Classical Mechanics
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The Laplace–Lunge–Renz Vector

Let q denote the difference in position of two particles in $\mathbb{R}^3$, $|q|$ the magnitude of q, and m the reduced mass. Supposing that the particles exert an inverse square force on one another, Newton's second law gives us that

$$m\ddot{q} = -\frac{k}{|q|^2}\hat{q} \tag{1}$$

where k is a known constant. Recall that the angular momentum of the system is given by

$$J = m(q \times \dot{q}).$$

1. Taking the cross product of (1) with J yields:

$$\begin{aligned} m(\ddot{q} \times J) &= -\frac{k}{|q|^3}(q \times J) \\ &= -\frac{mk}{|q|^3}[q \times (q \times \dot{q})] \\ &= -\frac{mk}{|q|^3}[(q \cdot \dot{q})q - |q|^2\dot{q}] \end{aligned} \tag{2}$$

where the last equality follows from the vector identity

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$$

Now, we compute $\dot{\hat{q}}$:

$$\dot{\hat{q}} = \frac{d}{dt}\left(\frac{q}{|q|}\right) = \frac{\dot{q}|q| - (q \cdot \dot{q})q/|q|}{|q|^2} = \frac{\dot{q}|q|^2 - (q \cdot \dot{q})q}{|q|^3}$$

and combine the result with (2) to obtain:

$$\ddot{q} \times J = k\dot{\hat{q}} \tag{3}$$

2. Now we note that the conservation of angular momentum gives:

$$\frac{d}{dt}(\dot{q} \times J) = \ddot{q} \times J + \dot{q} \times \dot{J} = \ddot{q} \times J$$

so that in conjunction with (3) we have

$$\frac{d}{dt}(\dot{q} \times J) = k\dot{\hat{q}}. \tag{4}$$

3. Integrating (4) with respect to t yields

$$\dot{q} \times J = k\hat{q} + x \tag{5}$$

where x is a fixed vector of integration. We define the **Runge–Lenz vector**:

$$A := \frac{x}{k} = \frac{\dot{q} \times J}{k} - \hat{q}.$$

4. Let's expand $\dot{q} \times J$ using the vector identity given in Exercise 1:

$$\dot{q} \times J = m[\dot{q} \times (q \times \dot{q})] = m[|\dot{q}|^2 q - (q \cdot \dot{q})\dot{q}].$$

Thus it follows that

$$(\dot{q} \times J) \cdot q = m[|\dot{q}|^2 |q|^2 - (q \cdot \dot{q})^2]$$

and if neither q nor $\dot{q}$ is zero we have that

$$(\dot{q} \times J) \cdot q = m|\dot{q}|^2 |q|^2 [1 - \cos^2 \varphi] = m|\dot{q}|^2 |q|^2 \sin^2 \varphi$$

where φ is the angle between q and $\dot{q}$. This last expression is exactly $J \cdot J/m$ so that we have shown:

$$(\dot{q} \times J) \cdot q = J \cdot J/m \tag{6}$$

(*if either q or $\dot{q}$ is zero, then J is zero, and the above expression remains valid*). Combining (6) with the definition of A yields:

$$A \cdot q = \frac{J \cdot J}{km} - |q|. \tag{7}$$

5. If we write $A \cdot q$ as $|A||q| \cos \theta$ where θ is the angle between A and q we see that (7) gives:

$$|A||q| \cos \theta = \frac{J \cdot J}{km} - |q|$$

which can be solved for $|q|$ to yield:

$$|q| = \frac{J \cdot J}{km} \frac{1}{1 + |A| \cos \theta}. \tag{8}$$