Classical Mechanics Homework
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The Euclidean Group

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map that preserves distances:

$$|f(x) - f(y)| = |x - y|$$

for all $x, y \in \mathbb{R}^n$. Show that

$$f(x) = R(x) + u$$

for some $(R, u) \in E(n)$.

First, let us define

$$L(x) = f(x) - f(0)$$

so that L is an isometry with $L(0) = 0$. Suppose that x, y and z are distinct, colinear, and further that

$$y = x + s(z - x) \qquad \text{and} \qquad y = z + (1 - s)(x - z)$$

for some $s \in (0, 1)$. (That is to say that y is somewhere *between* x and z.) Now, we note that the above representations of y give

$$|x - y| = s|z - x| \qquad \text{and} \qquad |z - y| = (1 - s)|z - x|$$

from which it directly follows that

$$|L(x) - L(y)| = s|L(z) - L(x)| \qquad \text{and} \qquad |L(z) - L(y)| = (1 - s)|L(z) - L(x)|,$$

and consequently

$$|L(x) - L(y)| + |L(z) - L(y)| = |L(z) - L(x)|.$$

The above equality says a number of things. The first thing it gives us is that $L(x), L(y)$ and $L(z)$ are distinct (L is clearly an injection) colinear points: noncolinear points could not generate equality due to the triangle inequality. The second thing it tells us is that $L(y)$ lies between $L(x)$ and $L(z)$. If $L(y)$ was not between these two points then either

$$|L(x) - L(y)| \qquad \text{or} \qquad |L(z) - L(y)|$$

would be strictly larger than $|L(x) - L(z)|$ contradicting the equality. Finally, we can conclude from the facts that $L(y)$ lies on the line between $L(x)$ and $L(z)$ and that $L(y)$ is s (parameter) units away from $L(x)$ gives

$$L(y) = L(x) + s(L(z) - L(x)), \tag{1}$$

and we have cleared the main obstacle to the remainder of the argument.

The remainder of the argument will center around unraveling (1). Take a nonzero $x \in \mathbb{R}^n$. Then $x, 0$ and $-x$ are distinct colinear points. From (1) we see that

$$L(0) = L(-x) + (1/2)(L(x) - L(-x)),$$

or more succinctly (since $L(0) = 0$):

$$L(-x) = -L(x). \tag{2}$$

Further, if $r \in (0, 1)$ and x is again some nonzero vector, then $0, rx$ and x are distinct colinear points with

$$L(rx) = L(0) + r(L(x) - L(0)) = rL(x).$$

If $r > 1$, then using the above we have

$$L(x) = L(r^{-1}(rx)) = r^{-1}L(rx).$$

The above in conjunction with (2) show that for any $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$ we get

$$L(rx) = rL(x). \tag{3}$$

Finally, take x and y distinct. Then (1) gives

$$L((x + y)/2) = L(x) + (1/2)(L(y) - L(x)),$$

which, when combined with (3) yields

$$L(x + y) = L(x) + L(y). \tag{4}$$

Equations (4) and (3) show that L is *linear*.

The function L is linear, so there is a matrix $R \in M_n(\mathbb{R})$ such that $L(x) = Rx$. Recall the polarization identity for the inner product on $\mathbb{R}^n$:

$$(u, v) = (1/4)(|u + v|^2 - |u - v|^2).$$

The fact that L is an isometry gives:

$$(Rx, Ry) = (x, y)$$

for all $x, y \in \mathbb{R}^n$. But then:

$$(x, (R^T R - I)y) = 0$$

for all x and y, and so $R^T R = I$ and we see that $R \in O(n)$.

We wrap all this up by noting that

$$f(x) = L(x) + f(0) = Rx + f(0)$$

and we wipe our hands and call it a day.

The Galilei Group

We define a map

$$i : G(n+1) \to \mathcal{M}$$

where $\mathcal{M}$ denotes the collection of functions from $\mathbb{R}^{n+1}$ to itself, by

$$X = (R, u, v, s) \mapsto F_X \qquad \text{with} \qquad F_X(x,t) = (Rx + u + tv, s + t).$$

Let's show that this map is an injection. Given $X = (R, u, v, s)$ and $Y = (R', u', v', s')$ with $i(X) = i(Y)$ we see that

$$F_X(0,0) = (u, s) = (u', s') = F_Y(0,0)$$

so $u = u'$ and $s = s'$. Computing

$$F_X(x,0) = (Rx + u, s) = (R'x + u', s') = F_Y(x,0)$$

we see that $Rx = R'x$ for all $x \in \mathbb{R}^n$ and we have $R = R'$. Finally, noting the previously obtained equalities together with the fact that $F_X(0,1) = F_Y(0,1)$ yields $v = v'$. Thus $X = Y$ and our map is an injection. We will henceforth identify $G(n+1)$ with its image in $\mathcal{M}$.

5. Given two elements $X = (R, u, v, s), Y = (R', u', v', s') \in G(n+1)$, compute

$$F_X \circ F_Y$$

and show that the composition is again in $G(n+1)$.

Back to computation:

$$\begin{aligned} F_X \circ F_Y(x,t) &= F_X(R'x + u' + tv', s' + t) \\ &= (R(R'x + u' + tv') + u + (s' + t)v, s + s' + t) \\ &= (RR'x + (Ru' + u + s'v) + t(Rv' + v), (s + s') + t) \\ &= F_Z(x,t) \end{aligned}$$

where

$$Z = (RR', Ru' + u + s'v, Rv' + v, s + s').$$

Thus $F_X \circ F_Y$ is in the image of $G(n+1)$ in $\mathcal{M}$. Because of remarks made above, Z must be unique.

6. Compute Y^{-1} for $Y \in G(n+1)$.

We are looking for a $X \in G(n+1)$ such that F_X satisfies $F_Y \circ F_X = (I, 0, 0, 0) = F_X \circ F_Y$. Using the notation above for X and Y this amounts to solving the following system of equations for R, u, v and s.

$$\begin{aligned} &RR' = I \\ &Ru' + u + s'v = 0 \\ &Rv' + v = 0 \\ &s + s' = 0 \end{aligned}$$

The solutions are readily obtained to be $R = R'^T$, $s = -s'$, $v = -R'^T v'$, and $u = -R'^T u' + s' R'^T v'$. So we claim that

$$(R,u,v,s)^{-1} = (R^T, -R^T u + sR^T v, -R^T v, -s).$$

A quick check (using the formula for Z in the above problem) shows that this is indeed the (double sided) inverse.

7. Show that $G(n+1)$ is a group.

We note that $G(n+1)$ is a subset of the group $\mathcal{M}$ (the collection of all mappings from $\mathbb{R}^{n+1}$ to itself) which is stable under composition (see problem number 5). Since every element X in $G(n+1)$ has an inverse mapping (see problem 6) X^{-1} in $G(n+1)$, then for X and Y in $G(n+1)$ we have $XY^{-1} \in G(n+1)$. It follows that $G(n+1)$ is a subgroup of $\mathcal{M}$.

The Free Particle

9 & 10. Construct a Galilean group action on $X = \mathbb{R}^n \times \mathbb{R}^n$.

Unless I'm missing something here, it seems like we have yet to define the Galilei group directly. In problems 5, 6, and 7 the Galilei group was defined implicitly by *first describing how it acted on* Galilean spacetime and from this action reverse engineering a group which accomplished said action. This technique (which was also employed for the Euclidean group) works through an identification of a given set of elements, say quadruples (R,u,v,s) of some prescribed form, with an (albeit very special) subgroup of the semigroup of all mappings on the set on which we wish to act. This last bit is very clever (not to mention fishy, sinister, and irritating from a pedagogical perspective), since *mapping subgroups come equipped with a free action*! Namely, given f and g, two maps from some space X to itself, we have by the definition of composition:

$$(f \circ g)(x) = f(g(x))$$

which is exactly the associativity portion of the action definition. Further, since we are working with a subgroup of mappings, the subgroup contains an identity map I and clearly

$$I(x) = x.$$

I must cry foul! From my ignorant position it seems that we have not been given the Galilei group, but rather a *certain collection of its actions*.

As things stand then, the real problem here seems *not* to be how to define the action, but rather, how to define the Galilei group! And since I am far too lazy for such things I will now do exactly that (sort of, indirectly, not really):

Let $Aut(X)$ denote the group of bijections from X to itself. You have given us four actions:

$$\begin{aligned} u = \pi_1 &: \mathbb{R}^n \to Aut(X) \\ R = \pi_2 &: O(n) \to Aut(X) \\ v = \pi_3 &: \mathbb{R}^n \to Aut(X) \\ s = \pi_4 &: \mathbb{R} \to Aut(X). \end{aligned}$$

It should be noted that all four actions are actually imbeddings. Further, the images of the actions have the property:

$$\operatorname{Im}(\pi_i) \bigcap \prod_{i \neq j} \operatorname{Im}(\pi_j) = \{I\}.$$

To see this, note that π_1 gives maps that shift the first coordinate of (q, p) by a fixed vector, π_3 gives maps that shift the second by a fixed vector and π_4 yields maps which mix the two coordinates so that both q *and* p appear in the first slot in the output; π_2 gives maps which *rotate* both coordinates. It is simply impossible for any three types of these of maps to accomplish the operation of the remaining type of map unless all maps being considered are the identity map! Think this through a bit...

Define the set $G = \mathbb{R}^n \times O(n) \times \mathbb{R}^n \times \mathbb{R}$. Let σ be any permutation on the set of four elements and define $F_\sigma : G \to Aut(X)$ by

$$F_\sigma(x_1, \dots, x_n) = \pi_{\sigma(1)}(x_{\sigma(1)}) \circ \cdots \circ \pi_{\sigma(4)}(x_{\sigma(4)}).$$

Suppose

$$F_\sigma(x_1, \dots, x_n) = F_\sigma(y_1, \dots, y_n).$$

Then it would follow that

$$\pi_{\sigma(1)}\left(x_{\sigma(1)}y^{-1}_{\sigma(1)}\right) \circ \cdots \circ \pi_{\sigma(4)}\left(x_{\sigma(4)}y^{-1}_{\sigma(4)}\right) = I.$$

Now, the above requires that the inverse of any given factor lie in the product of the images of the other three. By the argument about the images of the π_i, this is impossible unless all of the factors are the identity map. But then $x_i = y_i$ since each π_i was an imbedding. Whence F_σ is an injection from G to $Aut(X)$.

Finally, give G group structure by pullback along F_σ.[1] [2] With this structure we see that $F_\sigma : G \to Aut(X)$ is now an action of G on X and further that G can be identified with its image in $Aut(X)$ which, by the opening comments of this diatribe, comes with a de facto action on X. Oh, based on the work I have done, here is the formula:

$$(R, u, v, s)(q, p) = F_\sigma(R, u, v, s)(q, p)$$

which we admit, is not very helpful. So it goes...

[1] What's really going on here is that we're saying it doesn't matter which particular way we choose to combine the actions—there will be a group structure on G which will be compatible with this action, and not only that, but will reduce to the original four actions when we consider, say $O(n)$ as a subgroup of G.

[2] What's *really* going on here (I think) is that we are implicitly defining the Galilei group to be any one of the isomorphic copies of the semi direct product of $\mathbb{R}^n$, $\mathbb{R}^n$, $\mathbb{R}$ and $O(n)$ found in $AutX$.