A Spring in Imaginary Time
QG Homework 2 Fall 2006

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One of the stranger aspects of Lagrangian *dynamics* is how it turns into *statics* when we replace the time coordinate t by it - or in the jargon of physicists, when we 'Wick rotate' to 'imaginary time'! People usually take advantage of this to do interesting things in the context of *quantum mechanics*, but the basic ideas are already viable in *classical mechanics*. So, let's look at them!

1. Suppose you have a spring in $\mathbb{R}^n$ whose ends are held fixed, tracing out a curve

$$q : [s_0, s_1] \to \mathbb{R}^n$$

with endpoints

$$q(s_0) \;=\; a, \quad q(s_1) \;=\; b.$$

Suppose the spring is put into a potential

$$V : \mathbb{R}^n \to \mathbb{R}$$

(perhaps due to gravity, but not necessarily). What curve will the spring trace out when it is in equilibrium?

Hint: Hooke's law says that a stretched spring has energy proportional to the square of how much it is stretched. Here this is true of each little piece of the spring, so its total energy due to stretching will be

$$\frac{k}{2}\int_{s_0}^{s_1} \dot{q}(s)\dot{q}(s)ds$$

for some 'spring constant' k. But in addition, each little piece will aquire energy due to the potential V at that point, so the spring will also have potential energy

$$\int_{s_0}^{s_1} V(q(s))ds.$$

The total energy of the spring is thus:

$$E = \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s)\dot{q}(s) + V(q(s))\right) ds.$$

Our study of statics has taught us that in equilibrium, a static system minimizes its energy, or at least finds a critical point. So, set

$$\delta E = 0$$

for all allowed variations δq of the path, and work out the differential equation this implies for q.

Solution:
If we let q_μ=q+$\mu\delta$q, we want to find the curves that satisfy

$$\frac{d}{d\mu}E(q_\mu)\bigg|_{\mu=0} = 0,$$

$\forall$ δq:$[s_0, s_1]\rightarrow \mathbb{R}^n$ with $\delta\text{q}(s_0) = \delta\text{q}(s_1) = 0$.

$$\begin{aligned}
\frac{d}{d\mu}E(q_\mu)\bigg|_{\mu=0} &= \frac{d}{d\mu}\int_{s_0}^{s_1}\left(\frac{k}{2}\dot{q}_\mu(s)\dot{q}_\mu(s) + V(q_\mu(s))\right) ds\bigg|_{\mu=0} \\
&= \int_{s_0}^{s_1}\frac{d}{d\mu}\left(\frac{k}{2}\dot{q}_\mu(s)\dot{q}_\mu(s) + V(q_\mu(s))\right) ds\bigg|_{\mu=0} \\
&= \int_{s_0}^{s_1}\left(k\dot{q}_\mu(s)\frac{d}{d\mu}\dot{q}_\mu(s) + \nabla V(q_\mu(s))\frac{d}{d\mu}q_\mu(s)\right) ds\bigg|_{\mu=0} \\
&= \int_{s_0}^{s_1}\left(k\dot{q}_\mu(s)\frac{d}{ds}\frac{d}{d\mu}q_\mu(s) + \nabla V(q_\mu(s))\frac{d}{d\mu}q_\mu(s)\right) ds\bigg|_{\mu=0}
\end{aligned}$$

and using integration by parts

$$= k\dot{q}_\mu(s)\frac{d}{d\mu}q_\mu(s)\bigg]_{s_0}^{s_1} + \int_{s_0}^{s_1}\left(-k\frac{d}{d\mu}\dot{q}_\mu(s) + \nabla V(q_\mu(s))\right)\frac{d}{d\mu}q_\mu(s)ds\bigg|_{\mu=0}$$

and since the boundary terms are zero,

$$= \int_{s_0}^{s_1}\left(-k\frac{d}{d\mu}\dot{q}_\mu(s) + \nabla V(q_\mu(s))\frac{d}{d\mu}q_\mu(s)\right) ds\bigg|_{\mu=0}$$

which equals 0 for all δq if and only if the integrand is 0.

$$k\ddot{q}(s) = \nabla V(q(s))$$

2. Suppose the spring is in a constant downwards gravitational field in $\mathbb{R}^3$, so that

$$V(x, y, z) = mgz$$

where m is the mass density of the spring and g is the acceleration of gravity (9.8 meters/$second^2$). What sort of curve does the spring trace out in equilibrium?

Solution:
The solution to part (1) tells us that in equilibrium, the spring will trace out a path governed by the equation

$$k\ddot{q}(s) = \nabla V(q(s)).$$

Given that V(x,y,z) = mgz and viewing this equation in terms of its coordiantes, we get

$$k\ddot{q}(s) = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$= (0, 0, mg)$$

So,

$$\ddot{q}(s) = \left(0, 0, \frac{mg}{k}\right)$$

Which gives us equations

$$\begin{aligned}\ddot{q}_x &= 0\\ \ddot{q}_y &= 0\\ \ddot{q}_z &= \tfrac{mg}{k}.\end{aligned}$$

So, we know that the first two components of q are linear and the third is quadratic. The coefficients of these polynomials can be computed in terms of the fixed endpoints of the spring. In equilibrium the spring traces out a parabola.

3. The calculation in problem 1 should remind you strongly of the derivation of the Euler-Lagrange equations for a particle in a potential. To heighten this analogy, take the energy

$$E = \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s)\dot{q}(s) + V(q(s))\right) ds$$

and formally replace the parameter s by it, replacing the interval $[s_0, s_1] \subset \mathbb{R}$ by the imaginary interval $[t_0, t_1] \subset i\mathbb{R}$, where $it_i = s_i$. Show that up to some constant factor, the *energy* of the static spring becomes the *action* for a particle moving in a potential.

Solution:
If we formally replace s by it, the energy becomes

$$\begin{aligned}E &= \int_{t_0}^{t_1} \left(\frac{k}{2}\frac{d}{dt}q(it)\frac{d}{dt}q(it) + V(q(it))\right) d(it)\\ &= \int_{t_0}^{t_1} \left(\frac{k}{2}i\dot{q}(it)i\dot{q}(it) + V(q(it))\right) id(t)\\ &= -i\int_{t_0}^{t_1} \left(\frac{k}{2}\dot{q}(it)\dot{q}(it) - V(q(it))\right) d(t),\end{aligned}$$

where

$$\frac{k}{2}\dot{q}(it)\dot{q}(it) - V(q(it))$$

is the Lagrangian for a particle moving in a potential. Therefore,

$$E = -iS$$

4. Fill in the blanks in this analogy:

Statics	**Dynamics**
Principle of Least Energy	Principle of Least Action
spring	particle
energy	<u>action</u>
<u>stretching energy</u>	kinetic energy
<u>potential energy</u>	potential energy
spring constant k	<u>mass m</u>

5. What particular dynamics problem is the statics problem in 2 analogous to? How is the solution to the statics problem related to the statics problem related to the solution of the dynamics problem?

   Solution:
   The statics problem in part (2) is analogous to the dynamics problem of throwing an object of some mass. The curve satisfying the equations of motion for this problem is a downwards facing parabola, which is the solution to part (2) with an extra minus sign thrown in.

6. What does Newton's law $F = ma$ become if we formally replace t by $s = it$.
   Hint: by 'formally', I'm suggesting that you shouldn't think too much about what this actually means! It's a good thing to think about, but don't let that stop you from solving what's meant to be a quick and easy problem.

   Solution: The formal replacement of t by $s = it$ gives us two factors of i from differentiating q twice. Thus, Newton's Law becomes

$$F = -ma.$$