A Spring in Imaginary Time

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1. Suppose you have a spring in $\mathbb{R}^n$ with fixed endpoints, tracing out a curve

$$q : [s_0, s_1] \to \mathbb{R}^n, \qquad q(s_0) = a, q(s_1) = b.$$

If the spring is in a potential $V\mathbb{R}^n \to \mathbb{R}$, what curve will the spring trace out when it's in equilibrium?

Since the total spring energy is

$$E = \int_{s_0}^{s_1} \left[\frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds,$$

we set $\delta E = 0$ and investigate the implications.

$$\begin{aligned}
\delta E &= \delta \left(\int_{s_0}^{s_1} \left[\frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds \right) \\
&= \tfrac{\partial}{\partial \varepsilon} \left(\int_{s_0}^{s_1} \left[\frac{k}{2} \dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right] ds \right) \Big|_{\varepsilon=0} && \text{def of } \delta \\
&= \int_{s_0}^{s_1} \frac{k}{2} \tfrac{\partial}{\partial \varepsilon} \left[\dot{q}(s) \cdot \dot{q}(s) \right] + \tfrac{\partial}{\partial \varepsilon} \left[V(q(s)) \right] ds \Big|_{\varepsilon=0} && \text{linearity} \\
&= \int_{s_0}^{s_1} k \dot{q}_\varepsilon(s) \tfrac{\partial}{\partial \varepsilon} \left[\dot{q}_\varepsilon(s) \right] + \nabla V(q_\varepsilon(s)) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] ds \Big|_{\varepsilon=0} && \text{chain rule} \\
&= \int_{s_0}^{s_1} k \dot{q}_\varepsilon(s) \tfrac{\partial}{\partial t} \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] + \nabla V(q_\varepsilon(s)) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] ds \Big|_{\varepsilon=0} && \text{mixed partials} \\
&= \int_{s_0}^{s_1} -k \left(\tfrac{\partial}{\partial t} \dot{q}_\varepsilon(s) \right) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] + \nabla V(q_\varepsilon(s)) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] ds \Big|_{\varepsilon=0} && \text{IBP} \\
&= \int_{s_0}^{s_1} \Big(-k \ddot{q}_\varepsilon(s) + \nabla V(q_\varepsilon(s)) \Big) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right] ds \Big|_{\varepsilon=0} && \text{factoring} \\
&= \int_{s_0}^{s_1} \Big(-k \ddot{q}(s) + \nabla V(q(s)) \Big) \tfrac{\partial}{\partial \varepsilon} \left[q_\varepsilon(s) \right]_{\varepsilon=0} ds && \text{letting } \varepsilon = 0.
\end{aligned}$$

So if this is 0 for all allowable variations δq, we must have an integrand of 0, i.e.,

$$k\ddot{q}(s) = \nabla V(q(s)).$$

2. Suppose the spring is in a constant downwards gravitational field in $\mathbb{R}^3$, so that

$$V(x, y, z) = mgz,$$

where m is the mass density of the spring and g is the acceleration of gravity (9.8 m/s^2). What sort of curve does the spring trace out, in equilibrium?

Apply the answer from (1), $\nabla V = k\ddot{q}(s)$, and compute

$$\nabla V(x, y, z) = \nabla(mgz) = [0, 0, mg]$$

to obtain the system

$$\begin{cases} \ddot{q}_1(s) = 0 \\ \ddot{q}_2(s) = 0 \\ \ddot{q}_3(s) = \frac{mg}{k}. \end{cases}$$

All equations may be solved directly by successive integrations; the first two yield linear functions, and the third gives a polynomial in z:

$$\begin{cases} q_1(s) = (b_1 - a_1)s + a_1 \\ q_2(s) = (b_2 - a_2)s + a_2 \\ q_3(s) = \frac{mg}{2k}s^2 + \left(b_3 - a_3 - \frac{mg}{2k}\right) s + a_3, \end{cases}$$

where the values of the constants are deduced by comparison to the components of $q(s_0) = a, q(s_1) = b$.

Thus the spring traces out a parabola lying in the vertical plane whose intersection with the xy-plane is the straight line from (a_1, a_2) to (b_1, b_2).

3. Using the energy, as given previously, replace the parameter s by it and show that up to a constant, the energy of the static string becomes the action for a particle moving in a potential.

$$\begin{aligned} E &= \int_{s_0}^{s_1} \left[\frac{k}{2}\dot{q}(s) \cdot \dot{q}(s) + V(q(s))\right] ds \\ &= \int_{s_0}^{s_1} \left[\frac{k}{2}\frac{\partial}{\partial s}q(s) \cdot \frac{\partial}{\partial s}q(s) + V(q(s))\right] ds \\ &= \int_{t_0}^{t_1} \left[\frac{k}{2}\frac{\partial}{\partial t}q(it) \cdot \frac{\partial}{\partial t}q(it) + V(q(it))\right] d(it) \\ &= \int_{t_0}^{t_1} \left[\frac{k}{2}i\dot{q}(it) \cdot i\dot{q}(it) + V(q(it))\right] i\,dt \\ &= \int_{t_0}^{t_1} \left[-\frac{k}{2}\dot{q}(it) \cdot \dot{q}(it) + V(q(it))\right] i\,dt \\ &= -i\int_{t_0}^{t_1} \left[\frac{k}{2}\dot{q}(it) \cdot \dot{q}(it) - V(q(it))\right] dt \\ &= -iS(q) \end{aligned}$$

4. The analogy between statics and dynamics.

Principle of Least Energy	Principle of Least Action
spring	particle
energy	action
"tension" energy	kinetic energy
potential energy	potential energy
spring constant k	mass m

5. What particular dynamics problem (pun intended) is the statics problem in 2 analogous to? How is the solution to the statics problem related to the solution of this dynamics problem?

   The problem is: "What curve does a particle trace out when it moves through a (gravitational) potential, minimizing action?"

   The answer is again a parabola; the negative sign introduced during the rotation into imaginary time has the effect of flipping the parabola, so it open downwards, as is appropriate for the path of a projectile.

6. What does Newton's law $F = ma$ become if we formally replace t by $s = it$?

   Since

$$\begin{aligned} F &= ma \\ -\nabla V &= m \tfrac{\partial^2}{\partial t^2} q(it) \\ -\nabla V &= -m\ddot{q}(it) \\ \nabla V &= m\ddot{q}(it) \\ F &= -m\ddot{q}(it) \\ F &= -ma, \end{aligned}$$

   we see that in an imaginary world, Newton's law is reversed.