation as $\tilde{q}$, the reason being that Re is a real linear operator from complex-valued funtions to real-valued functions that commutes with differentiation!

So now that we've found the most general solution,

$$q(t) = A \cos \omega t + B \sin \omega t,$$

it only remains to determine A and B. We have

$$q(0) = A \cos 0 + B \sin 0 = A$$

and

$$\dot{q}(0) = -\omega A \sin 0 + \omega B \cos 0 = \omega B.$$

Our solution is thus

$$q(t) = q(0) \cos \omega t + \frac{\dot{q}(0)}{\omega} \sin \omega t.$$

Both of the trig functions in our solution have period

$$P = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

so q also has this period. q therefore oscillates with frequency

$$\omega = \sqrt{\frac{k}{m}}.$$