Classical Mechanics Homework
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John Baez homework by Brian Rolle

The Euclidean Group

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map that preserves distances: $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$. Then $f(x) = Rx + u$ for some $(R, u) \in E(n)$.

Solution: Since $\dfrac{|f(x) - f(y)|}{|x - y|} = 1$, we have that the determinate of the derivative of f is ± 1. So $f(x) = Rx + u$ where $R \in \mathrm{GL}(n)$. To show R is orthogonal, we note that $4(Rx, Ry) = |Rx + Ry|^2 - |Rx - Ry|^2 = |R(x + y) - R(0)|^2 - |Rx - Ry|^2 = |x + y|^2 - |x - y|^2 = 4(x, y)$, where $(\quad,\quad)$ is the normal inner product on $\mathbb{R}^n$. So computation on an orthonormal basis yields that the matrix for R is orthogonal.

The Galilei Group

5. Given $((R, u), v, s), ((R', u'), v', s') \in G(n+1)$, then $F_{((R,u),v,s)} \circ F_{((R',u'),v',s')} = F_{(RR',Ru'+u,Rv'+v,s+s')}$.

Solution: Computation gives $F_{((R,u),v,s)} \circ F_{((R',u'),v',s')}(x, t) = F_{((R,u),v,s)}(R'x + u' + v't, t + s') =$

$$(R(R'x+u'+v't)+u+vt, t+s'+s) = (RR'x+Ru'+u+Rv't+vt, t+s'+s) = F_{(RR',Ru'+u,v'+v,s+s')}(x, t).$$

6. Given $((R, u), v, s) \in G(n + 1)$, then $F^{-1}_{((R,u),v,s)} = F_{((R^*,-R^*u),-R^*v,-s)}$.

Solution: The computation gives $F_{((R,u),v,s)} \circ F_{((R^*,-R^*u),-R^*v,-s)}(x, t) =$ $(RR^*x - RR^*u + u - RR^*vt + vt, t - s + s) = (x, t)$. Similarly $F_{((R^*,-R^*u),-R^*v,-s)} \circ F_{((R,u),v,s)}(x, t) = (x, t)$.

7. With multiplication and inverses as defined, $G(n + 1)$ is a group.

Solution: Since the set of all invertible functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a group under composition, and since $G(n + 1)$ is nonempty, closed under composition and contains all inverses, it is subgroup, and therefore a group.