Classical Mechanics Homework
January 24, 2008
John Baez homework by Brian Rolle

The Kepler Problem

1. Show that the energy of a particle is given by $E = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2) + V(r)$ and the angular momentum J has z coordinate $j = mr^2\dot{\theta}$ and vanishing x and y coordinates.

Solution: We know $E = T + V$ where T is the kinetic energy and V is the potential energy. Kinetic energy is $T = \frac{1}{2}m\dot{q}(t) \cdot \dot{q}(t)$ where $q(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t), 0)$. Since $\dot{q}(t) = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta, 0)$, $\dot{q}\cdot\dot{q} = \dot{r}^2 + r^2\dot{\theta}^2$ and $T = \frac{1}{2}m(r^2\dot{\theta}^2 + \dot{r}^2)$. By assumption, V depends on r, since the force depends only on r.
The angular momentum is given $j = q(t) \times m\dot{q}(t)$. Since q and $\dot{q}$ lie in the xy plane, $q(t) \times \dot{q}(t) = (0, 0, r^2\dot{\theta})$ and so the z coordinate of the angular momentum is $mr^2\dot{\theta}$.

2. Use the angular momentum to solve for $\dot{\theta}$ and write E as $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$ where $V_{\text{eff}}(r) = V(r) + \frac{j^2}{2mr^2}$.

Solution: Since $j = mr^2\dot{\theta}$, we have $\dot{\theta} = \frac{j}{mr^2}$. The equation for T gives $T = \frac{1}{2}m\left(r^2\left(\frac{j}{mr^2}\right)^2 + \dot{r}^2\right)$ and $E = \frac{1}{2}m\dot{r}^2 + \frac{j^2}{2mr^2} + V(r) = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$, where $V_{\text{eff}}(r) = \frac{j^2}{2mr^2} + V(r)$.

3. Show that $\dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}$.

Solution: Solving $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$ for $\dot{r}^2$ gives $\dot{r}^2 = \frac{2}{m}(E - V_{\text{eff}}(r))$. Taking the square root gives $\dot{r} = \sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}$.

4. Show that $\frac{d\theta}{dr} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}$.

Solution: By the chain rule $\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr}$. Implicit differentiation (like in 9A) gives $\frac{dt}{dr} = \frac{1}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}$, provided $\dot{r} > 0$. Since $\frac{d\theta}{dt} = \frac{j}{mr^2}$, we have $\frac{d\theta}{dr} = \frac{j/mr^2}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}$.

Integrating this gives $\theta = \theta_0 + \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}}$.

5. Sketch the graph of $V_{\text{eff}}(r)$ when $V = -\frac{k}{r}$ and describe what a particle in this potential would do, depending on it's energy E.

Solution: We have $V_{\text{eff}}(r) = \frac{j^2}{2mr^2} - \frac{k}{r} = k\left(\frac{\frac{j^2}{2mk} - r}{r^2}\right)$. This is minimized when $r = \frac{j^2}{mk}$, and then $V_{\text{eff}} = -\frac{k^2 m}{2j^2}$. Since $\frac{1}{2}m\dot{r}^2$ must be nonnegative, we know that a particles total energy can never be less than $-\frac{k^2 m}{2j^2}$. If a particles total energy is positive, then r can go to infinity. If $E < 0$, then the particle will be stay in between to values of r, determined by when the energy equals V_{eff}.

6. Show $\theta = \theta_0 + \arccos \frac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}}$.

Solution: $\int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E - V_{\text{eff}}(r))}} = \int \frac{(j/mr^2)dr}{\sqrt{\frac{2}{m}(E + \frac{k}{r} - \frac{j^2}{2mr^2})}} = -\int \frac{dw}{\sqrt{\frac{2Em}{j^2} + \frac{2km}{j^2}w - w^2}}$, where $w = \frac{1}{r}$. Using the fact $\int \frac{dx}{\sqrt{ax^2 + bx + c}} = -\frac{1}{\sqrt{-a}} \arccos\left(\frac{-2ax - b}{\sqrt{b^2 - 4ac}}\right)$ when $a < 0$, we have $-\int \frac{dw}{\sqrt{\frac{2Em}{j^2} + \frac{2km}{j^2}w - \frac{1}{j}w^2}} = \arccos\left(\frac{2w - \frac{2km}{j^2}}{\sqrt{\frac{4k^2m^2}{j^4} + 4\frac{2Em}{j^2}}}\right) = \arccos\left(\frac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{k^2}{j^2} + \frac{2E}{m}}}\right)$. So $\theta = \theta_0 + \arccos \frac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}}$.

7. Letting $p = \dfrac{j^2}{km}$ and $e = \sqrt{1 + \dfrac{2Ej^2}{mk^2}}$, show $\theta = \theta_0 + \arccos\left(\dfrac{p/r - 1}{e}\right)$.

Solution: $\theta = \theta_0 + \arccos \dfrac{\frac{j}{mr} - \frac{k}{j}}{\sqrt{\frac{2E}{m} + \frac{k^2}{j^2}}} = \theta_0 + \arccos \dfrac{\frac{j}{mr} - \frac{k}{j}}{\frac{j}{k}\sqrt{\frac{2Ej^2}{mk^2} + 1}} = \theta_0 + \arccos \dfrac{\frac{j}{mr}\frac{j}{k} - 1}{\sqrt{\frac{2Ej^2}{mk^2} + 1}} =$ $\theta_0 + \arccos\left(\dfrac{p/r - 1}{e}\right)$. Solving for r gives $r = \dfrac{p}{1 + e\cos(\theta - \theta_0)}$.

8. The equation $r = \dfrac{p}{1 + e\cos(\theta - \theta_0)}$ describes an ellipse, parabola or hyperbola based on the value of e.

Solution: By a rotation, we can assume $\theta_0 = 0$. So we have $p = r + er\cos\theta$ or $p = \sqrt{x^2 + y^2} + ex$. So $x^2 + y^2 = p^2 - 2epx + x^2$, or $(1 - e^2)x^2 + 2epx + y^2 = p^2$.

If $e = 0$, we have $p^2 = x^2 + y^2$, the equation of a circle.

If $0 < e < 1$, let $k = \dfrac{ep}{1 - e^2}$. Then we have $x^2 + 2kx + \dfrac{y^2}{1 - e^2} = \dfrac{p^2}{1 - e^2} = (x - k)^2 + \dfrac{y^2}{1 - e^2} =$ $\dfrac{p^2}{1 - e^2} + k^2$, and ellipse.

If $e = 1$, we have $2px = -y^2 + p^2$, a parabola.

If $e > 1$, let $k = \dfrac{ep}{1 - e^2}$. Then we have $x^2 + 2kx + \dfrac{y^2}{1 - e^2} = \dfrac{p^2}{1 - e^2} = (x - k)^2 - \dfrac{y^2}{e^2 - 1} =$ $\dfrac{p^2}{1 - e^2} + k^2$, a hyperbola.

9. How are the 3 kinds of orbits related to energy?

Solution: The hyperbola corresponds to $e > 1$, which requires the energy to be positive. The parabola correspond to $e = 1$, which requires energy to be 0. The ellipse corresponds to $0 < e < 1$ which corresponds which requires energy to be negative.