1

I immediately have $m_1\ddot{\mathbf{q}}_1 = f(|\mathbf{q}|)\mathbf{q}/|\mathbf{q}|$ and $m_2\ddot{\mathbf{q}}_2 = -f(|\mathbf{q}|)\mathbf{q}/|\mathbf{q}|$. Multiplying these equations by m_2 and m_1 (respectively) and then subtracting, I get $m_1m_2(\ddot{\mathbf{q}}_1 - \ddot{\mathbf{q}}_2) = (m_2+m_1)f(|\mathbf{q}|)\mathbf{q}/|\mathbf{q}|$. The desired equation follows after dividing by m_1+m_2, which is licit since I assume $m_1, m_2 > 0$.

2

The kinetic energies of the particles are $T_1 = \frac{1}{2}m_1|\dot{\mathbf{q}}_1|^2$ and $T_2 = \frac{1}{2}m_2|\dot{\mathbf{q}}_2|^2$. Since $m_1\mathbf{q}_1+m_2\mathbf{q}_2 = 0$, I have $0 = |m_1\dot{\mathbf{q}}_1 + m_2\dot{\mathbf{q}}_2|^2 = {m_1}^2|\dot{\mathbf{q}}_1|^2+2m_1m_2\dot{\mathbf{q}}_1\cdot\dot{\mathbf{q}}_2+{m_2}^2|\dot{\mathbf{q}}_2|^2$. Adding this to $m_1m_2|\dot{\mathbf{q}}|^2 = m_1m_2|\dot{\mathbf{q}}_1|^2 - 2m_1m_2\dot{\mathbf{q}}_1\cdot\dot{\mathbf{q}}_2 + m_1m_2|\dot{\mathbf{q}}_2|^2$ yields $m_1m_2|\dot{\mathbf{q}}|^2 = (m_1+m_2)(m_1|\dot{\mathbf{q}}_1|^2 + m_2|\dot{\mathbf{q}}_2|^2)$. Thus the total kinetic energy is $T_1+T_2 = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{q}}|^2$, the first term of the desired formula. Meanwhile, the particles' potential energies are $V_1 = \frac{1}{2}V(|\mathbf{q}_1 - \mathbf{q}_2|)$ and $V_2 = \frac{1}{2}V(|\mathbf{q}_2 - \mathbf{q}_1|)$. Each of these is simply $\frac{1}{2}V(|\mathbf{q}|)$, so the total potential energy is the other term of the desired formula.

3

The angular momenta of the particles are $\mathbf{J}_1 = m_1\mathbf{q}_1\times\dot{\mathbf{q}}_1$ and $\mathbf{J}_2 = m_2\mathbf{q}_2\times\dot{\mathbf{q}}_2$. Since $m_1\mathbf{q}_1+m_2\mathbf{q}_2 = 0$, I have $0 = (m_1\mathbf{q}_1+m_2\mathbf{q}_2)\times(m_1\dot{\mathbf{q}}_1+m_2\dot{\mathbf{q}}_2) = {m_1}^2\mathbf{q}_1\times\dot{\mathbf{q}}_1 + m_1m_2(\mathbf{q}_1\times\dot{\mathbf{q}}_2+\mathbf{q}_2\times\dot{\mathbf{q}}_1) + {m_2}^2\mathbf{q}_2\times\dot{\mathbf{q}}_2$. Adding this to $m_1m_2\mathbf{q}\times\dot{\mathbf{q}} = m_1m_2\mathbf{q}_1\times\dot{\mathbf{q}}_1 - m_1m_2(\mathbf{q}_1\times\dot{\mathbf{q}}_2+\mathbf{q}_2\times\dot{\mathbf{q}}_1) + m_1m_2\mathbf{q}_2\times\dot{\mathbf{q}}_2$ yields $m_1m_2\mathbf{q}\times\dot{\mathbf{q}} = (m_1+m_2)(m_1\mathbf{q}_1\times\dot{\mathbf{q}}_1 + m_2\mathbf{q}_2\times\dot{\mathbf{q}}_2)$. Thus the total angular momentum is $\mathbf{J}_1+\mathbf{J}_2 = \frac{m_1m_2}{m_1+m_2}\mathbf{q}\times\dot{\mathbf{q}}$, which is the desired formula. Since the z components of $\mathbf{q}_1$ and $\mathbf{q}_2$ are zero, so the z components of $\mathbf{q}$ and $\dot{\mathbf{q}}$ are also zero. Thus, the x and y components of $\mathbf{q}\times\dot{\mathbf{q}}$ are zero.

4

I have $r^2 = |\mathbf{q}|^2$ and $r\cos\theta = \mathbf{q}\cdot\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector along the positive x axis. Differentiating these formulas, I get $2r\dot{r} = 2\mathbf{q}\cdot\dot{\mathbf{q}}$ and $\dot{r}\cos\theta - r\dot{\theta}\sin\theta = \dot{\mathbf{q}}\cdot\hat{\mathbf{x}}$. Multiplying the first of these formulæ by $\frac{1}{2}\hat{\mathbf{x}}$ and subtracting the other formula multiplied by $\mathbf{q}$, I get $\dot{\mathbf{q}}\times(\hat{\mathbf{x}}\times\mathbf{q}) = r\dot{r}\hat{\mathbf{x}} - \dot{r}\mathbf{q}\cos\theta + r\dot{\theta}\mathbf{q}\sin\theta$, where I've used the vector identity that $\dot{\mathbf{q}}\times(\hat{\mathbf{x}}\times\mathbf{q}) = (\dot{\mathbf{q}}\cdot\mathbf{q})\hat{\mathbf{x}} - (\dot{\mathbf{q}}\cdot\hat{\mathbf{x}})\mathbf{q}$. Now, $\hat{\mathbf{x}}\times\mathbf{q} = r\hat{\mathbf{z}}\sin\theta$, where $\hat{\mathbf{z}}$ is the unit vector along the positive z axis, and $|\dot{\mathbf{q}}\times\hat{\mathbf{z}}| = |\dot{\mathbf{q}}|$ since $\dot{\mathbf{q}}$ has zero z component. Thus I have

$$\begin{aligned}
r^2|\dot{\mathbf{q}}|^2\sin^2\theta &= |r\dot{r}\hat{\mathbf{x}} - \dot{r}\mathbf{q}\cos\theta + r\dot{\theta}\mathbf{q}\sin\theta|^2 \\
&= r^2\dot{r}^2 + \dot{r}^2|\mathbf{q}|^2\cos^2\theta + r^2\dot{\theta}^2|\mathbf{q}|^2\sin^2\theta \\
&\quad -2r\dot{r}^2\hat{\mathbf{x}}\cdot\mathbf{q}\cos\theta + 2r^2\dot{r}\dot{\theta}\hat{\mathbf{x}}\cdot\mathbf{q}\sin\theta - 2r\dot{r}\dot{\theta}|\mathbf{q}|^2\cos\theta\sin\theta, \quad (1)
\end{aligned}$$

where I've used that $|\hat{\mathbf{x}}| = 1$. Substituting r^2 for $|\mathbf{q}|^2$ and $r\cos\theta$ for $\hat{\mathbf{x}}\cdot\mathbf{q}$, this becomes

$$\begin{aligned} r^2|\dot{\mathbf{q}}|^2\sin^2\theta &= r^2\dot{r}^2 + r^2\dot{r}^2\cos^2\theta + r^4\dot{\theta}^2\sin^2\theta \\ &\quad -2r^2\dot{r}^2\cos^2\theta + 2r^3\dot{r}\dot{\theta}\cos\theta\sin\theta - 2r^3\dot{r}\dot{\theta}\cos\theta\sin\theta \\ &= r^2\dot{r}^2\sin^2\theta + r^4\dot{\theta}^2\sin^2\theta. \end{aligned} \tag{2}$$

Thus, $|\dot{\mathbf{q}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$, and the formula for the total energy is proved. (The conclusion is valid even when $\sin\theta = 0$, by continuity, since $\sin\theta$ can't be constantly 0 in the physically relevant situation where an orbit exists. Similarly, I've been assuming $r > 0$ all along.) As for the angular momentum,

$$\begin{aligned} |\mathbf{q}\times\dot{\mathbf{q}}|^2 &= |\hat{\mathbf{x}}\times(\mathbf{q}\times\dot{\mathbf{q}})|^2 = |(\hat{\mathbf{x}}\cdot\dot{\mathbf{q}})\mathbf{q} - (\hat{\mathbf{x}}\cdot\mathbf{q})\dot{\mathbf{q}}|^2 \\ &= |\dot{r}\mathbf{q}\cos\theta - r\dot{\theta}\mathbf{q}\sin\theta - r\dot{\mathbf{q}}\cos\theta|^2 \\ &= \dot{r}^2|\mathbf{q}|^2\cos^2\theta + r^2\dot{\theta}^2|\mathbf{q}|^2\sin^2\theta + r^2|\dot{\mathbf{q}}|^2\cos^2\theta \\ &\quad - 2r\dot{r}\dot{\theta}|\mathbf{q}|^2\cos\theta\sin\theta - 2r\dot{r}(\mathbf{q}\cdot\dot{\mathbf{q}})\cos^2\theta + 2r^2\dot{\theta}(\mathbf{q}\cdot\dot{\mathbf{q}})\sin\theta\cos\theta \\ &= r^2\dot{r}^2\cos^2\theta + r^4\dot{\theta}^2\sin^2\theta + r^2(\dot{r}^2 + r^2\dot{\theta}^2)\cos^2\theta \\ &\quad -2r^3\dot{r}\dot{\theta}\cos\theta\sin\theta - 2r^2\dot{r}^2\cos^2\theta + 2r^3\dot{r}\dot{\theta}\sin\theta\cos\theta \\ &= r^4\dot{\theta}^2, \end{aligned} \tag{3}$$

so $|j| = m|\mathbf{q}\times\dot{\mathbf{q}}| = mr^2|\dot{\theta}|$. I in fact have $j = mr^2\dot{\theta}$, since both sides are positive for counterclockwise motion, although ultimately I won't need the sign.

5

Note that the sign of $\dot{\theta}$ always equals the sign of the constant j in this formula, and the physical application requires that $\dot{\theta}$ not be constantly 0, so I can conclude that $j \neq 0$, which will be needed when I divide by j later on. To continue,

$$E = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + V(r) = \frac{1}{2}m(r^2\frac{j^2}{m^2r^4} + \dot{r}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + (V(r) + \frac{j^2}{2mr^2}),$$

as desired. By the way, note that the centrifugal force is $-(d/dr)(j^2/2mr^2) = j^2/mr^3$, not j^2/mr as stated.

6

Solving for $\dot{r}$ in the formula for E, I get $E - U(r) = \frac{1}{2}m\dot{r}^2$, or $\dot{r}^2 = \frac{2}{m}(E - U(r))$, so $\dot{r} = \pm\sqrt{\frac{2}{m}(E - U(r))}$. However, $\dot{r}$ may or may not be nonnegative.

7

If $\dot{r} \neq 0$, $\dot{\theta}/\dot{r} = \pm\frac{j}{mr^2}/\sqrt{\frac{2}{m}(E - U(r))}$, or $\dot{\theta} = \pm\frac{j}{mr^2}\dot{r}/\sqrt{\frac{2}{m}(E - U(r))}$, when $\dot{r} \neq 0$. Integrating this with respect to time, $\theta = \theta(0) \pm \int_0 (\frac{j}{mr^2}\dot{r}\, dt/\sqrt{\frac{2}{m}(E - U(r))})$.

Since $dr = \dot{r}\,dt$, this the desired formula except for the sign ambiguity, if $\theta_0 := \theta(0)$. This formula won't work, however, for an orbit of constant r, where the analysis can only be saved by arguing by continuity of the dependence of the solution on the parameters. My analysis, however, will avoid this problem entirely by not relying on any division by $\dot{r}$.

8

$U(r) = V(r) + j^2/2mr^2 = j^2/2mr^2 - k/r$. As r nears zero, $U(r)$ nears $|j^2/2m|\infty = \infty$. As r nears infinity, $U(r)$ nears 0. $U(r) = 0$ when $j^2/2mr = k$, or $r = j^2/2mk$. $U'(r) = 0$ when $k/r^2 - j^2/mr^3 = 0$, or $k = j^2/mr$, which is $r = j^2/mk$; $U(j^2/mk) = -mk^2/2j^2$. Finally, $U''(r) = 0$ when $3j^2/mr^4 - 2k/r^3 = 0$, or $3j^2/mr = 2k$, which is $r = 3j^2/2mk$; $U(3j^2/2mk) = -4mk^2/9j^2$. Now I have plenty of information to sketch a graph:

At an energy of $E > 0$, the effective particle will come in from infinity and then go back out. At an energy of $E = 0$, the effective particle will do this same thing, but extremally. At an energy of $-mk^2/2j^2 < E < 0$, the effective particle will follow a bound path. At an energy of $E = -mk^2/2j^2$, the effective particle will remain motionless at $r = j^2/mk$; of course, this doesn't mean that the original system is motionless then, only that the value of r will be constant. Finally, an energy of $E < -mk^2/2j^2$ is impossible; the potential energy alone is enough to rule that out, since the effective kinetic energy $\frac{1}{2}m\dot{r}^2$ can never be negative.

9

To avoid picking branch cuts for $\sqrt{}$ and arccos (or even for θ!), let me revert to a differential equation and look at $\dot{\theta}^2$. Since $\dot{\theta} \neq 0$ but I can't be so sure about $\dot{r}$, I'll use

$$\frac{\dot{r}^2}{\dot{\theta}^2} = \frac{\frac{2}{m}(E - U(r))}{j^2/m^2r^4} = \frac{2mr^4(E - V(r) - j^2/2mr^2)}{j^2} = \frac{r^2(2mEr^2 + 2mkr - j^2)}{j^2}.$$

Let me simplify the constants by introducing $p := j^2/mk$ and $e := \sqrt{1 + 2Ej^2/mk^2}$ now, so $j^2 = mkp$ and $mk^2e^2 = mk^2 + 2Ej^2 = mk^2 + 2Emkp$, or $2Ep = ke^2 - k$. Note that since $E \geq -mk^2/2j^2$, or $2Ej^2/mk^2 \geq -1$, the square root defining e is real. Then

$$\begin{aligned} \frac{\dot{r}^2}{\dot{\theta}^2} &= \frac{r^2(2mEr^2 + 2mkr - mkp)}{mkp} = \frac{r^2(2Epr^2 + 2kpr - kp^2)}{kp^2} \\ &= \frac{r^2(ke^2r^2 - kr^2 + 2kpr - kp^2)}{kp^2} = \frac{r^2(e^2r^2 - r^2 + 2pr - p^2)}{p^2}. \end{aligned} \quad (4)$$

The numerator now contains the subtraction of a square, which suggests the use of trigonometry. Since the left side of this equation must be nonnegative, I can conclude that $e^2r^2 \geq r^2 - 2pr + p^2 = (p - r)^2$. Thus, there must always be an angle α such that $p - r = er\cos\alpha$. In terms of this angle, then, I have $\dot{r}^2/\dot{\theta}^2 = r^2(e^2r^2 - e^2r^2\cos^2\alpha)/p^2 = e^2r^4\sin^2\alpha/p^2$. If $\dot{r} \neq 0$, then, I can now write this as $\dot{\theta}/\dot{r} = \pm p/er^2\sin\alpha$, or $\dot{\theta} = \pm p\dot{r}/er^2\sin\alpha$. In order to integrate this, I'll want to understand $\dot{\alpha}$, so differentiate the equation $p - r = er\cos\alpha$ to get $-\dot{r} = e\dot{r}\cos\alpha - er\dot{\alpha}\sin\alpha$; since $e\cos\alpha = p/r - 1$, this becomes $-\dot{r} = p\dot{r}/r - \dot{r} - er\dot{\alpha}\sin\alpha$, or $\dot{\alpha} = p\dot{r}/er^2\sin\alpha$. I need analyse this no further, for now I see that $\dot{\theta} = \pm\dot{\alpha}$, so $\theta = \pm\alpha + \theta_0$ for some constant θ_0. In particular, $\cos(\theta - \theta_0) = \cos(\pm\alpha) = \cos\alpha = (\frac{p}{r} - 1)/e$, so I may write $\theta = \theta_0 + \arccos\big((\frac{p}{r} - 1)/e\big)$ for some branch of θ and arccos. In preparation for writing this in terms of the original parameters, let me multiply both sides of the fraction in the arccosine by k/j. Then

$$\begin{aligned} \theta &= \theta_0 + \arccos\left(\frac{\frac{kp}{jr} - \frac{k}{j}}{\frac{ke}{j}}\right) = \theta_0 + \arccos\left(\frac{\frac{kj^2}{kmjr} - \frac{k}{j}}{\frac{k}{j}\sqrt{1 + \frac{2Ej^2}{mk^2}}}\right) \\ &= \theta_0 + \arccos\left(\frac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{k^2}{j^2} + \frac{2E}{m}}}\right), \end{aligned} \quad (5)$$

which is the desired formula. Note that this θ_0 may not be the $\theta(0)$ from above, but that could be fixed by adjusting the indefinite integral.

10

I already reduced the clutter in the previous problem. However, to make sure of the formula for r, I should check also the situation when $\dot{r} = 0$. I know from the qualitative analysis that this occurs only when $E = -mk^2/2j^2$ and that $r = j^2/mk$ then. In terms of e and p, these are equations are, irrespectively, $r = p$ and $mk^2(e^2-1)/2j^2 = -mk^2/2j^2$, or $e = 0$. Thus, the equation $r = p/(1 + e\cos(\theta - \theta_0))$ holds then as well.

11

You used the translation and Galilean symmetries by setting the origin at the centre of mass for all time, and you used some of the rotational symmetry by placing the problem within the x, y plane. But there remains a rotational symmetry within that plane, so I use it to set θ_0 to 0. (There is also a symmetry of reflection through the z axis, which could be set by requiring $j > 0$, and a scaling symmetry, which could be set by requiring $p = 1$. But I will not do this yet.) Then the equation for the orbit becomes $p = r + er\cos\theta$. Now, $r\cos\theta = x$, so I get $r = ex - p$, or $r^2 = e^2x^2 - 2epx + p^2$. Since $r^2 = x^2 + y^2$, this becomes $x^2 - e^2x^2 + 2epx + y^2 = p^2$, as desired. If $e = 0$, then this equation is $x^2 + y^2 = p^2$, a circle of radius p centred at the origin. If $0 < e < 1$, then prepare to complete the square by writing $(1-e^2)^2x^2 + 2ep(1-e^2)x + (1-e^2)y^2 = p^2(1-e^2)$, or $(1-e^2)^2x^2 + 2ep(1-e^2)x + e^2p^2 + (1-e^2)y^2 = p^2(1-e^2) + e^2p^2$, so $\big((1-e^2)x + ep\big)^2 + (1-e^2)y^2 = p^2$, which becomes $(x + \frac{ep}{1-e^2})^2/\frac{p^2}{(1-e^2)^2} + y^2/\frac{p^2}{1-e^2} = 1$, which is an ellipse centred at $(-ep/(1-e^2), 0)$ with minor radius $p/(1-e^2)$ along the x axis and major radius $p/\sqrt{1-e^2}$ along the y axis. If $e = 1$, then the equation is $2px + y^2 = p^2$, or $x = p/2 - y^2/2p$, which is a parabola centred along the x axis, opening to the left, and with a vertex at $(p/2, 0)$. Finally, if $e > 1$, then I can complete the square as in the elliptic case, only now the equation should be written $(x - \frac{ep}{e^2-1})^2/\frac{p^2}{(e^2-1)^2} - y^2/\frac{p^2}{e^2-1} = 1$, which is a hyperbola centred at $(ep/(e^2-1), 0)$ with minimal radius $p/(e^2-1)$ along the x axis and asymptotes $x = \pm(e^2-1)y$.