

# The Kepler Problem Revisited: The Runge–Lenz Vector

Math 241 Homework  
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Whenever we have two particles interacting by a central force in 3d Euclidean space, we have conservation of energy, momentum, and angular momentum. However, when the force is gravity — or more precisely, whenever the force goes like  $1/r^2$  — there is an extra conserved quantity. This is often called the **Runge–Lenz vector**, but it was originally discovered by Laplace. Its existence can be seen in the fact that in the gravitational 2-body problem, each particle orbits the center of mass in an ellipse (or parabola, or hyperbola) whose *perihelion does not change with time*. The Runge–Lenz vector points in the direction of the perihelion! If the force went like  $1/r^{2.1}$ , or something like that, the orbit could still be roughly elliptical, but the perihelion would ‘precess’.

Indeed, the first piece of experimental evidence that Newtonian gravity was not quite correct was the precession of the perihelion of Mercury. Most of this precession is due to the pull of other planets and other effects, but about 43 arcseconds per century remained unexplained until Einstein invented general relativity.

In fact, we can use the Runge–Lenz vector to simplify the proof that gravitational 2-body problem gives motion in ellipses, hyperbolas or parabolas. Here’s how it goes. As before, let’s work with the relative position vector

$$\mathbf{q}(t) = \mathbf{q}_1(t) - \mathbf{q}_2(t)$$

where  $\mathbf{q}_1, \mathbf{q}_2: \mathbb{R} \rightarrow \mathbb{R}^3$  are the positions of the two bodies as a function of time. In what follows we will use  $q$  to stand for the magnitude of the vector  $\mathbf{q}$ , and  $\hat{\mathbf{q}}$  to stand for a normalized vector that points in the direction of  $\mathbf{q}$ :

$$q = |\mathbf{q}|, \quad \hat{\mathbf{q}} = \frac{\mathbf{q}}{q}.$$

In a previous homework we saw that  $\mathbf{q}(t)$  satisfies

$$m\ddot{\mathbf{q}} = f(q)\hat{\mathbf{q}}$$

where  $f$  is the force as a function of distance, and  $m$  is the so-called ‘reduced mass’. Since the force of gravity goes like  $1/r^2$ , we have

$$f(q) = -k/q^2$$

where  $k$  is the same constant as in the previous homework. We thus have

$$m\ddot{\mathbf{q}} = -k\hat{\mathbf{q}}/q^2 \tag{1}$$

and this will be our starting-point for all that follows.

1. As a warmup, define the angular momentum vector  $\mathbf{J}$  in the usual way:

$$\mathbf{J} = m\mathbf{q} \times \dot{\mathbf{q}}$$

Just for fun, use equation (1) to check that angular momentum is conserved, even though we already know it must be:

$$\dot{\mathbf{J}} = 0.$$

2. Take the cross product of both sides of equation (1) with the vector  $\mathbf{J}$ . Simplify the right-hand side and show that

$$\ddot{\mathbf{q}} \times \mathbf{J} = k\dot{\hat{\mathbf{q}}} \tag{2}$$

Hint: you will probably need to use the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

(which I won't force you to prove, since you're supposed to already know it). You will also probably need to prove that

$$\dot{\hat{\mathbf{q}}} = \frac{(\mathbf{q} \cdot \mathbf{q})\dot{\mathbf{q}} - (\mathbf{q} \cdot \dot{\mathbf{q}}) \cdot \mathbf{q}}{q^3}.$$

3. Use parts 1 and 2 to show that

$$\frac{d}{dt}(\dot{\mathbf{q}} \times \mathbf{J}) = k \frac{d}{dt} \hat{\mathbf{q}}.$$

4. Use part 3 to show that

$$\dot{\mathbf{q}} \times \mathbf{J} = k\hat{\mathbf{q}} + \mathbf{x}$$

for some vector  $\mathbf{x} \in \mathbb{R}^3$  which is independent of time. It will be handy to divide this vector by  $k$ , obtaining the **Runge-Lenz vector**:

$$\mathbf{A} = \frac{\mathbf{x}}{k}$$

which clearly is also independent of time. In other words, the Runge-Lenz vector

$$\mathbf{A} = \frac{\dot{\mathbf{q}} \times \mathbf{J}}{k} - \hat{\mathbf{q}} \tag{3}$$

is a conserved quantity for the Kepler problem:

$$\dot{\mathbf{A}} = 0.$$

5. Using equation (3) show that

$$\mathbf{A} \cdot \mathbf{q} = \frac{\mathbf{J} \cdot \mathbf{J}}{km} - q. \tag{4}$$

Hint: you will probably need to use another vector identity — but anyway, you're supposed to know these identities.

6. Now write

$$\mathbf{A} \cdot \mathbf{q} = Aq \cos \theta$$

where  $A = |\mathbf{A}|$  is the magnitude of the Runge–Lenz vector and  $\theta$  is the angle between  $\mathbf{q}$  and the Runge–Lenz vector. Using equation (4), show that

$$q = \frac{\mathbf{J} \cdot \mathbf{J}}{km} \frac{1}{1 + A \cos \theta}.$$

***And now your work is done. Let me explain what you have achieved!***

The above equation looks almost like equation (6) in our previous homework about the Kepler problem:

$$r = \frac{p}{1 + e \cos(\theta - \theta_0)}.$$

And indeed, they are really just different ways of writing the same equation, except that now we have rotated our polar coordinate system so that  $\theta_0 = 0$ . Another way of saying this is that in these coordinates,  $\theta$  is zero at the perihelion of the orbit.

So, you've just given a new proof that the orbit in the Kepler problem must be an ellipse, parabola or hyperbola! Moreover, comparing the two equations above we see that

$$e = A,$$

so *the magnitude of the Runge–Lenz vector is the eccentricity of the orbit*. We also see that *the Runge–Lenz vector points in the direction of the orbit's perihelion*. Thus the conservation of the Runge–Lenz vector is just a way of saying the eccentricity and perihelion don't change with time! Finally, we see that

$$p = \frac{\mathbf{J} \cdot \mathbf{J}}{km}$$

which we already saw last time.