QG F06a Homework 1
Mike Stay

1. *Suppose you have a spring in $\mathbb{R}^n$ whose ends are held fixed, tracing out a curve*

$$q : [s_0, s_1] \to \mathbb{R}^n$$

with endpoints

$$q(s_0) = a, \quad q(s_1) = b.$$

Suppose the spring is put into a potential

$$V : \mathbb{R}^n \to \mathbb{R}$$

(perhaps due to gravity, but not necessarily). What curve will the spring trace out when it is in equilibrium?

$$\begin{aligned}
\delta E &= \delta \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right) ds \\
&= \int_{s_0}^{s_1} \left(\delta \frac{k}{2}\dot{q}(s) \cdot \dot{q}(s) + \delta V(q(s)) \right) ds \\
&= \int_{s_0}^{s_1} \Big(k\dot{q}(s) \cdot \delta\dot{q}(s) + \nabla V(q(s)) \cdot \delta q(s) \Big) ds \\
&= \int_{s_0}^{s_1} \Big(k\dot{q}(s) \cdot \delta\dot{q}(s) \Big) ds + \int_{s_0}^{s_1} \Big(\nabla V(q(s)) \cdot \delta q(s) \Big) ds \\
&= \int_{s_0}^{s_1} \left(k\dot{q}(s) \cdot \frac{d}{ds}\delta q(s) \right) ds + \int_{s_0}^{s_1} \Big(\nabla V(q(s)) \cdot \delta q(s) \Big) ds \\
&= k\dot{q}(s) \cdot \delta q(s) \Big]_{s_0}^{s_1} - \int_{s_0}^{s_1} \left(k\frac{d}{ds}\dot{q}(s) \cdot \delta q(s) \right) ds + \int_{s_0}^{s_1} \Big(\nabla V(q(s)) \cdot \delta q(s) \Big) ds \\
&= 0 - \int_{s_0}^{s_1} \Big(k\ddot{q}(s) \cdot \delta q(s) \Big) ds + \int_{s_0}^{s_1} \Big(\nabla V(q(s)) \cdot \delta q(s) \Big) ds
\end{aligned}$$

which is zero only when

$$k\ddot{q}(s) = \nabla V(q(s)).$$

2. *Suppose the spring is in a constant downwards gravitational field in $\mathbb{R}^3$, so that*

$$V(x, y, z) = mgz,$$

where m is the mass density of the spring and g is the acceleration of gravity ($9.8 meters/second^2$). What sort of curve does the spring trace out, in equilibrium?

$\nabla V = (0, 0, mg)$, so the second derivative of the path with respect to s is $\ddot{q} = mg/k$, a constant. Thus the curve must be a parabola.

3. *The calculation in problem 1 should remind you strongly of the derivation of the Euler-Lagrange equations for a particle in a potential. To heighten this analogy, take the energy*

$$E = \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right) ds$$

and formally replace the parameter s by it, replacing the real interval $[s_0, s_1] \subset R$ by the imaginary interval $[t_0, t_1] \subset iR$, where $it_j = s_j$. Show that up to some constant factor, the energy of the static spring becomes the action for a particle moving in a potential.

$$\begin{aligned}
E &= \int_{s_0}^{s_1} \left(\frac{k}{2}\dot{q}(s) \cdot \dot{q}(s) + V(q(s)) \right) ds \\
&= \int_{s_0}^{s_1} \left(\frac{k}{2}\frac{d}{ds}q(s) \cdot \frac{d}{ds}q(s) + V(q(s)) \right) ds \\
&= \int_{t_0}^{t_1} \left(\frac{k}{2}\frac{d}{dt}q(it) \cdot \frac{d}{dt}q(it) + V(q(it)) \right) d(it) \\
&= \int_{t_0}^{t_1} \left(\frac{k}{2}i\frac{d}{d(it)}q(it) \cdot i\frac{d}{d(it)}q(it) + V(q(it)) \right) idt \\
&= -i\int_{t_0}^{t_1} \left(\frac{k}{2}\dot{q}(it) \cdot \dot{q}(it) - V(q(it)) \right) dt \\
&= -iS
\end{aligned}$$

4. *Fill in the blanks in this analogy:*

STATICS	DYNAMICS
Principle of Least Energy	Principle of Least Action
spring	particle
energy $= T + V$	*action* $= \int T - V dt$
stretching energy T	~~kinetic energy~~ *kinetic action, i.e.* $\int T dt$
potential energy V	~~potential energy~~ *potential action, i.e.* $-\int V dt$
spring constant k	*mass*

5. *a) What particular dynamics problem is the statics problem in 2 analogous to? b) How is the solution to the statics problem related to the solution of this dynamics problem?*

a) A particle moving in a constant gravitational field.

b) The result is the same up to a sign change.

6. *What does Newton's law $F = ma$ become if we formally replace t by $s = it$?*

$$
\begin{aligned}
F &= ma = m\frac{d^2x}{dt^2} \\
\Rightarrow F &= m\frac{d^2x}{d(it)^2} \\
&= -m\frac{d^2x}{dt^2} \\
&= -ma
\end{aligned}
$$