The Euclidean Group

1 $f_{R,u}(f_{R',u'}(x)) = f_{R,u}(R'x + u') = R(R'x + u') + u = (RR')x + (Ru' + u) = f_{RR',Ru'+u}(x)$, so $(R'', u'') = (RR', Ru' + u)$ will work and is uniquely determined by the composed function.

2 If $x' = f_{R,u}(x) = Rx + u$, then $x = R^{-1}(x' - u) = R^{-1}x' + (-R^{-1}u) = f_{R^{-1},-R^{-1}u}(x)$, so $(R', u') = (R^{-1}, -R^{-1}u)$ will work and is uniquely determined by the composed function.

3 There is of course a group of all invertible functions on $\mathbb{R}^n$, the permutation group $\mathbb{R}^n!$. The previous problems prove that $\mathrm{E}(n)$ is a subset of $\mathbb{R}^n!$ that is closed under the operations of composition and inversion. Furthermore, $f_{1,0}(x) = 1x + 0 = x$, so $\mathrm{E}(n)$ also owns the identity function. Therefore, $\mathrm{E}(n)$ is a subgroup of $\mathbb{R}^n!$, in particular a group.

4 G is constructed as the range of a function from $\mathbb{R}^n$ to $\mathrm{E}(n)$, which is invertible by the fundamental property of ordered pairs. First, $(1, u)(1, u') = (11, 1u' + u) = (1, u + u')$, so G is closed under multiplication and the correspondence between G and $\mathbb{R}^n$ preserves that operation. Next, $(1, u)^{-1} = (1^{-1}, -1^{-1}u) = (1, -u)$, so G is closed under inverses and the correspondence between G and $\mathbb{R}^n$ preserves that operation. Finally, the identity element of $\mathrm{E}(n)$ is $(1, 0)$, so G owns the identity element and it corresponds to the identity element 0 of $\mathbb{R}^n$. Therefore, G is a subgroup of $\mathrm{E}(n)$ that is isomorphic to $\mathbb{R}^n$. Also, $(R, u') \times (1, u)(R, u')^{-1} = (R1, Ru + u')(R^{-1}, -R^{-1}u') = (RR^{-1}, -RR^{-1}u' + Ru + u') = (1, Ru)$, which belongs to G, so the subgroup G is normal.

H is constructed as the range of a function from $\mathrm{O}(n)$ to $\mathrm{E}(n)$, which is invertible by the fundamental property of ordered pairs. First, $(R, 0)(R', 0) = (RR', R0 + 0) = (RR', 0)$, so H is closed under multiplication and the correspondence between H and $\mathrm{O}(n)$ preserves that operation. Next, $(R, 0)^{-1} = (R^{-1}, -R^{-1}0) = (R^{-1}, 0)$, so H is closed under inverses and the correspondence between H and $\mathrm{O}(n)$ preserves that operation. Finally, the identity element of $\mathrm{E}(n)$ is $(1, 0)$, so H owns the identity element and it corresponds to the identity element 1 of $\mathrm{O}(n)$. Therefore, H is a subgroup of $\mathrm{E}(n)$ that is isomorphic to $\mathrm{O}(n)$.

$(1, u)(R, 0) = (1R, 10 + u) = (R, u)$, so by the fundamental property of ordered pairs, every element of $\mathrm{E}(n)$ is a unique product of an element of G and an element of H.

5 Let u be $f(0)$, and let Rx be $f(x) - u$ for $x \in \mathbb{R}^n$. Note that $R0 = u - u = 0$ and

$$|Rx - Ry| = |(f(x) - u) - (f(y) - u)| = |f(x) - f(y)| = |x - y|,$$

so R preserves the origin and lengths. Therefore, R must be an element of $\mathrm{O}(n)$. Since $f(x) = Rx + u$, the desired result follows.

The Galilei Group

6 Set $f = f_{R,u}$ and $f' = f_{R',u'}$. Then

$$\begin{aligned} F_{f,v,s}(F_{f',v',s'}(x,t)) &= F_{f,v,s}(R'x + u' + v't, t + s') = (R(R'x + u' + v't) + u + v(t + s'), t + s' + s) \\ &= ((RR')x + (Ru' + u + vs') + (Rv' + v)t, t + (s' + s)) = F_{f_{RR',Ru'+u+vs'},Rv'+v,s'+s}, \end{aligned}$$

so $(f'', v'', s'') = (f_{RR',Ru'+u+vs'}, Rv' + v, s' + s)$ will work and is uniquely determined by the composed function.

7 Set $f = f_{R,u}$. Then if $(x', t') = F_{f,v,s}(x, t) = (f(x) + vt, t + s) = (Rx + u + vt, t + s)$, then $t = t' - s = t' + (-s)$ and $x = R^{-1}(x' - u - vt) = R^{-1}(x' - u - v(t' - s)) = R^{-1}x' + (sR^{-1}v - R^{-1}u) + (-R^{-1}v)t'$, so $(x, t) = F_{f_{R^{-1},sR^{-1}v-R^{-1}u},-R^{-1}v,-s}$, so $(f', v', s') = (f_{R^{-1},sR^{-1}v-R^{-1}u}, -R^{-1}v, -s)$ will work and is uniquely determined by the composed function.

8 There is of course a group of all invertible functions on $\mathbb{R}^{n+1}$, the permutation group $\mathbb{R}^{n+1}!$. The previous problems prove that $\mathrm{G}(n + 1)$ is a subset of $\mathbb{R}^{n+1}!$ that is closed under the operations of composition and inversion. Furthermore, $F_{1,0,0}(x, t) = (x + 0t, t + 0) = (x, t)$, so $\mathrm{G}(n + 1)$ also owns the identity function. Therefore, $\mathrm{G}(n + 1)$ is a subgroup of $\mathbb{R}^{n+1}!$, in particular a group.

9 Given points $p, q, r \in \mathbb{R}^{n+1}$, define vectors $v = q - p$ and $w = r - p$ tangent to $\mathbb{R}^{n+1}$ at p. Write $v = (v_x, v_t)$ and $w = (w_x, w_t)$. I claim that Galilean transformations preserve the time difference $w_t - v_t$ and the relative speed $|w_x/w_t - v_x/v_t|$. Now, the latter is not defined if v_t or w_t is 0, but since these denominators are also time differences, $|v_t w_x - w_t v_x|$ should be preserved as well. Furthermore, I claim that any transformation that preserves these is Galilean. Thus in terms of the original points $p = (p_x, p_t)$, $q = (q_x, q_t)$, and $r = (r_x, r_t)$, the claimed complete invariant is the ordered pair $(|q_t r_x - p_t r_x - r_t q_x + p_t q_x + r_t p_x - q_t p_x|, r_t - q_t)$.

First, let me verify that this is an invariant. Applying $F_{f,v,s}$, where $f = f_{R,u}$, the first component of the claimed invariant becomes

$$\begin{aligned}&|(q_t + s)(Rr_x + u + vr_t) - (p_t + s)(Rr_x + u + vr_t) - (r_t + s)(Rq_x + u + vq_t)\\&\quad + (p_t + s)(Rq_x + u + vq_t) + (r_t + s)(Rp_x + u + vp_t) - (q_t + s)(Rp_x + u + vp_t)|\\&= |q_t Rr_x - p_t Rr_x - r_t Rq_x + p_t Rq_x + r_t Rp_x - q_t Rp_x|\\&= |R(q_t r_x - p_t r_x - r_t q_x + p_t q_x + r_t p_x - q_t p_x)|\\&= |q_t r_x - p_t r_x - r_t q_x + p_t q_x + r_t p_x - q_t p_x|,\end{aligned}$$

since $R \in \mathrm{O}(n)$ is linear and preserves lengths. Meanwhile, the second component becomes $(r_t + s) - (q_t + s) = r_t - q_t$. Thus, this is indeed an invariant.

Now suppose that $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ preserves this invariant. Let s be the second component of $F(0,0)$. Setting $q = (0,0)$ and $r = (x,t)$, I see from the second component of the invariant that $t + s$ is the second component of $F(x,t)$. Thus let $F_x(x,t)$ be the first component of $F(x,t)$, so that $F(x,t) = (F_x(x,t), t + s)$. Now let u be $F_x(0,0)$, let v be $F_x(0,1) - u$, and let Rx be $F_x(x,1) - u - v$ for $x \in \mathbb{R}^n$. Note that $R0 = (v + u) - u - v = 0$. Setting $p = (0,0)$, $q = (y,1)$, and $r = (x,1)$,

$$\begin{aligned}|Rx - Ry| &= |(F_x(x,1) - u - v) - (F_x(y,1) - u - v)| = |F_x(x,1) - F_x(y,1)|\\&= |(1+s)F_x(x,1) - sF_x(x,1) - (1+s)F_x(y,1) + sF_x(y,1) + (1+s)u - (1+s)u|.\end{aligned}$$

Applying the first component of the invariant, this equals $|1x - 0x - 1y + 0y + 1\cdot 0 - 1\cdot 0| = |x - y|$. Thus, R preserves the origin and lengths, so it must be an element of $\mathrm{O}(n)$. Now setting $p = (0,0)$, $q = (x/t, 1)$, and $r = (x,t)$, the first component of the invariant says that

$$\begin{aligned}&|1x - 0x - t(x/t) + 0(x/t) + t0 - 1\cdot 0|\\&\quad = |(1+s)F_x(x,t) - sF_x(x,t) - (t+s)(R(x/t) + v + u) + s(R(x/t) + v + u) + (t+s)u - (1+s)u|,\end{aligned}$$

or $0 = |F_x(x,t) - tR(x/t) - tv - u|$. Since every element of $\mathrm{O}(n)$ is linear, it follows that $F_x(x,t)$ must be $(Rx + u + vt, t + s)$. Thus $F = F_{f_{R,u},v,s} \in \mathrm{G}(n+1)$.

The Free Particle

Since our Galilean transformations have been passive, not active, time translation acts as $s(x,p) = (q - sp/m, p)$, not $(q + sp/m, p)$.

10 In $\mathrm{G}(n+1)$, $(f,v,s) = (1,v,s)(f,0,0) = (1,0,s)(1,v,0)(f,0,0)$, while I already know that $(R,u) = (1,u) \times (0,R)$ in $\mathrm{E}(n)$. That is, $(R,u,v,s) = (1,0,0,s)(1,0,v,0)(1,u,0,0)(R,0,0,0)$. Then

$$\begin{aligned}(R,u,v,s)(q,p) &= (1,0,0,s)(1,0,v,0)(1,u,0,0)(R,0,0,0)(q,p) = (1,0,0,s)(1,0,v,0)(1,u,0,0)(Rq,Rp)\\&= (1,0,0,s)(1,0,v,0)(Rq + u, Rp) = (1,0,0,s)(Rq + u, Rp + mv)\\&= (Rq + u - s(Rp + mv)/m, Rp + mv) = (Rq + u - sRp/m - sv, Rp + mv).\end{aligned}$$

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$$\begin{aligned}&(R,u,v,s)\big((R',u',v',s')(q,p)\big) = (R,u,v,s)(R'q + u' - s'R'p/m - s'v', R'p + mv')\\&\quad = (R(R'q + u' - s'R'p/m - s'v') + u - sR(R'p + mv')/m - sv, R(R'p + mv') + mv)\\&\quad = (RR'q + Ru' - s'RR'p/m - s'Rv' + u - sRR'p/m - sRv' - sv, RR'p + mRv' + mv)\\&\quad = (RR'q + Ru' + u + s'v - s'RR'p/m - sRR'p/m - s'Rv' - s'v - sRv' - sv, RR'p + mRv' + mv)\\&\quad = ((RR')q + (Ru' + u + s'v) - (s' + s)(RR')p/m - (s' + s)(Rv' + v), (RR')p + m(Rv' + v))\\&\quad = (RR', Ru' + u + s'v, Rv' + v, s' + s)(q,p) = \big((R,u,v,s)(R',u',v',s')\big)(q,p).\end{aligned}$$

Also, if $(q',p') = (R,u,v,s)(q,p) = (Rq + u - sRp/m - sv, Rp + mv)$, then $p = R^{-1}(p' - mv) = R^{-1}p' - mR^{-1}v = R^{-1}p' + m(-R^{-1}v)$ and

$$\begin{aligned} q &= R^{-1}(q' - u + sRp/m + sv) = R^{-1}(q' - u + sR(R^{-1}p' - mR^{-1}v)/m + sv) \\ &= R^{-1}q' - R^{-1}u + sR^{-1}p'/m - sv + sv = R^{-1}q' + sR^{-1}v - R^{-1}u + sR^{-1}p'/m - sR^{-1}v \\ &= R^{-1}q' + (sR^{-1}v - R^{-1}u) - (-s)R^{-1}p'/m - (-s)(-R^{-1}v), \end{aligned}$$

so $(q,p) = (R^{-1}, R^{-1}sv - R^{-1}u, -R^{-1}v, -s)(q',p') = (R,u,v,s)^{-1}(q',p')$. Finally,

$$(1,0,0,0)(q,p) = (1q + 0 - 0 \cdot 1p/m - 0 \cdot 0, 1p + m0) = (q,p).$$

Therefore, this is an action of $\mathrm{G}(n+1)$ on $\mathbb{R}^{2n}$.