Classical Mechanics Homework
January 24, 2008
John Baez
homework by: Scot Childress

The Kepler Problem

(**Background**) Suppose we have a particle moving in a central force. Its position is a function of time, say $q\colon \mathbb{R} \to \mathbb{R}^3$, satisfying Newton's law:

$$m\ddot{q} = f(|q|)\frac{q}{|q|}$$

Here m is its masses, and the force is described by some smooth function $f\colon (0,\infty) \to \mathbb{R}$. Let's write the force in terms of a potential as follows:

$$f(r) = -\frac{dV}{dr}.$$

Using conservation of angular momentum we can choose coordinates where the particle lies in the xy plane at all times. Thus we may assume the z component of $q(t)$ and $\ddot{q}(t)$ vanish for all t. In short, we have reduced the problem to a 2-dimensional problem!

Now let's work in polar coordinates: the point q lies in the xy plane so write it in polar coordinates as (r,θ). As usual, let's write time derivatives with dots:

$$\dot{r} = \frac{dr}{dt}, \qquad \dot{\theta} = \frac{d\theta}{dt}.$$

1. *Show that the energy E of the particle is given by*

$$E = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + V(r) \tag{1}$$

and the angular momentum J is a vector with vanishing x and y components, and z component given by

$$j = mr^2\dot{\theta}. \tag{2}$$

Recall that the energy of such a particle is given by

$$E = \frac{1}{2}m\dot{q}(t)^2 + V(|q(t)|). \tag{3}$$

Noting that in polar coordinates

$$\dot{q} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta), \tag{4}$$

we see that

$$\dot{q}(t)^2 = \dot{r}^2\cos^2\theta - r\dot{r}\dot{\theta}\sin 2\theta + r^2\dot{\theta}^2\sin\theta + \dot{r}^2\sin^2\theta + r\dot{r}\dot{\theta}\sin 2\theta + r^2\dot{\theta}^2\cos^2\theta$$

which reduces nicely to $\dot{q}(t)^2 = \dot{r}^2 + r^2\dot{\theta}^2$. Substitution of this last expression for $\dot{q}(t)^2$ into (3) and noting that $r = |q(t)|$ yields (1).

Now we will show that the angular momentum J is a vector with vanishing x and y components with the z component given by (2). The angular momentum is

$$J = mq \times \dot{q}$$

and if we use the expression for $\dot{q}$ obtained in (4), we have

$$\begin{aligned} q \times \dot{q} &= (r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath}) \times [(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{\imath} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{\jmath}] \\ &= [r^2\dot{\theta}\cos^2\theta + r\dot{r}\cos\theta\sin\theta - r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)]\hat{k} \\ &= r^2\dot{\theta}\hat{k}, \end{aligned}$$

so that $J = mr^2\dot{\theta}\hat{k}$.

2. *We use equation (2) to solve for $\dot{\theta}$ in terms of r:*

$$\dot{\theta} = \frac{j}{mr^2} \tag{5}$$

Combining this and equation (1) we express E in terms of r:

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) \tag{6}$$

where

$$V_{\text{eff}}(r) = V(r) + \frac{j^2}{2mr^2}.$$

The only thing to note here is that $\dot{\theta}^2 = j^2/m^2r^4$.

3. We solve (6) for $\dot{r}$ to obtain

$$\dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}. \tag{7}$$

It should be noted that in our use of the symbol for the positive square root *we are not asserting that $\dot{r}$ is positive*! It is entirely possible that the above root is negative! This, as we will discuss below (in # 5) will not effect the form of our solution for r in terms of θ.

4. *Using* (5) *and* (7) *show that*

$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}.$$

By the chain rule, we have that

$$\dot{\theta} = \frac{d\theta}{dr}\dot{r},$$

which when combined with (7) (and subsequently (5)) gives:

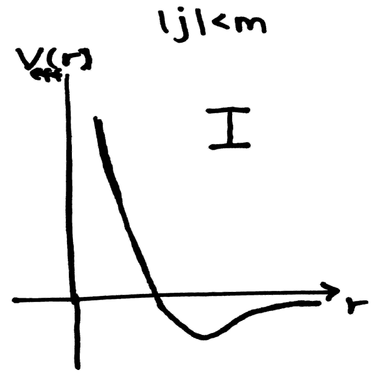
$$\frac{d\theta}{dr} = \frac{\dot{\theta}}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}.$$

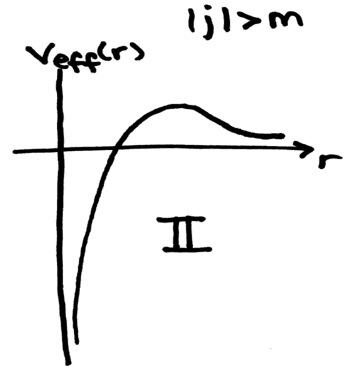
Upon integration we arrive at

$$\theta = \theta_0 + \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}. \tag{8}$$

Now let's specialize to the case of gravity, where $f(r) = -k/r^2$ ***and thus*** $V(r) = -k/r$ ***for some constant*** k***.***

5. *Sketch a graph of the effective potential* $V_{\text{eff}}(r)$ *in this case, and say what a particle moving in this potential would do, depending on its energy* E.





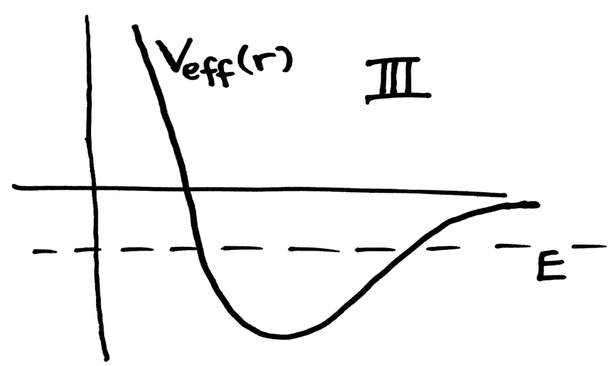


Figure I shows a sketch of V_{eff} in the case that $|j| > m$ (this is the case where $V'_{\text{eff}}(r) < 0$ for $r < j^2/2mk$) and II shows V_{eff} where $|j| < m$ (where $V'_{\text{eff}}(r) > 0$ for $r < j^2/2mk$.) In both sketches, the zero is at $r = j^2/2mk$ and the r-axis is a horizontal asymptote as $r \to \infty$.

Let us briefly discuss the behavior of a particle with energy $E < 0$ with $|j| > m$. Such a particle is shown in III. As was discussed in the example in class, the particles radius r would oscillate within the classically allowed region (the r values lying between the intersection points of E and $V_{\text{eff}}(r)$). The particle would be moving fastest at the minimum value of V_{eff} and would change from moving away from the origin to moving towards it (or vice a versa) at the intersection points.

6. *Carry out the integration in* (8).

We must compute

$$\int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E + k/r - j^2/2mr^2)}}.$$

Too much has been made of this bugaboo! Let's put this "beast" to rest by an elementary trigonometric substitution:

$$\frac{j}{m}\left(\frac{1}{r}-\frac{mk}{j^2}\right)=\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}\cos u.$$

(The sign of the radical here is chosen to match the sign of the radical in # 3) This substitution comes from completing the square under the radical—a simple and computationally economical process—and recalling the pythagorean identity for sine and cosine. All showboating aside, we see that

$$(j/mr^2)dr=\sin u\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}du,$$

and upon substitution, the integral becomes

$$\int udu=u$$

(the constant of integration already being accounted for in θ_0, and any sign changes from radicals canceling). Reversing the trignometric substitution we see that u and hence the sought after antiderivative is

$$\arccos\frac{\frac{j}{mr}-\frac{k}{j}}{\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}}.$$

Whence,

$$\theta=\theta_0+\arccos\frac{\frac{j}{mr}-\frac{k}{j}}{\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}}.\tag{9}$$

7. *Reduce the clutter in* (9) *by defining*

$$p=j^2/km,\qquad e=\sqrt{1+\frac{2Ej^2}{mk^2}}.$$

Note that

$$\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}=\frac{k}{j}e,$$

so that

$$\frac{\frac{j}{mr}-\frac{k}{j}}{\sqrt{\frac{2E}{m}+\frac{k^2}{j^2}}}=\frac{j}{k}\frac{\frac{j}{mr}-\frac{k}{j}}{e}=\frac{p/r-1}{e},$$

from whence it follows that

$$\theta=\theta_0+\arccos\left(\frac{p/r-1}{e}\right).$$

Solving for r yields:

$$r=\frac{p}{1+e\cos(\theta-\theta_0)}.\tag{10}$$

We should note that if the sign of the radical for $\dot{r}$

8. *Show that equation* (10) *describes an ellipse, parabola, or hyperbola in polar coordinates, depending on the value of the parameter* e*, which we call the* ***eccentricity***.

Begin by making a shift (a rotation) of θ_0 in θ. We will call the new coordinates that result from this shift r' and θ'. We have that

$$r' = \frac{p}{1 + e\cos\theta'}$$

by (10), or equivalently

$$r' + er'\cos\theta' = p.$$

Making the standard change to cartesian coordinates, the above reads

$$\sqrt{x^2 + y^2} + ex = p.$$

Now a little algebra yields

$$x^2 + y^2 = p^2 - 2ex + e^2x^2,$$

or put a little differently,

$$(1 - e^2)x^2 + 2ex + y^2 = p^2; \tag{11}$$

which we immediately recognize as the equation of a conic.

The particular conic that (11) describes will be determined by the value of e. If $e = 0$, for instance, then (11) reduces to

$$x^2 + y^2 = p^2,$$

a circle centered at the origin with radius p. If $e = 1$, then (11) reduces to

$$2(x - p^2/2) = y^2,$$

a parabola with vertex (in the original polar coordinates) $(p^2/2, \theta_0)$ opening *towards* the origin.

Let's exhaust all of the cases. If $e \neq 0$ or 1, then we may rewrite (11) as

$$\frac{\left(x + \frac{ep}{1-e^2}\right)^2}{p^2\frac{1+e^2}{1-e^2}} + \frac{y^2}{p^2(1+e^2)} = 1. \tag{12}$$

We see that in this case (12) represents either a hyperbola ($e > 1$) with vertices (in rotated cartesian coordinates)

$$\left(\frac{-ep}{1-e^2}, \pm p(1-e^2)^{1/2}\right)$$

opening in the y direction or an ellipse ($0 < e < 1$) with center (again in rotated cartesian coordinates)

$$\left(\frac{-ep}{1-e^2}, 0\right).$$

9. *How are the three kinds of orbits related to the energy* E?

Recall that e is given by

$$e = \sqrt{1 + \frac{2Ej^2}{mk^2}},$$

so that

$$E = \frac{mk^2}{2j^2}(e^2 - 1).$$

Using this, we compile the following chart:

Orbit(type)	Energy
Circular	$E = -mk^2/2j^2$
Parabolic	$E = 0$
Hyperbolic	$E > 0$
Elliptic	$-mk^2/2j^2 < E < 0$