Classical Mechanics Homework
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John Baez homework by Brian Rolle

The Kepler Problem Revisited

1. Define $J = mq \times \dot{q}$. Then, using $m\ddot{q} = -k\hat{q}/q^2$, we have $\ddot{q} \times J = k\dot{\hat{q}}$

Solution: Using the vector identities given, we have $\ddot{q} \times J = \ddot{q} \times (mq \times \dot{q}) = -(k\hat{q}/q^2) \times (q \times \dot{q}) = (-k/q^2)\hat{q} \times (q \times \dot{q}) = (-k/q^2)((\hat{q} \cdot \dot{q})q - (\hat{q} \cdot q)\dot{q}) = -k\dfrac{(q \cdot \dot{q})q - (q \cdot q)\dot{q}}{q^3} = k\dot{\hat{q}}$

2. Using the above $\dfrac{d}{dt}(\dot{q} \times J) = k\dot{\hat{q}}$.

Solution: We have $\dfrac{d}{dt}(\dot{q} \times J) = \ddot{q} \times J + \dot{q} \times \dot{J} = \ddot{q} \times J = k\dot{\hat{q}}$ since $\dot{J} = 0$.

3. Using the above $\dot{q} \times J = k\hat{q} + x$, where x is independent of time.

Solution: Since $\dfrac{d}{dt}(\dot{q} \times J - k\hat{q}) = \dfrac{d}{dt}(\dot{q} \times J) - k\dot{\hat{q}} = 0$, we have $\dot{q} \times J - k\hat{q} = x$, where x is independent of time.

4. Define $A = \dfrac{x}{k} = \dfrac{\dot{q} \times J}{k} - \hat{q}$. Then we have $A \cdot q = \dfrac{J \cdot J}{km} - |q|$

Solution: Since $\hat{q} \cdot q = |q|$, we have $A \cdot q = q \cdot \left(\dfrac{\dot{q} \times J}{k}\right) - |q| = J \cdot \left(\dfrac{q \times \dot{q}}{k}\right) - |q| = J \cdot \left(\dfrac{mq \times \dot{q}}{mk}\right) - |q| = \dfrac{J \cdot J}{km} - |q|$, using the identity $a \cdot (b \times c) = c \cdot (a \times b)$.

5. If θ is the angle between A and q, we have $A \cdot q = |A||q| \cos\theta$. Then $|q| = \dfrac{J \cdot J}{km} \dfrac{1}{1 + |A| \cos\theta}$.

Solution: We have $\dfrac{J \cdot J}{km} = A \cdot q + |q| = |A||q| \cos\theta + |q| = |q|(1 + |A| \cos\theta)$. So $|q| = \dfrac{J \cdot J}{km} \dfrac{1}{1 + |A| \cos\theta}$.