The Euclidean Group, the Galilei Group and the Free Particle

Math 241 Homework
John Baez

This homework gives you some choice in which problems you do, depending on how ambitious you feel. In the section on the Euclidean group you can either do problems 1–4, or if you're feeling more ambitious, problem 5. In the section on the Galilei group you can either do problems 6–8, or if you're feeling more ambitious, problem 9. Everyone has to do problems 10–11.

The Euclidean Group

Define an element of the **Euclidean group** $\mathrm{E}(n)$ to be a pair (R, u), where $R \in \mathrm{O}(n)$ is a linear transformation of $\mathbb{R}^n$ with $RR^* = 1$, and $u \in \mathbb{R}^n$. Any element (R, u) gives a transformation of n-dimensional Euclidean space

$$f_{(R,u)} \colon \mathbb{R}^n \to \mathbb{R}^n$$

defined by

$$f_{(R,u)}(x) = Rx + u.$$

The map $f_{(R,u)}$ uniquely determines R and u, so we can also think of $\mathrm{E}(n)$ as a set of maps.

1. Given two elements $(R, u), (R', u') \in \mathrm{E}(n)$ show that

$$f_{(R,u)} \circ f_{(R',u')} = f_{(R'',u'')}$$

for some unique $(R'', u'') \in \mathrm{E}(n)$. Work out the explicit formula for (R'', u'').

This formula lets us define a 'multiplication' operation on $\mathrm{E}(n)$ *by:* $(R, u)(R', u') = (R'', u'')$.

2. Given an element $(R, u) \in \mathrm{E}(n)$ show that

$$f^{-1}_{(R,u)} = f_{(R',u')}$$

for some unique $(R', u') \in \mathrm{E}(n)$. Work out the explicit formula for (R', u').

This formula lets us define an 'inverse' operation on $\mathrm{E}(n)$ *by:* $(R, u)^{-1} = (R', u')$.

3. Using these formulas for the multiplication and inverse operations on $\mathrm{E}(n)$, show that $\mathrm{E}(n)$ becomes a group. (Hint: the good way to do this requires almost no calculation.)

Note that as a set we have $\mathrm{E}(n) = \mathrm{O}(n) \times \mathbb{R}^n$. *However, as a group* $\mathrm{E}(n)$ *is not the direct product of the groups* $\mathrm{O}(n)$ *and* $\mathbb{R}^n$, *because the formulas for multiplication and inverse are not just*

$$(R, u)(R', u') = (RR', u + u'), \qquad (R, u)^{-1} = (R^{-1}, -u).$$

Instead, the formulas involve the action of $\mathrm{O}(n)$ *on* $\mathbb{R}^n$, *so we say* $\mathrm{E}(n)$ *is a 'semidirect' product of* $\mathrm{O}(n)$ *and* $\mathbb{R}^n$. *More precisely:*

4. Show that the translations

$$G = \{(1, u) \colon u \in \mathbb{R}^n\} \subseteq \mathrm{E}(n)$$

form a normal subgroup of the Euclidean group that is isomorphic to $\mathbb{R}^n$. Show that the rotations/reflections

$$H = \{(R, 0) \colon R \in \mathrm{O}(n)\} \subseteq \mathrm{E}(n)$$

form a subgroup of the Euclidean group that is isomorphic to $\mathrm{O}(n)$. Also show that that every element of $\mathrm{E}(n)$ is of the form gh for a unique $g \in G$ and $h \in H$.

Whenever we have a group with a normal subgroup G and a subgroup H such that every group element is of the form gh for a unique $g \in G$ and $h \in H$, we call this group **a semidirect product** *of G and H and write it as $G \rtimes H$. In this situation there is always an action of H on G by conjugation, and the formulas for multiplication and inverse in $G \rtimes H$ depend on this action. That's why we speak of 'a' semidirect product rather than 'the' semidirect product. The direct product is a special case of a semidirect product.*

5. Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is a map that preserves distances:

$$|f(x) - f(y)| = |x - y|$$

for all $x, y \in \mathbb{R}^n$. Show that

$$f(x) = Rx + u$$

for some $(R, u) \in \mathrm{E}(n)$. Thus we can more elegantly define the Euclidean group to be the group of all distance-preserving transformations of Euclidean space!

The Galilei Group

Define an element of the **Galilei group** $\mathrm{G}(n+1)$ to be an triple (f, v, s) where $f \in \mathrm{E}(n)$, $v \in \mathbb{R}^n$ and $s \in \mathbb{R}$. Any element (f, v, s) gives a transformation of $(n+1)$-dimensional spacetime

$$F_{(f,v,s)} \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

defined by

$$F_{(f,v,s)}(x, t) = (f(x) + vt, t + s)$$

for all $(x, t) \in \mathbb{R}^{n+1}$. The map $F_{(f,v,s)}$ uniquely determines f, v and s, so we can also think of $\mathrm{G}(n+1)$ as a set of maps.

6. Given two elements $(f, v, s), (f', v', s') \in \mathrm{G}(n+1)$ show that

$$F_{(f,v,s)} \circ F_{(f',v',s')} = F_{(f'',v'',s'')}$$

for some unique $(f'', v'', s'') \in \mathrm{G}(n+1)$. Work out the explicit formula for (f'', v'', s'').

This formula lets us define a 'multiplication' operation on $\mathrm{G}(n+1)$ *by:* $(f, v, s)(f', v', s') = (f'', v'', s'')$.

7. Given an element $(f, v, s) \in \mathrm{G}(n+1)$ show that

$$F^{-1}_{(f,v,s)} = F_{(f',v',s')}$$

for some unique $(f', v', s') \in \mathrm{G}(n+1)$. Work out the explicit formula for (f', v', s').

This formula lets us define an 'inverse' operation on $\mathrm{G}(n+1)$ *by:* $(f, v, s)^{-1} = (f', v', s')$.

8. Using these formulas for the multiplication and inverse operations on $\mathrm{G}(n+1)$, show that $\mathrm{G}(n+1)$ becomes a group. (Again, the good way to do this requires almost no calculation.)

As a set we have $\mathrm{G}(n+1) = \mathrm{E}(n) \times \mathbb{R}^n \times \mathbb{R}$. *However, it is again not the direct product of these groups, but only a semidirect product.*

9. Describe some structure on $\mathbb{R}^{n+1}$ such that $\mathrm{G}(n+1)$ is precisely the group of all maps $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that preserve this structure. Prove that this is indeed the case.

More generally, we could axiomatically define an $(n+1)$*-dimensional* **Galilean spacetime** *and prove that the symmetry group of any such thing is isomorphic to* $\mathrm{G}(n+1)$.

The Free Particle

Recall that a group G **acts** on a set X if for any $g \in G$ and $x \in X$ we get an element $gx \in X$, and

$$g(g'x) = (gg')(x), \qquad 1x = x \tag{1}$$

for all $g, g' \in G$ and $x \in X$. We have just described how the Euclidean group acts on Euclidean space and how the Galilei group acts on Galilean spacetime. Now we will figure out how the Galilei group acts on the phase space of a free particle! Recall that the phase space of a particle in n-dimensional Euclidean space is $X = \mathbb{R}^n \times \mathbb{R}^n$, where a point $(q,p) \in X$ describes the particle's position and momentum. I will tell you how various subgroups of the Galilei group act on X, and you will use that information to figure out how the the whole group acts on X.

• The translation group $\mathbb{R}^n$ is a subgroup of $\mathrm{E}(n)$ and thus $\mathrm{G}(n+1)$ in an obvious way, and it acts on X as follows:

$$u(q,p) = (q+u, p) \qquad\qquad u \in \mathbb{R}^n.$$

In other words, to translate a particle we translate its position but leave its momentum alone!

• The rotation/reflection group $\mathrm{O}(n)$ is also a subgroup of $\mathrm{E}(n)$ and thus $\mathrm{G}(n+1)$ in an obvious way, and it acts on X as follows:

$$R(q,p) = (Rq, Rp) \qquad\qquad R \in \mathrm{O}(n).$$

In other words, to rotate a particle we rotate both its position and momentum!

• The group of Galilei boosts $\mathbb{R}^n$ is a subgroup of $\mathrm{G}(n+1)$ in an obvious way, and it acts on X as follows:

$$v(q,p) = (q, p+mv) \qquad\qquad v \in \mathbb{R}^n.$$

In other words, to boost a particle's velocity by v we add mv to its momentum but leave its position alone!

• Finally, the time translation group $\mathbb{R}$ is a subgroup of $\mathrm{G}(n+1)$ in an obvious way, and it acts on X as follows:

$$s(q,p) = (q - sp/m, p) \qquad\qquad s \in \mathbb{R}.$$

This is where we are assuming the particle is **free**. The sign here is a funny thing... (I SHOULD EXPLAIN THIS BETTER: stuff about active versus passive transformations).

10. Assuming that all these group actions fit together to define an action of the whole Galilei group on X, figure out how the whole Galilei group acts on X.

Hint: you'll probably want to use formula (1) and also some results from problems 1–4 and 6–8. An element of the Galilei group is a triple $(f, v, s) \in \mathrm{E}(n) \times \mathbb{R}^n \times \mathbb{R}$, but here it's best to think of it as a quadruple $(R, u, v, s) \in \mathrm{O}(n) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, using the fact that $f = (R, u)$. I want you to give me a formula like

$$(R, u, v, s)(q, p) = \cdots$$

11. Finally, check that you *really have defined an action* of $\mathrm{G}(n+1)$ on X. That is, check equation (1) for all $g = (R, u, v, s)$ and $h = (R', u', v', s')$ in the Galilei group and all $x = (q, p) \in X$.