Lectures on Classical Mechanics

by

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Preface

These are notes for a mathematics graduate course on classical mechanics at U.C. Riverside. I've taught this course three times recently. Twice I focused on the Hamiltonian approach. In 2005 I started with the Lagrangian approach, with a heavy emphasis on action principles, and derived the Hamiltonian approach from that. This approach seems more coherent.

Derek Wise took beautiful handwritten notes on the 2005 course, which can be found on my website:

`http://math.ucr.edu/home/baez/classical/`

Later, Blair Smith from Louisiana State University miraculously appeared and volunteered to turn the notes into LaTeX . While not yet the book I'd eventually like to write, the result may already be helpful for people interested in the mathematics of classical mechanics.

The chapters in this LaTeX version are in the same order as the weekly lectures, but I've merged weeks together, and sometimes split them over chapter, to obtain a more textbook feel to these notes. For reference, the weekly lectures are outlined here.

Week 1: (Mar. 28, 30, Apr. 1)—The Lagrangian approach to classical mechanics: deriving $F = ma$ from the requirement that the particle's path be a critical point of the action. The prehistory of the Lagrangian approach: D'Alembert's "principle of least energy" in statics, Fermat's "principle of least time" in optics, and how D'Alembert generalized his principle from statics to dynamics using the concept of "inertia force".

Week 2: (Apr. 4, 6, 8)—Deriving the Euler-Lagrange equations for a particle on an arbitrary manifold. Generalized momentum and force. Noether's theorem on conserved quantities coming from symmetries. Examples of conserved quantities: energy, momentum and angular momentum.

Week 3 (Apr. 11, 13, 15)—Example problems: (1) The Atwood machine. (2) A frictionless mass on a table attached to a string threaded through a hole in the table, with a mass hanging on the string. (3) A special-relativistic free particle: two Lagrangians, one with reparametrization invariance as a gauge symmetry. (4) A special-relativistic charged particle in an electromagnetic field.

Week 4 (Apr. 18, 20, 22)—More example problems: (4) A special-relativistic charged particle in an electromagnetic field in special relativity, continued. (5) A general-relativistic free particle.

Week 5 (Apr. 25, 27, 29)—How Jacobi unified Fermat's principle of least time and Lagrange's principle of least action by seeing the classical mechanics of a particle in a potential as a special case of optics with a position-dependent index of refraction. The ubiquity of geodesic motion. Kaluza-Klein theory. From Lagrangians to Hamiltonians.

Week 6 (May 2, 4, 6)—From Lagrangians to Hamiltonians, continued. Regular and strongly regular Lagrangians. The cotangent bundle as phase space. Hamilton's equations. Getting Hamilton's equations directly from a least action principle.

Week 7 (May 9, 11, 13)—Waves versus particles: the Hamilton-Jacobi equation. Hamilton's principal function and extended phase space. How the Hamilton-Jacobi equation foreshadows quantum mechanics.

Week 8 (May 16, 18, 20)—Towards symplectic geometry. The canonical 1-form and the symplectic 2-form on the cotangent bundle. Hamilton's equations on a symplectic manifold. Darboux's theorem.

Week 9 (May 23, 25, 27)—Poisson brackets. The Schrödinger picture versus the Heisenberg picture in classical mechanics. The Hamiltonian version of Noether's theorem. Poisson algebras and Poisson manifolds. A Poisson manifold that is not symplectic. Liouville's theorem. Weil's formula.

Week 10 (June 1, 3, 5)—A taste of geometric quantization. Kähler manifolds.

If you find errors in these notes, please email me! I thank Sheeyun Park and Curtis Vinson for catching lots of errors.

Contents

Chapter 1

From Newton's Laws to Langrange's Equations

(Week 1, March 28, 30, April 1.)

Classical mechanics is a very peculiar branch of physics. It used to be considered the sum total of our theoretical knowledge of the physical universe (Laplace's daemon, the Newtonian clockwork), but now it is known as an idealization, a toy model if you will. The astounding thing is that probably all professional applied physicists still use classical mechanics. So it is still an indispensable part of any physicist's or engineer's education.

It is so useful because the more accurate theories that we know of (general relativity and quantum mechanics) make corrections to classical mechanics generally only in extreme situations (black holes, neutron stars, atomic structure, superconductivity, and so forth). Given that GR and QM are much harder theories to use and apply it is no wonder that scientists will revert to classical mechanics whenever possible.

So, what is classical mechanics?

1.1 Lagrangian and Newtonian Approaches

We begin by comparing the Newtonian approach to mechanics to the subtler approach of Lagrangian mechanics. Recall Newton's law:

$$F = ma \tag{1.1}$$

wherein we consider a particle moving in $\mathbb{R}^n$. Its position, say q, depends on time $t \in \mathbb{R}$, so it defines a function,

$$q : \mathbb{R} \longrightarrow \mathbb{R}^n.$$

From this function we can define **velocity**,

$$v = \dot{q} : \mathbb{R} \longrightarrow \mathbb{R}^n$$

where $\dot{q} = \frac{dq}{dt}$, and also **acceleration**,

$$a = \ddot{q} : \mathbb{R} \longrightarrow \mathbb{R}^n.$$

Now let $m > 0$ be the **mass** of the particle, and let F be a vector field on $\mathbb{R}^n$ called the **force**. Newton claimed that the particle satisfies $F = ma$. That is:

$$m\,a(t) = F\left(q(t)\right). \tag{1.2}$$

This is a 2nd-order differential equation for $q\colon \mathbb{R} \to \mathbb{R}^n$ which will have a unique solution given some $q(t_0)$ and $\dot{q}(t_0)$, provided the vector field F is 'nice' — by which we technically mean smooth and bounded (i.e., $|F(x)| < B$ for some $B > 0$, for all $x \in \mathbb{R}^n$).

We can then define a quantity called **kinetic energy**:

$$K(t) \;:=\; \frac{1}{2}m\,v(t)\cdot v(t) \tag{1.3}$$

This quantity is interesting because

$$\begin{aligned} \frac{d}{dt}K(t) &= m\,v(t)\cdot a(t) \\ &= F(q(t))\cdot v(t) \end{aligned}$$

So, kinetic energy goes up when you push an object in the direction of its velocity, and goes down when you push it in the opposite direction. Moreover,

$$\begin{aligned} K(t_1) - K(t_0) &= \int_{t_0}^{t_1} F(q(t))\cdot v(t)\,dt \\ &= \int_{t_0}^{t_1} F(q(t))\cdot \dot{q}(t)\,dt \end{aligned}$$

So, the change of kinetic energy is equal to the **work** done by the force, that is, the integral of F along the curve $q : [t_0, t_1] \to \mathbb{R}^n$. This implies (by Stokes's theorem relating line integrals to surface integrals of the curl) that the change in kinetic energy $K(t_1) - K(t_0)$ is independent of the curve going from $q(t_0) = a$ to $q(t_1) = b$ iff

$$\boldsymbol{\nabla}\times F = 0.$$

This in turn is true iff

$$F = -\boldsymbol{\nabla} V \tag{1.4}$$

for some function $V\colon \mathbb{R}^n \to \mathbb{R}$. This function is then unique up to an additive constant; we call it the **potential**. A force with this property is called **conservative**. Why? Because in this case we can define the **total energy** of the particle by

$$E(t) \;:=\; K(t) + V(q(t)) \tag{1.5}$$

where $V(t) := V(q(t))$ is called the **potential energy** of the particle, and then we can show that E is **conserved**: that is, constant as a function of time. To see this, note that $F = ma$ implies

$$\begin{aligned} \frac{d}{dt}\left[K(t) + V(q(t))\right] &= F(q(t)) \cdot v(t) + \boldsymbol{\nabla} V(q(t)) \cdot v(t) \\ &= 0, \qquad (\text{because } F = -\boldsymbol{\nabla} V). \end{aligned}$$

Conservative forces let us apply a whole bunch of cool techniques. In the Lagrangian approach we define a quantity

$$L \;:=\; K(t) - V(q(t)) \tag{1.6}$$

called the **Lagrangian**, and for any curve $q : [t_0, t_1] \to \mathbb{R}^n$ with $q(t_0) = a$, $q(t_1) = b$, we define the **action** to be

$$S(q) \;:=\; \int_{t_0}^{t_1} L(t)\, dt \tag{1.7}$$

From here one can go in two directions. One is to claim that nature causes particles to follow paths of least action, and derive Newton's equations from that principle. The other is to start with Newton's principles and find out what conditions, if any, on $S(q)$ follow from this. We will use the shortcut of hindsight, bypass the philosophy, and simply use the mathematics of variational calculus to show that particles follow paths that are 'critical points' of the action $S(q)$ if and only if Newton's law $F = ma$ holds. To do this,

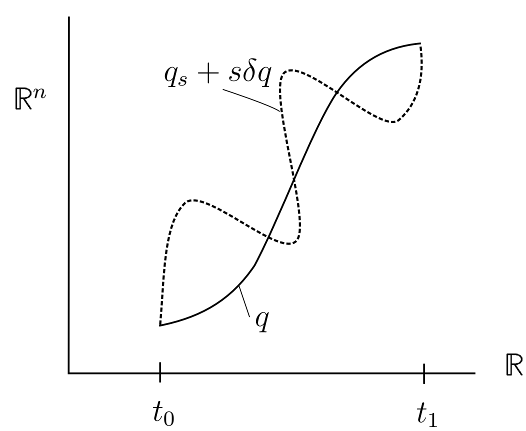


Figure 1.1: A particle can sniff out the path of least action.

let us look for curves (like the solid line in Fig. 1.1) that are **critical points** of S, namely:

$$\frac{d}{ds} S(q_s)|_{s=0} = 0 \tag{1.8}$$

where

$$q_s = q + s\delta q$$

for all $\delta q : [t_0, t_1] \to \mathbb{R}^n$ with,

$$\delta q(t_0) = \delta q(t_1) = 0.$$

To show that

$$F = ma \quad \Leftrightarrow \quad \frac{d}{ds} S(q_s)|_{t=0} = 0 \text{ for all } \delta q \colon [t_0, t_1] \to \mathbb{R}^n \text{ with } \delta q(t_0) = \delta q(t_1) = 0 \quad (1.9)$$

we start by using integration by parts on the definition of the action, and first noting that $dq_s/ds = \delta q(t)$ is the variation in the path:

$$\begin{aligned}
\frac{d}{ds} S(q_s)\bigg|_{s=0} &= \frac{d}{ds} \int_{t_0}^{t_1} \frac{1}{2} m\dot{q}_s(t) \cdot \dot{q}_s(t) - V(q_s(t))\, dt \bigg|_{s=0} \\
&= \int_{t_0}^{t_1} \frac{d}{ds} \left[\frac{1}{2} m\dot{q}_s(t) \cdot \dot{q}_s(t) - V(q_s(t))\right] dt \bigg|_{s=0} \\
&= \int_{t_0}^{t_1} \left[m\dot{q}_s \cdot \frac{d}{ds}\dot{q}_s(t) - \boldsymbol{\nabla} V(q_s(t)) \cdot \frac{d}{ds} q_s(t)\right] dt \bigg|_{s=0}
\end{aligned}$$

Note that

$$\frac{d}{ds}\dot{q}_s(t) = \frac{d}{ds}\frac{d}{dt} q_s(t) = \frac{d}{dt}\frac{d}{ds} q_s(t)$$

and when we set $s = 0$ this quantity becomes just:

$$\frac{d}{dt}\delta q(t)$$

So,

$$\frac{d}{ds} S(q_s)\bigg|_{s=0} = \int_{t_0}^{t_1} \left[m\dot{q} \cdot \frac{d}{dt}\delta q(t) - \boldsymbol{\nabla} V(q(t)) \cdot \delta q(t)\right] dt$$

Then we can integrate by parts, noting the boundary terms vanish because $\delta q = 0$ at t_1 and t_0:

$$\frac{d}{ds} S(q_s)|_{s=0} = \int_{t_0}^{t_1} [-m\ddot{q}(t) - \boldsymbol{\nabla} V(q(t))] \cdot \delta q(t) dt$$

It follows that variation in the action is zero for *all* variations δq iff the term in brackets is identically zero, that is,

$$-m\ddot{q}(t) - \boldsymbol{\nabla} V(q(t)) = 0$$

So, the path q is a critical point of the action S iff

$$F = ma. \qquad \square$$

The above result applies only for conservative forces, i.e., forces that can be written as the minus the gradient of some potential. However, this seems to be true of the most fundamental forces that we know of in our universe. It is a simplifying assumption that has withstood the test of time and experiment.

1.1.1 Lagrangian versus Hamiltonian Approaches

I am not sure where to mention this, but before launching into the history of the Lagrangian approach may be as good a time as any. In later chapters we will describe another approach to classical mechanics: the Hamiltonian approach. Why do we need two approaches, Lagrangian and Hamiltonian?

They both have their own advantages. In the simplest terms, the Hamiltonian approach focuses on *position and momentum*, while the Lagrangian approach focuses on *position and velocity*. The Hamiltonian approach focuses on *energy*, which is a function of position and momentum — indeed, 'Hamiltonian' is just a fancy word for energy. The Lagrangian approach focuses on the *Lagrangian*, which is a function of position and velocity. Our first task in understanding Lagrangian mechanics is to get a gut feeling for what the Lagrangian means. The key is to understand the integral of the Lagrangian over time – the 'action', S. We shall see that this describes the 'total amount that happened' from one moment to another as a particle traces out a path. And, peeking ahead to quantum mechanics, the quantity $\exp(iS/\hbar)$, where $\hbar$ is Planck's constant, will describe the 'change in phase' of a *quantum* system as it traces out this path.

In short, while the Lagrangian approach takes a while to get used to, it provides invaluable insights into classical mechanics and its relation to quantum mechanics. We shall see this in more detail soon.

1.2 Prehistory of the Lagrangian Approach

We've seen that a particle going from point a at time t_0 to a point b at time t_1 follows a path that is a critical point of the action,

$$S = \int_{t_0}^{t_1} K - V \, dt$$

so that slight changes in its path do not change the action (to first order). Often, though not always, the action is minimized, so this is called the **Principle of Least Action**.

Suppose we did not have the hindsight afforded by the Newtonian picture. Then we might ask, "Why does nature like to minimize the action? And why *this* action $\int K - V\, dt$? Why not some other action?"

'Why' questions are always tough. Indeed, some people say that scientists should never ask 'why'. This seems too extreme: a more reasonable attitude is that we should only ask a 'why' question if we expect to learn something scientifically interesting in our attempt to answer it.

There are certainly some interseting things to learn from the question "why is action minimized?" First, note that total energy is conserved, so energy can slosh back and forth between kinetic and potential forms. The Lagrangian $L = K - V$ is big when most of the energy is in kinetic form, and small when most of the energy is in potential form.

Kinetic energy measures how much is 'happening' — how much our system is moving around. Potential energy measures how much *could* happen, but isn't yet — that's what the word 'potential' means. (Imagine a big rock sitting on top of a cliff, with the potential to fall down.) So, the Lagrangian measures something we could vaguely refer to as the 'activity' or 'liveliness' of a system: the higher the kinetic energy the more lively the system, the higher the potential energy the less lively. So, we're being told that nature likes to minimize the total of 'liveliness' over time: that is, the total action.

In other words, nature is as lazy as possible!

For example, consider the path of a thrown rock in the Earth's gravitational field, as in Fig. 1.2. The rock traces out a parabola, and we can think of it as doing this in order

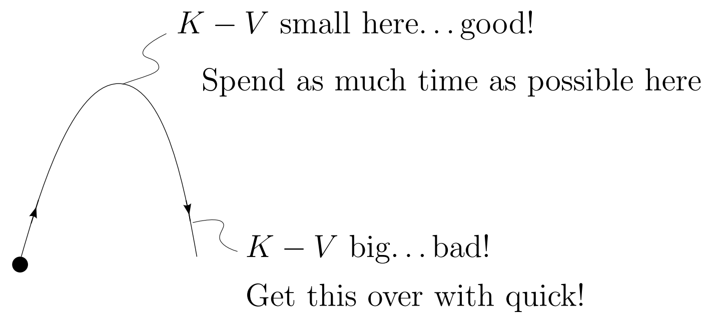


Figure 1.2: A particle's "lazy" motion, minimizes the action.

to minimize its action. On the one hand, it wants to spend a lot much time near the top of its trajectory, since this is where the kinetic energy is least and the potential energy is greatest. On the other hand, if it spends *too* much time near the top of its trajectory, it will need to really rush to get up there and get back down, and this will take a lot of action. The perfect compromise is a parabolic path!

Here we are anthropomorphizing the rock by saying that it 'wants' to minimize its action. This is okay if we don't take it too seriously. Indeed, one of the virtues of the Principle of Least Action is that it lets us put ourselves in the position of some physical system and imagine what we would do to minimize the action.

There is another way to make progress on understanding 'why' action is minimized: history. Historically there were two principles that were fairly easy to deduce from observations of nature: (i) the principle of minimum energy used in statics, and (ii) the principle of least time, used in optics. By putting these together, we can guess the principle of least action. So, let us recall these earlier minimum principles.

1.2.1 The Principle of Minimum Energy

Before physicists really got going in studying dynamical systems they used to study statics. **Statics** is the study of objects at rest, or in equilibrium. Archimedes studied the laws of

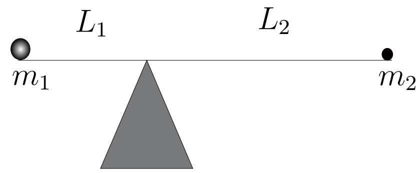


Figure 1.3: A principle of energy minimization determines a lever's balance.

a see-saw or lever (Fig. 1.3), and he found that this would be in equilibrium if

$$m_1 L_1 = m_2 L_2.$$

Later D'Alembert understood this using his "principle of virtual work". He considered moving the lever slightly, i.e., infinitesimally, He claimed that in equilibrium the infinites-

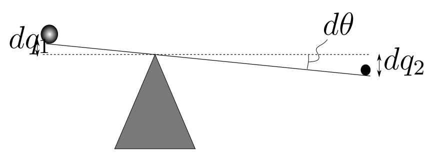


Figure 1.4: A principle of energy minimization determines a lever's balance.

imal work done by this motion is zero! He also claimed that the work done on the i^{th} body is,

$$dW_i = F_i dq_i$$

and gravity pulls down with a force $m_i g$ so,

$$\begin{aligned} dW_i &= (0, 0, -mg) \cdot (0, 0, -L_1 d\theta) \\ &= m_1 g L_1 \, d\theta \end{aligned}$$

and similarly,

$$dW_2 = -m_2 g L_2 \, d\theta$$

Now D'Alembert's principle says that equilibrium occurs when the "virtual work" $dW = dW_1 + dW_2$ vanishes for all $d\theta$ (that is, for all possible infinitesimal motions). This happens when

$$m_1 L_1 - m_2 L_2 = 0$$

which is just as Archimedes wrote.

1.2.2 D'Alembert's Principle and Lagrange's Equations

Let's go over the above analysis in more detail. I'll try to make it clear what we mean by **virtual work**.

The forces and constraints on a system may be time dependent. So equal small infinitesimal displacements of the system might result in the forces $\mathbf{F}_i$ acting on the system doing different amounts of work at different times. To *displace a system by* $\delta\mathbf{r}_i$ *for each position coordinate, and yet remain consistent with all the constraints and forces at a given instant of time* t, without any time interval passing is called a *virtual displacement*. It's called 'virtual' because it cannot be realized: any actual displacement would occur over a finite time interval and possibly during which the forces and constraints might change. Now call the work done by this hypothetical virtual displacement, $\mathbf{F}_i \cdot \delta\mathbf{r}_i$, the **virtual work**. Consider a system in the special state of being in equilibrium, i.e.,. when $\sum \mathbf{F}_i = 0$. Then because by definition the virtual displacements do not change the forces, we must deduce that the virtual work vanishes for a system in equilibrium,

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0, \qquad \text{(when in equilibrium)} \tag{1.10}$$

Note that in the above example we have two particles in $\mathbb{R}^3$ subject to a constraint (they are pinned to the lever arm). However, a number n of particles in $\mathbb{R}^3$ can be treated as a single quasi-particle in $\mathbb{R}^{3n}$, and if there are constraints it can move in some submanifold of $\mathbb{R}^{3n}$. So ultimately we need to study a particle on an arbitrary *manifold*. But, we'll postpone such sophistication for a while.

For a particle in $\mathbb{R}^n$, D'Alembert's principle simply says,

$$\begin{array}{lll}
q(t) = q_0 \quad \text{satisfies} \quad F = ma, & & \text{(it's in equilibrium)} \\
\qquad \Updownarrow & & \\
dW = F \cdot dq \quad \text{vanishes for all} \quad dq \in \mathbb{R}^n, & & \text{(virtual work is zero for } \delta q \to 0) \\
\qquad \Updownarrow & & \\
\quad F = 0, & & \text{(no force on it!)}
\end{array}$$

If the force is conservative ($F = -\boldsymbol{\nabla} V$) then this is also equivalent to,

$$\boldsymbol{\nabla} V(q_0) = 0$$

that is, we have equilibrium at a critical point of the potential. The equilibrium will be **stable** if q_0 is a local minimum of the potential V.

We can summarize all the above by proclaiming that we have a "principle of least energy" governing stable equilibria. We also have an analogy between statics and dynamics,

Statics	Dynamics
equilibrium, $a = 0$	$F = ma$
potential, V	action, $S = \int_{t_0}^{t_1} K - V\,dt$
critical points of V	critical points of S

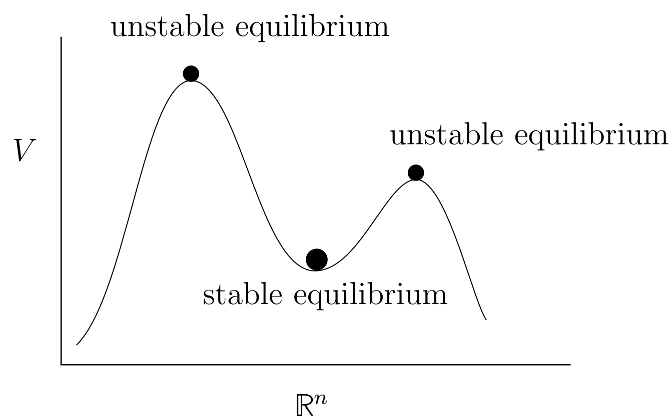


Figure 1.5: A principle of energy minimization determines a lever's balance.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{1.11}$$

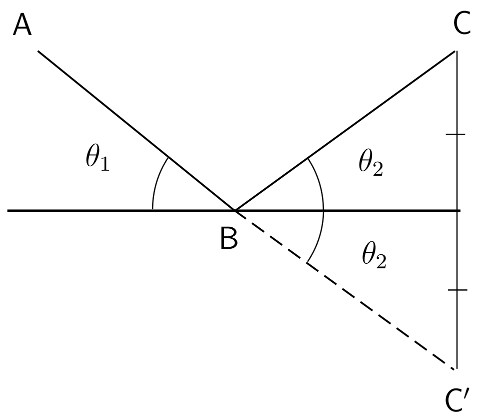
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{1.12}$$

D'Alembert's principle is an expression for Newton's second law under conditions where the virtual work done by the forces of constraint is zero.

1.2.3 The Principle of Least Time

Time now to look at the second piece of history surrounding the principles of Lagrangian mechanics. As well as hints from statics, there were also hints from the behavior of light, hints again that nature likes to minimize effort. In a vacuum light moves in straight lines, which in Euclidean space is the minimum distance. But more interesting than straight lines are piecewise straight paths and curves. Consider reflection of light from a mirror,

What path does the light take? The empirical answer was known since antiquity, it chooses B such that $\theta_1 = \theta_2$, so the angle of incidence equals the angle of reflection. But this is precisely the path that minimizes the distance of the trajectory subject to the condition that it must hit the mirror (at least at one point). In fact light traveling from A to B takes *both* the straight paths ABC and AC. Why is ABC the shortest path hitting

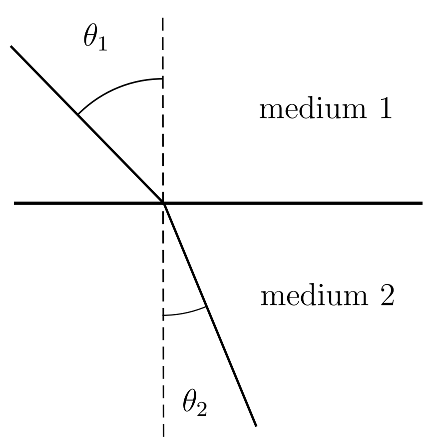


the mirror? This follows from some basic Euclidean geometry:

$$\begin{aligned} B \text{ minimizes } AB + BC &\Leftrightarrow B \text{ minimizes } AB + BC' \\ &\Leftrightarrow A, B, C' \text{ lie on a line} \\ &\Leftrightarrow \theta_1 = \theta_2 \end{aligned}$$

Note the introduction of the fictitious image C′ "behind" the mirror, this is a trick often used in solving electrostatic problems (a conducting surface can be replaced by fictitious mirror image charges to satisfy the boundary conditions), it is also used in geophysics when one has a geological fault, and in hydrodynamics when there is a boundary between two media (using mirror image sources and sinks).

The big clue leading to D'Alembert's principle however came from refraction of light. Snell (and predecessors) noted that each medium has some number n associated with it,



called the **index of refraction**, such that,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

(having normalized n so that for a vacuum $n = 1$). Someone guessed the explanation, realizing that if the speed of light in a medium is proportional to $1/n$, then light will satisfy Snell's law if the light minimizes the *time* it takes to get from A to C. In the case of refraction it is the *time* that is important, not just the path distance. But the same is true for the law of reflection, since in that case the path of minimum length gives the same results as the path of minimum time.

So, not only is light the fastest thing around, it's also always taking the quickest path from here to there!

1.2.4 How D'Alembert and Others Got to the Truth

Sometimes laws of physics are just guessed using a bit of intuition and a gut feeling that nature must be beautiful or elegantly simple (though occasionally awesomely complex in beauty). One way to make good guesses is to *generalize.*

D'Alembert's principle of virtual work for statics says that equilibrium occurs when

$$F(q_0) \cdot \delta q = 0, \qquad \forall \delta q \in \mathbb{R}^n$$

D'Alembert generalized this to dynamics by inventing what he called the "inertia force"$=-m\,a$, and he postulated that in dynamics equilibrium occurs when the **total force**$=F+$inertia force, vanishes. Or symbolically when,

$$\big(F(q(t)) - ma(t)\big) \cdot \delta q(t) = 0 \tag{1.13}$$

We then take a variational path parameterized by s,

$$q_s(t) = q(t) + s\,\delta q(t)$$

where

$$\delta q(t_0) = \delta q(t_1) = 0$$

and with these paths, for any function f on the space of paths we can define the **variational** derivative,

$$\delta f \;:=\; \frac{d}{ds} f(q_s)\Big|_{s=0} \tag{1.14}$$

Then D'Alembert's principle of virtual work implies

$$\int_{t_0}^{t_1} \Big(F(q) - m\ddot{q}\Big) \cdot \delta q\, dt = 0$$

for all δq, so if $F = -\boldsymbol{\nabla} V$, we get

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \Big(-\boldsymbol{\nabla} V(q) - m\ddot{q}\Big) \cdot \delta q\, dt \\ &= \int_{t_0}^{t_1} \Big(-\boldsymbol{\nabla} V(q) \cdot \delta q + m\dot{q}\delta\dot{q}\Big)\, dt \end{aligned}$$

using

$$\frac{d}{ds}V(q_s(t))\bigg|_{s=0} = \frac{dV}{dq}\,\frac{dq_s(t)}{ds}\bigg|_{s=0}$$

and

$$\delta(\dot{q}^2) = \frac{d\dot{q}_s^2(t)}{ds}$$

then we have

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} \Big(-\frac{dV}{dq}\frac{dq_s}{dt} + \frac{m}{2}\frac{d\dot{q}_s(t)}{ds}\Big)\,dt \\
&= \frac{d}{ds}\int_{t_0}^{t_1} \Big(-\frac{d}{dt}V\big(q_s(t)\big) + \frac{m}{2}\dot{q}_s(t)\Big)\bigg|_{s=0} dt \\
&= \delta\bigg(\int_{t_0}^{t_1} \Big(-\frac{d}{dt}V\big(q_s(t)\big) + \frac{m}{2}\dot{q}_s(t)^2\Big)\,dt\bigg)
\end{aligned}$$

therefore

$$\delta\Big(\int_{t_0}^{t_1} \big(-V(q) + K\big)\,dt\Big) = 0$$

so the path taken by the particle is a critical point of the action,

$$S(q) = \int (K - V)\,dt \tag{1.15}$$

We've described how D'Alembert might have arrived at the principle of least action by generalizing previously known energy minimization and least time principles. Still, there's something unsatisfying about the treatment so far. We do not really understand why one must introduce the 'inertia force'. We only see that it's necessary to obtain agreement with Newtonian mechanics (which is manifest in Eq.(1.13)).

We conclude with a few more words about this mystery. Recall from undergraduate physics that in an accelerating coordinate system there is a fictional force $= ma$, which is called the centrifugal force. We use it, for example, to analyze simple physics in a rotating reference frame. If you are *inside* the rotating system and you throw a ball straight ahead it will appear to curve away from your target, and if you did not know that you were rotating relative to the rest of the universe then you'd think there was a force on the ball equal to the centrifugal force. If you are inside a big rapidly rotating drum then you'll also feel pinned to the walls. This is an example of an inertia force which comes from using a funny coordinate system. In general relativity, one sees that — in a certain sense — *gravity* is an inertia force!

Chapter 2

Equations of Motion

(Week 2, April 4, 6, 8.)

In this chapter we'll start to look at the Lagrangian equations of motion in more depth. We'll look at some specific examples of problem solving using the Euler-Lagrange equations. First we'll show how the equations are derived.

2.1 The Euler-Lagrange Equations

We are going to start thinking of a general classical system as a set of points in an abstract *configuration space* or phase space[1]. So consider an arbitrary classical system as living in a space of points in some manifold Q. For example, the space for a spherical double pendulum would look like Fig. 2.1, where $Q = S^2 \times S^2$. So our system is "a particle in

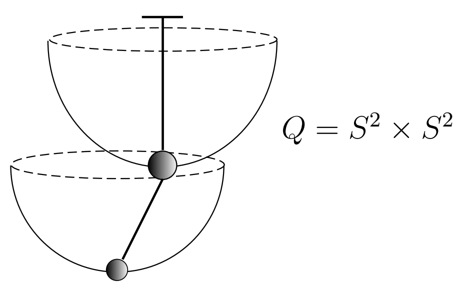


Figure 2.1: Double pendulum configuration space.

Q", which means you have to disabuse yourself of the notion that we're dealing with real

[1]The tangent bundle TQ will be referred to as *configuration space*, later on when we get to the chapter on Hamiltonian mechanics we'll find a use for the cotangent bundle T^*Q, and normally we call this the *phase space*.

particles, we are not, we are dealing with a *single quasi-particle* in an abstract higher dimensional space. The single quasi-particle represents two real particles if we are talking about the classical system in Fig. 2.1. Sometimes to make this clear we'll talk about "the system taking a path", instead of "the particle taking a path". It is then clear that when we say, "the system follows a path $q(t)$" that we're referring to the point q in configuration space Q that represents all of the particles in the real system.

So as time passes "the system" traces out a path

$$q : [t_0, t_1] \longrightarrow Q$$

and we define it's *velocity*,

$$\dot{q}(t) \in T_{q(t)}Q$$

to be the tangent vector at $q(t)$ given by the equivalence class $[\sigma]$ of curves through $q(t)$ with derivatives $\dot{\sigma}(t) = dq(s)/ds|_{s=t}$. We'll just write is as $\dot{q}(t)$.

Let Γ be the space of smooth paths from $a \in Q$ to $b \in Q$,

$$\Gamma = \{q : [t_0, t_1] \to Q | q(t_0) = a, q(t_1) = b\}$$

(Γ is an infinite dimensional manifold, but we won't go into that for now.) Let the *Lagrangian*$=L$ for the system be *any* smooth function of position and velocity (not explicitly of time, for simplicity),

$$L : TQ \longrightarrow \mathbb{R}$$

and define the *action*, S:

$$S : \Gamma \longrightarrow \mathbb{R}$$

by

$$S(q) := \int_{t_0}^{t_1} L(q, \dot{q})\, dt \tag{2.1}$$

The path that the quasi particle will actually take is a critical point of S, in accord with D'Alembert's principle of least action. In other words, a path $q \in \Gamma$ such that for any smooth 1-parameter family of paths $q_s \in \Gamma$ with $q_0 = q_1$, we have

$$\frac{d}{ds} S(q_s)\Big|_{s=0} = 0 \tag{2.2}$$

We write,

$$\frac{d}{ds}\Big|_{s=0} \quad \text{as} \quad \text{“}\delta''$$

so Eq.(2.2) can be rewritten

$$\delta S = 0 \tag{2.3}$$

2.1.1 Comments

What is a "1-parameter family of paths"? Well, a path is a curve, or a 1D manifold. So the 1-parameter family is nothing more nor less than a set of well-defined paths $\{q_s\}$, each one labeled by a parameter s. So a smooth 1-parameter family of paths will have $q(s)$ everywhere infinitesimally close to $q(s+\epsilon)$ for an infinitesimal hyperreal ϵ. So in Fig. 2.2

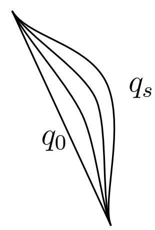


Figure 2.2: Schematic of a 1-parameter family of curves.

we can go from q_0 to q_s by smoothly varying s from $s = 0$ to $s = s$

What does the condition $\delta S = 0$ imply? Patience, we are just getting to that. We will now start to explore what $\delta S = 0$ means for our Lagrangian.

2.1.2 Lagrangian Dynamics

We were given that Q is a manifold, so it admits a covering of coordinate charts. For now, let's pick coordinates in a neighborhood U of some point $q(t) \in Q$. Next, consider only variations q_s such that $q_s = q$ outside U. A cartoon of this looks like Fig. 2.3 Then

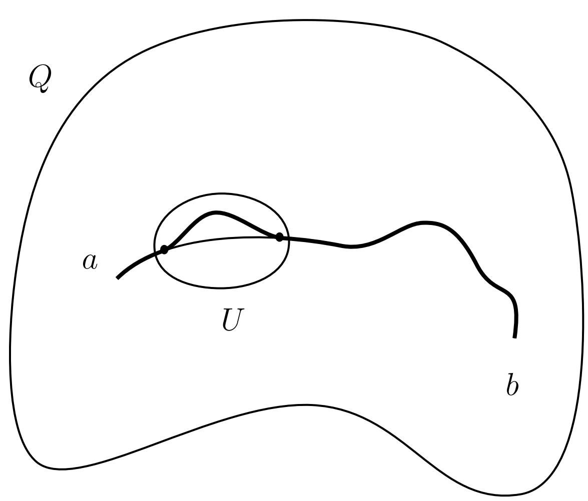


Figure 2.3: Local path variation.

we restrict attention to a subinterval $[t_0', t_1'] \subseteq [t_0, t_1]$ such that $q_s(t) \in U$ for $t_0' \leq t \leq t_1'$.

Let's just go ahead and rename t_0' and t_1' as " t_0 and t_1" to drop the primes. We can use the coordinate charts on U,

$$\begin{aligned} \varphi : U &\longrightarrow \mathbb{R}^n \\ x &\longmapsto \varphi(x) = (x^1, x^2, \ldots, x^n) \end{aligned}$$

and we also have coordinates for the 1-forms,

$$\begin{aligned} d\varphi : TU &\longrightarrow T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \\ (x, y) &\longmapsto d\varphi(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n) \end{aligned}$$

where $y \in T_x Q$. We restrict $L : T\mathcal{M} \to \mathbb{R}$ to $TU \subseteq T\mathcal{M}$, in our case the manifold is $\mathcal{M} = Q$, and then we can describe it, L, using the coordinates x^i, y^i on TU. The ^{i} are generalized *position* coordinates, the y^i are the associated generalized *velocity* coordinates. (Velocity and position are in the abstract configuration space Q). Using these coordinates we get

$$\begin{aligned} \delta S &= \delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t))\, dt \\ &= \int_{t_0}^{t_1} \delta L(q, \dot{q})\, dt \\ &= \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial x^i} \delta q^i + \frac{\partial L}{\partial y^i} \delta \dot{q}^i \Big)\, dt \end{aligned}$$

where we've used the given smoothness of L and the Einstein summation convention for repeated indices i. Note that we can write δL as above using a local coordinate patch because the path variations δq are entirely trivial outside the patch for U. Continuing, using the Leibniz rule

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial y} \delta q \Big) = \frac{d}{dt} \frac{\partial L}{\partial y} \delta q + \frac{\partial L}{\partial y} \delta \dot{q}$$

we have,

$$\begin{aligned} \delta S &= \int_{t_0}^{t_1} \Big(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} \Big) \delta q^i(t)\, dt \\ &= 0. \end{aligned}$$

If this integral is to vanish as demanded by $\delta S = 0$, then it must vanish for all path variations δq, further, the boundary terms vanish because we deliberately chose δq that vanish at the endpoints t_0 and t_1 inside U. That means the term in brackets must be identically zero, or

$$\frac{d}{dt}\frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i} = 0 \tag{2.4}$$

This is necessary to get $\delta S = 0$, for all δq, but in fact it's also sufficient. Physicists always give the coordinates x^i, y^i on TU the symbols "q^i" and "$\dot{q}^i$", despite the fact that these also have another meaning, namely the x^i and y^i coordinates of the quantity,

$$\big(q(t), \dot{q}(t)\big) \in TU.$$

So in any case, physicists write,

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}}$$

and they call these the Euler-Lagrange equations.

2.2 Interpretation of Terms

The derivation of the Euler-Lagrange equations above was fairly abstract, the terms "position" and "velocity" were used but were not assumed to be the usual kinematic notions that we are used to in physics, indeed the only reason we used those terms was for their analogical appeal. Now we'll try to illuminate the E-L equations a bit by casting them into the usual position and velocity terms.

So consider,

$$\begin{aligned} Q &= \mathbb{R} \\ L(q, \dot{q}) &= \frac{1}{2} m \dot{q} \cdot \dot{q} - V(q) \\ &= \frac{1}{2} m \dot{q}^i \dot{q}_i - V(q) \end{aligned}$$

TERMS	MEANING (in this example)	MEANING (in general)
$\frac{\partial L}{\partial \dot{q}^i}$	$m\dot{q}$	the momentum p_i
$\frac{\partial L}{\partial q^i}$	$-(\boldsymbol{\nabla} V)_i$	the force F_i

When we write $\partial V / \partial q^i = \boldsymbol{\nabla} V$ we're assuming the q^i are Cartesian coordinates on Q.

So translating our example into general terms, if we conjure up some abstract Lagrangian then we can think of the independent variables as generalized positions and velocities, and then the Euler-Lagrange equations can be interpreted as equations relating generalized concepts of *momentum* and *force*, and they say that

$$\dot{p} = F \tag{2.5}$$

So there's no surprise that in the mundane case of a single particle moving in $\mathbb{R}^3$ under time t this just recovers Newton II. Of course we can do all of our classical mechanics with Newton's laws, it's just a pain in the neck to deal with the redundancies in $F = ma$ when we could use symmetry principles to vastly simplify many examples. It turns out that the Euler-Lagrange equations are one of the reformulations of Newtonian physics that make it highly convenient for introducing symmetries and consequent simplifications. Simplifications generally mean quicker, shorter solutions and more transparent analysis or at least more chance at insight into the characteristics of the system. The main thing is that when we use symmetry to simplify the equations we are reducing the number of independent variables, so it gets closer to the fundamental degrees of freedom of the system and so we cut out a lot of the wheat and chaff (so to speak) with the full redundant Newton equations.

One can of course introduce simplifications when solving Newton's equations, it's just that it's easier to do this when working with the Euler-Lagrange equations. Another good reason to learn Lagrangian (or Hamiltonian) mechanics is that it translates better into quantum mechanics.

Chapter 3

Lagrangians and Noether's Theorem

If the form of a system of dynamical equations does not change under spatial translations then the momentum is a conserved quantity. When the form of the equations is similarly invariant under time translations then the total energy is a conserved quantity (a constant of the equations of motion). Time and space translations are examples of 1-parameter groups of transformations. *Invariance under a group of transformations* is precisely what we mean by a symmetry in group theory. So symmetries of a dynamical system give conserved quantities or conservation laws. The rigorous statement of all this is the content of *Noether's theorem.*

3.1 Time Translation

To handle time translations we need to replace our paths $q : [t_0, t_1] \to Q$ by paths $q : \mathbb{R} \to Q$, and then define a new space of paths,

$$\Gamma = \{q : \mathbb{R} \to Q\}.$$

The bad news is that the action

$$S(q) = \int_{-\infty}^{\infty} L\Big(q(t), \dot{q}(t)\Big)\, dt$$

typically will not converge, so S is then no longer a function of the space of paths. Nevertheless, if $\delta q = 0$ outside of some finite interval, then the functional variation,

$$\delta S \;:= \int_{-\infty}^{\infty} \frac{d}{ds} L\Big(q_s(t), \dot{q}_s(t)\Big)\Big|_{s=0} \; dt$$

will converge, since the integral is smooth and vanishes outside this interval. Moreover, demanding that this δS vanishes for all such variations δq is enough to imply the Euler-

Lagrange equations:

$$\begin{aligned}
\delta S &= \int_{-\infty}^{\infty} \frac{d}{ds} L\Big(q_s(t), \dot{q}_s(t)\Big)\Big|_{s=0} \, dt \\
&= \int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \\
&= \int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt
\end{aligned}$$

where again the boundary terms have vanished since $\delta q = 0$ near $t = \pm\infty$. To be explicit, the first term in

$$\frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q$$

vanishes when we integrate. Then the whole thing vanishes for all compactly supported smooth δq iff

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

Recall that,

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}_i} &= p_i, \quad \text{is the generalized momentum, by defn.} \\
\frac{\partial L}{\partial q_i} &= \dot{p}_i, \quad \text{is the force, by the E-L eqns.}
\end{aligned}$$

Note the similarity to Hamilton's equations—if you change L to H you need to stick in a minus sign, and change variables from $\dot{q}$ to p_i and eliminate $\dot{p}_i$.

3.1.1 Canonical and Generalized Coordinates

In light of this noted similarity with the Hamilton equations of motion, let's spend a few moments clearing up some terminology (I hate using jargon, but sometimes it's unavoidable, and sometimes it can be efficient—provided everyone is clued in).

Generalized Coordinates

For Lagrangian mechanics we have been using *generalized coordinates*, these are the $\{q_i, \dot{q}_i\}$. The q_i are generalized positions, and the $\dot{q}_i$ are generalized velocities. The full set of independent generalized coordinates represent the degrees of freedom of a particle, or system of particles. So if we have N particles then we'd typically have $6N$ generalized coordinates (the "6" is for 3 space dimensions, and at each point a position and a momentum). These can be in any reference frame or system of axes, so for example, in a Cartesian frame, with two particles, in 3D space we'd have the $2 \times 3 = 6$ position coordinates, and $2 \times 3 = 6$ velocities,

$$\{x_1, y_1, z_1, x_2, y_2, z_2\}, \{u_1, v_1, w_1, u_2, v_2, w_2\}$$

where say $u = v_x$, $v = v_y$, $w = v_z$ are the Cartesian velocity components. This makes $12 = 6 \times 2 = 6N$ coordinates, matching the total degrees of freedom as claimed. If we constrain the particles to move in a plane (say place them on a table in a gravitational field) then we get $2N$ fewer degrees of freedom, and so $4N$ d.o.f. overall. By judicious choice of coordinate frame we can eliminate one velocity component and one position component for each particle.

It is also handy to respect other symmetries of a system, maybe the particles move on a sphere for example, one can then define new positions and momenta with a consequent reduction in the number of these generalized coordinates needed to describe the system.

Canonical Coordinates

In Hamiltonian mechanics (which we have not yet fully introduced) we will find it more useful to transform from generalized coordinates to canonical coordinates. The canonical coordinates are a special set of coordinates on the cotangent bundle of the configuration space manifold Q. They are usually written as a set of (q^i, p_j) or (x^i, p_j) with the x's or q's denoting the coordinates on the underlying manifold and the p's denoting the conjugate momentum, which are 1-forms in the cotangent bundle at the point q in the manifold.

It turns out that the q^i together with the p_j, form a coordinate system on the cotangent bundle T^*Q of the configuration space Q, hence these coordinates are called the *canonical coordinates.*

We will not discuss this here, but if you care to know, later on we'll see that the relation between the generalized coordinates and the canonical coordinates is given by the Hamilton-Jacobi equations for a system.

3.2 Symmetry and Noether's Theorem

First, let's give a useful definition that will make it easy to refer to a type of dynamical system symmetry. We want to refer to symmetry transformations (of the Lagrangian) governed by a single parameter.

Definition 3.1 (one-parameter family of symmetries)**.** *A 1-parameter family of symmetries of a Lagrangian system $L : TQ \to \mathbb{R}$ is a smooth map,*

$$\begin{aligned} F : \mathbb{R} \times \Gamma &\longrightarrow \Gamma \\ (s, q) &\longmapsto q_s, \qquad \textit{with } q_0 = q \end{aligned}$$

such that there exists a function $\ell(q, \dot{q})$ for which

$$\delta L = \frac{d\ell}{dt}$$

for some $\ell : TQ \to \mathbb{R}$, that is,

$$\frac{d}{ds} L\Big(q_s(t), \dot{q}_s(t)\Big)\Big|_{s=0} = \frac{d}{dt}\ell\Big(q_s(t), \dot{q}_s(t)\Big)$$

for all paths q.

Remark: The simplest case is $\delta L = 0$, in which case we really have a way of moving paths around ($q \mapsto q_s$) that doesn't change the Lagrangian—i.e., a symmetry of L in the most obvious way. But $\delta L = \frac{d}{dt}\ell$ is a sneaky generalization whose usefulness will become clear.

3.2.1 Noether's Theorem

Here's a statement of the theorem. Note that ℓ in this theorem is the function associated with F in definition 3.1.

Theorem 3.1 (Noether's Theorem)**.** *Suppose F is a one-parameter family of symmetries of the Lagrangian system, $L : TQ \to \mathbb{R}$. Then,*

$$p^i \delta q_i - \ell$$

is conserved, that is, it's time derivative is zero for any path $q \in \Gamma$ satisfying the Euler-Lagrange equations. In other words, in boring detail,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial y^i}\big(q(s)\dot{q}(s)\big)\ \frac{d}{ds} q_s^i(t)\Big|_{s=0} - \ell\big(q(t), \dot{q}(t)\big)\right] = 0$$

Proof.

$$\begin{aligned}\frac{d}{dt}\Big(p_i \delta q^i - \ell\Big) &= \dot{p}_i \delta q^i + p_i \delta \dot{q}^i - \frac{d}{dt}\ell \\ &= \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i - \delta L \\ &= \delta L - \delta L = 0.\end{aligned}$$

□

"Okay, big deal" you might say. Before this can be of any use we'd need to find a symmetry F. Then we'd need to find out what this $p^i \delta q_i - \ell$ business is that is conserved. So let's look at some examples.

Example

1. **Conservation of Energy.** (The most important example!)

   All of our Lagrangian systems will have time translation invariance (because the laws of physics do not change with time, at least not to any extent that we can tell). So we have a one-parameter family of symmetries

   $$q_s(t) = q(t+s)$$

   This indeed gives,

   $$\delta L = \dot{L}$$

   for

   $$\frac{d}{ds}L(q_s)\bigg|_{s=0} = \frac{d}{dt}L = \dot{L}$$

   so here we take $\ell = L$ simply! We then get the conserved quantity

   $$p_i \delta q^i - \ell = p_i \dot{q}^i - L$$

   which we normally call the *energy*. For example, if $Q = \mathbb{R}^n$, and if

   $$L = \frac{1}{2}m\dot{q}^2 - V(q)$$

   then this quantity is

   $$m\dot{q}\cdot\dot{q} - \left(\frac{1}{2}m\dot{q}\cdot\dot{q} - V\right) = \frac{1}{2}m\dot{q}^2 + V(q)$$

   The term in parentheses is $K - V$, and the left-hand side is $K + V$.

Let's repeat this example, this time with a specific Lagrangian. It doesn't matter what the Lagrangian is, if it has 1-parameter families of symmetries then it'll have conserved quantities, guaranteed. The trick in physics is to write down a correct Lagrangian in the first place! (Something that will accurately describe the system of interest.)

3.3 Conserved Quantities from Symmetries

We've seen that any 1-parameter family

$$\begin{aligned} F_s : \Gamma &\longrightarrow \Gamma \\ q &\longmapsto q_s \end{aligned}$$

which satisfies

$$\delta L = \dot{\ell}$$

for some function $\ell = \ell(q, \dot{q})$ gives a conserved quantity

$$p_i \delta q^i - \ell$$

As usual we've defined

$$\delta L := \frac{d}{ds} L\Big(q_s(t), \dot{q}_s(t)\Big)\Big|_{s=0}$$

Let's see how we arrive at a conserved quantity from a symmetry.

3.3.1 Time Translation Symmetry

For any Lagrangian system, $L : TQ \to \mathbb{R}$, we have a 1-parameter family of symmetries

$$q_s(t) = q(t + s)$$

because

$$\delta L = \dot{L}$$

so we get a conserved quantity called the *total energy* or *Hamiltonian*,

$$H = p_i \dot{q}^i - L \tag{3.1}$$

(You might prefer "Hamiltonian" to "total energy" because in general we are not in the same configuration space as Newtonian mechanics, if you are doing Newtonian mechanics then "total energy" is appropriate.)

For example: a particle on $\mathbb{R}^n$ in a potential V has $Q = \mathbb{R}^n$, $L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q)$. This system has

$$p_i \dot{q}^i = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i = m \dot{q}^2 = 2K$$

so

$$H = p_i \dot{q}^i - L = 2K - (K - V) = K + V$$

as you'd have hoped.

3.3.2 Space Translation Symmetry

For a free particle in $\mathbb{R}^n$, we have $Q = \mathbb{R}^n$ and $L = K = \frac{1}{2} m \dot{q}^2$. This has *spatial translation* symmetries, so that for any $v \in \mathbb{R}^n$ we have the symmetry

$$q_s(t) = q(t) + s\, v$$

with

$$\delta L = 0$$

because $\delta \dot{q} = 0$ and L depends only on $\dot{q}$ not on q in this particular case. (Since L does not depend upon q^i we'll call q^i an *ignorable coordinate*; as above, these ignorables always give symmetries, hence conserved quantities. It is often useful therefore, to change coordinates so as to make some of them ignorable if possible!)

In this example we get a conserved quantity called *momentum in the v direction*:

$$p_i \delta q^i = m \dot{q}_i v^i = m \dot{q} \cdot v$$

Aside: Note the subtle difference between two uses of the term "momentum"; here it is a conserved quantity derived from space translation invariance, but earlier it was a different thing, namely the momentum $\partial L / \partial \dot{q}^i = p_i$ conjugate to q^i. These two different "momentum's" happen to be the same in this example!

Since this is conserved for all v we say that $m\dot{q} \in \mathbb{R}^n$ is conserved. (In fact that whole Lie group $G = \mathbb{R}^n$ is acting as a translation symmetry group, and we're getting a $\mathfrak{q}(= \mathbb{R}^n)$-valued conserved quantity!)

3.3.3 Rotational Symmetry

The free particle in $\mathbb{R}^n$ also has *rotation* symmetry. Consider any $X \in \mathfrak{so}(n)$ (that is a skew-symmetric $n \times n$ matrix), then for all $s \in \mathbb{R}$ the matrix e^{sX} is in $SO(n)$, that is, it describes a rotation. This gives a 1-parameter family of symmetries

$$q_s(t) = e^{sX} q(t)$$

which has

$$\delta L = \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i = m \dot{q}_i \delta \dot{q}^i$$

now q_i is ignorable and so $\partial L / \partial q^i = 0$, and $\partial L / \partial \dot{q}^i = p_i$, and

$$\begin{aligned}
\delta \dot{q}^i &= \frac{d}{ds} \dot{q}^i_s \bigg|_{s=0} \\
&= \frac{d}{ds} \frac{d}{dt} \Big(e^{sX} q \Big) \bigg|_{s=0} \\
&= \frac{d}{dt} X\, q \\
&= X\, \dot{q}
\end{aligned}$$

So,

$$\begin{aligned}\delta L &= m\dot{q}_i X^i_j \dot{q}^j \\ &= m\dot{q} \cdot (X\,\dot{q}) \\ &= 0\end{aligned}$$

since X is skew symmetric as stated previously ($X \in \mathfrak{so}(n)$). So we get a conserved quantity, the *angular momentum* in the X direction.

(Note: this whole bunch of math above for δL just says that the kinetic energy doesn't change when the velocity is *rotated*, without changing its magnitude.)

We write,

$$p_i \delta q^i = m\dot{q}_i \cdot (X\,q)^i$$

($\delta q^i = Xq$ just as $\delta \dot{q}^i = X\dot{q}$ in our previous calculation), or if X has zero entries except in ij and ji positions, where it's ± 1, then we get

$$m(\dot{q}_i q^j - \dot{q}_j q^i)$$

the "ij component of angular momentum". If $n = 3$ we write these as,

$$m\dot{\mathbf{q}}{\times}\mathbf{q}$$

Note that above we have assumed one can construct a basis for $\mathfrak{so}(n)$ using matrices of the form assumed for X, i.e., skew symmetric with ± 1 in the respectively ij and ji elements, otherwise zero.

I mentioned earlier that we can do mechanics with any Lagrangian, but if we want to be useful we'd better pick a Lagrangian that actually describes a real system. But how do we do that? All this theory is fine but is useless unless you know how to apply it. The above examples were for a particularly simple system, a free particle, for which the Lagrangian is just the kinetic energy, since there is no potential energy variation for a free particle. We'd like to know how to solve more complicated dynamics.

The general idea is to guess the kinetic energy and potential energy of the particle (as functions of your generalized positions and velocities) and then let,

$$L = K - V$$

So we are not using Lagrangians directly to tell us what the fundamental physical laws should be, instead we plug in some assumed physics and use the Lagrangian approch to solve the system of equations. If we like, we can then compare our answers with experiments, which indirectly tells us something about the physical laws—but only provided the Lagrangian formulation of mechanics is itself a valid procedure in the first place.

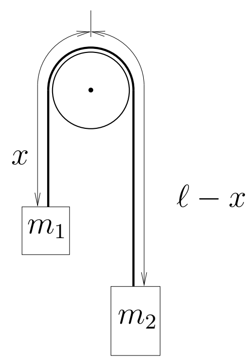
3.4 Example Problems

(Week 3, Apr. 11, 13, 15.)

To see how the formalisms in this chapter function in practise, let's do some problems. It's vastly superior to the simplistic $F = ma$ formulation of mechanics. The Lagrangian formulation allows the configuration space to be any manifold, and allows us to easily use any coordinates we wish.

3.4.1 The Atwood Machine

A frictionless pulley with two masses, m_1 and m_2, hanging from it. We have



$$K = \frac{1}{2}(m_1 + m_2)(\frac{d}{dt}(\ell - x))^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$
$$V = -m_1 g x - m_2 g(\ell - x)$$

so

$$L = K - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g(\ell - x)$$

The configuration space is $Q = (0, \ell)$, and $x \in (0, \ell)$ (we could use the "owns" symbol $\ni$ here and write $Q = (0, \ell) \ni x$). Moreover $TQ = (0, \ell) \times \mathbb{R} \ni (x, \dot{x})$. As usual $L : TQ \to \mathbb{R}$. Note that solutions of the Euler-Lagrange equations will only be defined for *some* time $t \in \mathbb{R}$, as eventually the solutions reaches the "edge" of Q.

The momentum is:

$$p = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

and the force is:

$$F = \frac{\partial L}{\partial x} = (m_1 - m_2)g$$

The Euler-Lagrange equations say

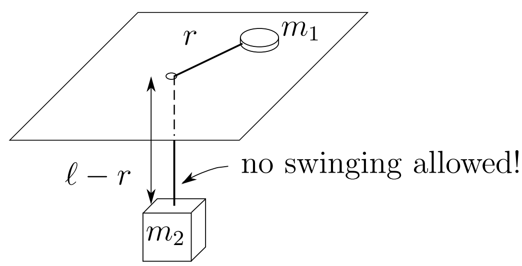
$$\begin{aligned} \dot{p} &= F \\ (m_1+m_2)\ddot{x} &= (m_1-m_2)g \\ \ddot{x} &= \frac{m_1-m_2}{m_1+m_2}g \end{aligned}$$

So this is like a falling object in a downwards gravitational acceleration $a = \left(\frac{m_1-m_2}{m_1+m_2}\right) g$.

It is trivial to integrate the expression for $\ddot{x}$ twice (feeding in some appropriate initial conditions) to obtain the complete solution to the motion $x(t)$ and $\dot{x}(t)$. Note that $\ddot{x} = 0$ when $m_1 = m_2$, and $\ddot{x} = g$ if $m_2 = 0$.

3.4.2 Disk Pulled by Falling Mass

Consider next a disk pulled across a table by a falling mass. The disk is free to move on a frictionless surface, and it can thus whirl around the hole to which it is tethered to the mass below.



$$\begin{aligned} \text{Here } Q = &\text{ open disk of radius } \ell\text{, minus it's center} \\ &= (0,\ell)\times S^1 \ni (r,\theta) \\ TQ &= (0,\ell)\times S^1 \times \mathbb{R}\times\mathbb{R} \ni (r,\theta,\dot{r},\dot{\theta}) \end{aligned}$$

$$\begin{aligned} K &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\left(\frac{d}{dt}(\ell - r)\right)^2 \\ V &= gm_2(r-\ell) \\ L &= \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 + gm_2(\ell - r) \end{aligned}$$

having noted that ℓ is constant so $d/dt(\ell - r) = -\dot{r}$. For the momenta we get,

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = (m_1+m_2)\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\dot{\theta}. \end{aligned}$$

Note that θ is an "ignorable coordinate"—it doesn't appear in L—so there's a symmetry, rotational symmetry, and p_θ, the conjugate momentum, is conserved.

The forces are,

$$F_r = \frac{\partial L}{\partial r} = m_1 r\dot{\theta}^2 - gm_2$$
$$F_\theta = \frac{\partial L}{\partial \theta} = 0, \quad (\theta \text{ is ignorable})$$

Note: in F_r the term $m_1 r\dot{\theta}^2$ is recognizable as a centrifugal force, pushing m_1 radially *out*, while the term $-gm_2$ is gravity pulling m_2 down and thus pulling m_1 radially *in*.

So, the Euler-Lagrange equations give,

$$\dot{p}_r = F_r, \quad (m_1 + m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g$$
$$\dot{p}_\theta = 0, \quad p_\theta = m_1 r^2\dot{\theta} = J = \text{ a constant.}$$

Let's use our conservation law here to eliminate $\dot{\theta}$ from the first equation:

$$\dot{\theta} = \frac{J}{m_1 r^2}$$

so

$$(m_1 + m_2)\ddot{r} = \frac{J^2}{m_1 r^3} - m_2 g$$

Thus effectively we have a particle on $(0, \ell)$ of mass $m = m_1 + m_2$ feeling a force

$$F_r = \frac{J^2}{m_1 r^3} - m_2 g$$

which could come from an "effective potential" $V(r)$ such that $dV/dr = -F_r$. So integrate $-F_r$ to find $V(r)$:

$$V(r) = \frac{J^2}{2m_1 r^2} + m_2 gr$$

this is a sum of two terms that look like Fig. 3.1

If $\dot{\theta}(t = 0) = 0$ then there is no centrifugal force and the disk will be pulled into the hole until it gets stuck. At that time the disk reaches the hole, which is topologically the center of the disk that has been removed from Q, so then we've hit the boundary of Q and our solution is broken.

At $r = r_0$, the minimum of $V(r)$, our disc mass m_1 will be in a stable circular orbit of radius r_0 (which depends upon J). Otherwise we get orbits like Fig. 3.2.

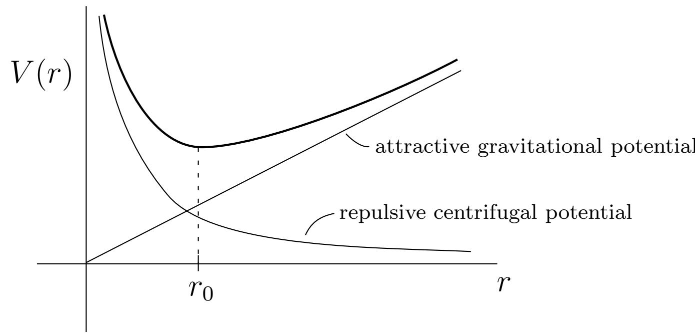


Figure 3.1: Potential function for disk pulled by gravitating mass.

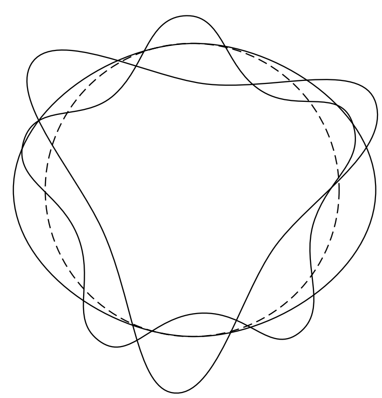


Figure 3.2: Orbits for the disc and gravitating mass system.

3.4.3 Free Particle in Special Relativity

In relativistic dynamics the parameter coordinate that parametrizes the particle's path in Minkowski spacetime need not be the "time coordinate", indeed in special relativity there are many allowed time coordinates.

Minkowski spacetime is,

$$\mathbb{R}^{n+1} \ni (x^0, x^1, \ldots, x^n)$$

if space is n-dimensional. We normally take x^0 as "time", and $(x^1, \ldots, x^n)$ as "space", but of course this is all relative to one's reference frame. Someone else travelling at some high velocity relative to us will have to make a Lorentz transformation to translate from our coordinates to theirs.

This has a Lorentzian metric

$$\begin{aligned} g(v, w) &= v^0 w^0 - v^1 w^1 - \ldots - v^n w^n \\ &= \eta_{\mu\nu} v^\mu w^\nu \end{aligned}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}$$

In special relativity we take *spacetime* to be the configuration space of a single point particle, so we let Q be Minkowski spacetime, i.e., $\mathbb{R}^{n+1} \ni (x^0, \dots, x^n)$ with the metric $\eta_{\mu\nu}$ defined above. Then the path of the particle is,

$$q : \mathbb{R}(\ni t) \longrightarrow Q$$

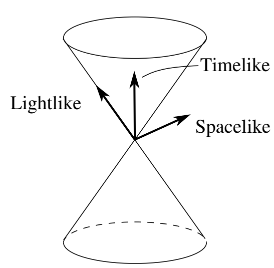
where t is a completely arbitrary parameter for the path, not necessarily x^0, and not necessarily *proper time* either. We want some Lagrangian $L : TQ \to \mathbb{R}$, i.e., $L(q^i, \dot{q}^i)$ such that the Euler-Lagrange equations will dictate how our free particle moves at a constant velocity. Many Lagrangians do this, but the "best" ones(s) give an action that is *independent* of the parameterization of the path—since the parameterization is "unphysical" (it can't be measured). So the action

$$S(q) = \int_{t_0}^{t_1} L\Big(q^i(t), \dot{q}^i(t)\Big)\, dt$$

for $q : [t_0, t_1] \to Q$, should be independent of t. The obvious candidate for S is mass times arclength,

$$S = m \int_{t_0}^{t_1} \sqrt{\eta_{ij}\dot{q}^i(t)\dot{q}^j(t)}\, dt$$

or rather the Minkowski analogue of arclength, called *proper time*, at least when $\dot{q}$ is a timelike vector, i.e., $\eta_{ij}\dot{q}^i\dot{q}^j > 0$, which says $\dot{q}$ points into the future (or past) lightcone and makes S *real*, in fact it's then the time ticked off by a clock moving along the path $q : [t_0, t_1] \to Q$. By "obvious candidate" we are appealing somewhat to physical intuition and



generalization. In Euclidean space, free particles follow straight paths, so the arclength or pathlength variation is an extremum, and we expect the same behavior in Minkowski spacetime. Also, the arclength does not depend upon the parameterization, and lastly, the mass m merely provides the correct units for 'action'.

So let's take

$$L = m\sqrt{\eta_{ij}\dot{q}^i\dot{q}^j} \tag{3.2}$$

and work out the Euler-Lagrange equations. We have

$$\begin{aligned} p_i = \frac{\partial L}{\partial \dot{q}^i} &= m\frac{\partial}{\partial \dot{q}^i}\sqrt{\eta_{ij}\dot{q}^i\dot{q}^j} \\ &= m\frac{2\eta_{ij}\dot{q}^j}{2\sqrt{\eta_{ij}\dot{q}^i\dot{q}^j}} \\ &= m\frac{\eta_{ij}\dot{q}^j}{\sqrt{\eta_{ij}\dot{q}^i\dot{q}^j}} = \frac{m\dot{q}_i}{\|\dot{q}\|} \end{aligned}$$

(Note the numerator is "mass times 4-velocity", at least when $n = 3$ for a real single particle system, but we're actually in a more general $n+1$-dim spacetime, so it's more like the "mass times $n+1$-velocity"). Now note that this p_i doesn't change when we change the parameter to accomplish $\dot{q} \mapsto \alpha\dot{q}$. The Euler-Lagrange equations say,

$$\dot{p}_i = F_i = \frac{\partial L}{\partial q^i} = 0$$

The meaning of this becomes clearer if we use "proper time" as our parameter (like parameterizing a curve by its arclength) so that

$$\int_{t_0}^{t_1} \|\dot{q}\| dt = t_1 - t_0, \quad \forall\, t_0, t_1$$

which fixes the parametrization up to an additive constant. This implies $\|\dot{q}\| = 1$, so that

$$p_i = m\frac{\dot{q}_i}{\|\dot{q}\|} = m\dot{q}_i$$

and the Euler-Lagrange equations say

$$\dot{p}_i = 0 \Rightarrow m\ddot{q}_i = 0$$

so our (free) particle moves unaccelerated along a straight line, which is as we desired (expected).

Comments

This Lagrangian from Eq.(3.2) has lots of symmetries coming from *reparameterizing the path*, so Noether's theorem yields lots of conserved quantities for the relativistic free particle. This is in fact called "the problem of time" in general relativity. Here we see it starting to show up in special relativity.

These reparameterization symmetries work as follows. Consider any (smooth) 1-parameter family of reparameterizations, i.e., diffeomorphisms

$$f_s : \mathbb{R} \longrightarrow \mathbb{R}$$

with $f_0 = \mathbb{1}_{\mathbb{R}}$. These act on the space of paths $\Gamma = \{q : \mathbb{R} \to Q\}$ as follows: given any $q \in \Gamma$ we get

$$q_s(t) = q\big(f_s(t)\big)$$

where we should note that q_s is *physically indistinguishable* from q. Let's show that

$$\delta L = \dot{\ell}, \quad \text{(when E-L eqns. hold)}$$

so that Noether's theorem gives a conserved quantity

$$p_i \delta q^i - \ell$$

Here we go then:

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \\
&= p_i \delta \dot{q}^i \\
&= \frac{m\dot{q}_i}{\|\dot{q}\|}\;\frac{d}{ds}\dot{q}^i\Big(f_s(t)\Big)\bigg|_{s=0} \\
&= \frac{m\dot{q}_i}{\|\dot{q}\|}\;\frac{d}{dt}\frac{d}{ds}q^i\Big(f_s(t)\Big)\bigg|_{s=0} \\
&= \frac{m\dot{q}_i}{\|\dot{q}\|}\;\frac{d}{dt}\dot{q}^i\Big(f_s(t)\Big)\frac{f_s(t)}{ds}\bigg|_{s=0} \\
&= \frac{m\dot{q}_i}{\|\dot{q}\|}\frac{d}{dt}\left(\dot{q}^i \delta f_s\right) \\
&= \frac{d}{dt}\left(p_i \dot{q}^i \delta f\right)
\end{aligned}$$

where in the last step we used the E-L eqns., i.e. $\frac{d}{dt}p_i = 0$, so $\delta L = \dot{\ell}$ with $\ell = p_i \dot{q}^i \delta f$.

So to recap a little: we saw the free relativistic particle has

$$L = m\|\dot{q}\| = m\sqrt{\eta_{ij}\dot{q}^i \dot{q}^j}$$

and we've considered reparameterization symmetries

$$q_s(t) = q\Big(f_s(t)\Big), \qquad f_s : \mathbb{R} \to \mathbb{R}$$

we've used the fact that

$$\delta q^i \ := \ \frac{d}{ds} q^i \Big(f_s(t) \Big) \Big|_{s=0} = \dot{q}^i \delta f$$

so (repeating a bit of the above)

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \\
&= p_i \delta \dot{q}^i, \quad \text{(since } \partial L / \partial q^i = 0 \text{, and } \partial L / \partial \dot{q}^i = p) \\
&= p_i \delta \dot{q}^i \\
&= p_i \frac{d}{dt} \delta q^i \\
&= p_i \frac{d}{dt} \dot{q}^i \delta f \\
&= \frac{d}{dt} p_i \dot{q}^i \delta f, \quad \text{and set } p_i \dot{q}^i \delta f = \ell
\end{aligned}$$

so Noether's theorem gives a conserved quantity

$$\begin{aligned}
p_i \delta q^i - \ell &= p_i \dot{q}^i \delta f - p_i \dot{q}^i \delta f \\
&= 0
\end{aligned}$$

So these conserved quantities *vanish*! In short, we're seeing an example of what physicists call *gauge symmetries*. This is a good topic for starting a new section.

3.5 Electrodynamics and Relativistic Lagrangians

We will continue the story of symmetry and Noether's theorem from the last section with a few more examples. We use principles of least action to conjure up Lagrangians for our systems, realizing that a given system may not have a unique Lagrangian but will often have an obvious natural Lagrangian. Given a Lagrangian we derive equations of motion from the Euler-Lagrange equations. Symmetries of L guide us in finding conserved quantities, in particular Hamiltonians from time translation invariance, via Noether's theorem.

This section also introduces gauge symmetry, and this is where we begin.

3.5.1 Gauge Symmetry and Relativistic Hamiltonian

What are gauge symmetries?

1. These are symmetries that permute different mathematical descriptions of the same physical situation—in this case reparameterizations of a path.

2. These symmetries make it impossible to compute $q(t)$ given $q(0)$ and $\dot{q}(0)$: since if $q(t)$ is a solution so is $q(f(t))$ for any reparameterization $f : \mathbb{R} \to \mathbb{R}$. We have a high degree of non-uniqueness of solutions to the Euler-Lagrange equations.

3. These symmetries give conserved quantities that work out to equal zero!

Note that (1) is a subjective criterion, (2) and (3) are objective, and (3) is easy to test, so we often use (3) to distinguish *gauge* symmetries from *physical* symmetries.

3.5.2 Relativistic Hamiltonian

What then is the Hamiltonian for special relativity theory? We're continuing here with the example problem of §3.4.3. Well, the Hamiltonian comes from Noether's theorem from *time translation* symmetry,

$$q_s(t) = q(t + s)$$

and this is an example of a reparametrization (with $\delta f = 1$), so we see from the previous results that the Hamiltonian is *zero*!

$$H = 0.$$

Explicitly, $H = p_i \delta \dot{q}^i - \ell$ where under $q(t) \to q(t+s)$ we have $\delta \dot{q}^i = \dot{q}^i \delta f$, and so $\delta L = d\ell/dt$, which implies $\ell = p_i \delta q^i$. The result $H = 0$ follows.

Now you know why people talk about "the problem of time" in general relativity theory, it's glimmerings are seen in the flat Minkowski spacetime of special relativity. You may think it's nice and simple to have $H = 0$, but in fact it means that there is no temporal evolution possible! So we can't establish a dynamical theory on this footing! That's bad news. (Because it means you might have to solve the static equations for the 4D universe as a whole, and that's impossible!)

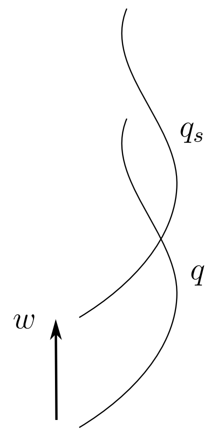
But there is another conserved quantity deserving the title of "energy" which is *not* zero, and it comes from the symmetry,

$$q_s(t) = q(t) + s\,w$$

where $w \in \mathbb{R}^{n+1}$ and w points in some timelike direction.

In fact *any* vector w gives a conserved quantity,

$$\begin{aligned}
\delta L &= \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i \\
&= p_i \delta \dot{q}^i, \quad (\text{since } \partial L/\partial q^i = 0 \text{ and } \partial L/\partial \dot{q}^i = p_i) \\
&= p_i 0 \;= 0
\end{aligned}$$



since $\delta q^i = w^i$, $\delta \dot{q}^i = \dot{w}^i = 0$. This is our $\dot{\ell}$ from Noether's theorem with $\ell = 0$, so Noether's theorem says that we get a conserved quantity

$$p_i \delta q^i - \ell = p_w^i$$

namely, the *momentum in the w direction*. We know $\dot{p} = 0$ from the Euler-Lagrange equations, for our free particle, but here we see it coming from spacetime translation symmetry;

$$\begin{aligned} p =& (p_0, p_1, \ldots, p_n) \\ & p_0 \text{ is energy}, \quad (p_1, \ldots, p_n) \text{ is spatial momentum.} \end{aligned}$$

We've just about exhausted all the basic stuff that we can learn from the free particle. So next we'll add some external force via an electromagnetic field.

3.6 Relativistic Particle in an Electromagnetic Field

The electromagnetic field is described by a 1-form A on spacetime, A is the *vector potential*, such that

$$dA = \mathsf{F} \tag{3.3}$$

is a 2-form containing the electric and magnetic fields,

$$\mathsf{F}_{\mu\nu} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \tag{3.4}$$

We'd write (for Q having local charts to $\mathbb{R}^{n+1}$),

$$A = A_0 dx^0 + A_1 dx^1 + \ldots A_n dx^n$$

and then because $d^2 = 0$

$$dA = dA_0 dx^0 + dA_1 dx^1 + \dots dA_n dx^n$$

and since the "A_i" are just functions,

$$dA_i = \partial_\mu A_i dx^\mu$$

using the summation convention and $\partial_\mu := \partial/\partial x^\mu$. The student can easily check that the components for $F = F_{01}dx^0 \wedge dx^1 + F_{02}dx^0 \wedge dx^2 + \dots$, agrees with the matrix expression below (at least for 4D).

So, for example, in 4D spacetime

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

where E is the electric field and B is the magnetic field. The action for a particle of charge e is

$$S = m \int_{t_0}^{t_1} \|\dot{q}\| \, dt + e \int_q A$$

here

$$\int_{t_0}^{t_1} \|\dot{q}\| \, dt = \text{ proper time,}$$
$$\int_q A = \text{ integral of } A \text{ along the path } q.$$

Note that since A is a 1-form it can be integrated (it is a linear combination of some basis 1-forms like the $\{dx^i\}$).

(Week 4, April 18, 20, 22.)

Note that since A is a 1-form we can integrate it over an oriented manifold, but one can also write the path integral using time t as a parameter, with $A_i\dot{q}^i \, dt$ the differential, after $dq^i = \dot{q}^i dt$.

The Lagrangian in the above action, for a charge e with mass m in an electromagnetic potential A is

$$L(q, \dot{q}) = m\|\dot{q}\| + eA_i\dot{q}^i \tag{3.5}$$

so we can work out the Euler-Lagrange equations:

$$\begin{aligned} p_i = \frac{\partial L}{\partial \dot{q}^i} &= m\frac{\dot{q}_i}{\|\dot{q}\|} + eA_i \\ &= mv_i + e\,A_i \end{aligned}$$

where $v \in \mathbb{R}^{n+1}$ is the velocity, normalized so that $\|v\| = 1$. Note that *now momentum is no longer mass times velocity*! That's because we're in $n+1$-d spacetime, the momentum is an $n+1$-vector. Continuing the analysis, we find the force

$$\begin{aligned} F_i = \frac{\partial L}{\partial q^i} &= \frac{\partial}{\partial q^i}\Big(e\,A_j \dot{q}^j\Big) \\ &= e\frac{\partial A_j}{\partial q^i}\dot{q}^j \end{aligned}$$

So the Euler-Lagrange equations say (noting that $A_i = A_j\Big(q(t)\Big)$:

$$\begin{aligned} \dot{p} &= F \\ \frac{d}{dt}\Big(mv_i + eA_i\Big) &= e\frac{\partial A_j}{\partial q^i}\dot{q}^j \\ m\frac{dv_i}{dt} &= e\frac{\partial A_j}{\partial q^i}\dot{q}^j - e\frac{dA_i}{dt} \\ m\frac{dv_i}{dt} &= e\frac{\partial A_j}{\partial q^i}\dot{q}^j - e\frac{\partial A_i}{\partial q^j}\dot{q}^j \\ &= e\left(\frac{\partial A_j}{\partial q^i} - \frac{\partial A_i}{\partial q^j}\right)\dot{q}^j \end{aligned}$$

the term in parentheses is $\mathsf{F}_{ij} =$ the electromagnetic field, $F = dA$. So we get the following equations of motion

$$\boxed{m\frac{dv_i}{dt} = e\mathsf{F}_{ij}\dot{q}^j, \quad \text{(Lorentz force law)}} \tag{3.6}$$

(Usually called the "Lorenz" force law.)

3.7 Alternative Lagrangians

We'll soon discuss a charged particle Lagrangian that is free of the reparameterization symmetry. First a paragraph on objects other than point particles!

3.7.1 Lagrangian for a String

So we've looked at a point particle and tried

$$S = m \cdot (\text{arclength}) + \int A$$

or with 'proper time' instead of 'arclength', where the 1-from A can be integrated over a 1-dimensional path. A generalization (or specialization, depending on how you look at it) would be to consider a Lagrangian for an extended object.

In string theory we boost the dimension by +1 and consider a string tracing out a 2D surface as time passes (Fig. 3.3).

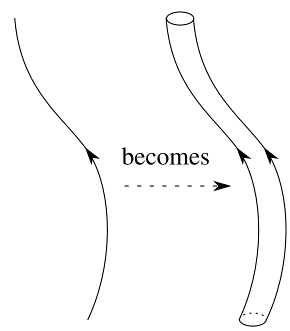


Figure 3.3: Worldtube of a closed string.

Can you infer an appropriate action for this system? Remember, the physical or physico-philosophical principle we've been using is that the path followed by physical objects minimizes the "activity" or "aliveness" of the system. Given that we presumably cannot tamper with the length of the closed string, then the worldtube quantity analogous to arclength or proper time would be the area of the worldtube (or worldsheet for an open string). If the string is also assumed to be a source of electromagnetic field then we need a 2-form to integrate over the 2D worldtube analogous to the 1-form integrated over the pathline of the point particle. In string theory this is usually the "Kalb-Ramond field", call it B. To recover electrodynamic interactions it should be antisymmetric like A, but it's tensor components will have two indices since it's a 2-form. The string action can then be written

$$S = \alpha \cdot (area) + e \int B \tag{3.7}$$

We've also replaced the point particle mass by the string tension α [mass·length^{-1}] to obtain the correct units for the action (since replacing arclength by area meant we had to compensate for the extra length dimension in the first term of the above string action).

This may still seem like we've pulled a rabbit out of a hat. But we haven't checked that this action yields sensible dynamics yet! But supposing it does, then would it justify our guesswork and intuition in arriving at Eq.(3.7)? Well by now you've probably realized that one can have more than one form of action or Lagrangian that yields the same dynamics. So provided we supply reasonabe physically realistic heuristics then whatever Lagrangian or action that we come up with will stand a good chance of describing some system with a healthy measure of physical verisimilitude.

That's enough about string for now. The point was to illustrate the type of reasoning that one can use in conjuring up a Lagrangian. It's particularly useful when Newtonian theory cannot give us a head start, i.e., in relativistic dynamics and in the physics of extended particles.

3.7.2 Alternate Lagrangian for Relativistic Electrodynamics

In § 3.6, Eq.(3.5) we saw an example of a Lagrangian for relativistic electrodynamics that had awkward reparametrization symmetries, meaning that $H = 0$ and there were non-unique solutions to the Euler-Lagrange equations arising from applying gauge transformations. This freedom to change the gauge can be avoided.

Recall Eq.(3.5), which was a Lagrangian for a charged particle with reparametrization symmetry

$$L = m\|\dot{q}\| + eA_i\dot{q}^i$$

just as for an uncharged relativistic particle. But there's another Lagrangian we can use that doesn't have this gauge symmetry:

$$L = \frac{1}{2}m\dot{q}\cdot\dot{q} + eA_i\dot{q}^i \tag{3.8}$$

This one even has some nice features.

- It looks formally like "$\frac{1}{2}mv^2$", familiar from nonrelativistic mechanics.
- There's no ugly square root, so it's everywhere differentiable, and there's no trouble with paths being timelike or spacelike in direction, they are handled the same.

What Euler-Lagrange equations does this Lagrangian yield?

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}^i} = m\dot{q}_i + eA_i \\ F_i &= \frac{\partial L}{\partial q^i} = e\frac{\partial A_j}{\partial q^i}\dot{q}^j \end{aligned}$$

Very similar to before! The E-L eqns. then say

$$\begin{aligned} \frac{d}{dt}\Big(m\dot{q}_i + eA_i\Big) &= e\frac{\partial A_j}{\partial q^i}\dot{q}^j \\ m\ddot{q}_i &= e\mathsf{F}_{ij}\dot{q}^j \end{aligned}$$

almost as before. (I've taken to using F here for the electromagnetic field tensor to avoid clashing with F for the generalized force.) The only difference is that we have $m\ddot{q}_i$ instead of $m\dot{v}_i$ where $v_i = \dot{q}_i/\|\dot{q}\|$. So the old Euler-Lagrange equations of motion reduce to the new ones if we pick a parametrization with $\|\dot{q}\| = 1$, which would be a parametrization by proper time for example.

Let's work out the Hamiltonian for this

$$L = \frac{1}{2} m \dot{q} \cdot \dot{q} + e A_i \dot{q}^i$$

for the relativistic charged particle in an electromagnetic field. Recall that for our reparametrization-invariant Lagrangian

$$L = m\sqrt{\dot{q}_i \dot{q}^i} + e A_i q^i$$

we got $H = 0$, time translation was a gauge symmetry. With the new Lagrangian it's not! Indeed

$$H = p_i \dot{q}^i - L$$

and now

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m\dot{q}^i + eA_i$$

so

$$\begin{aligned} H &= (m\dot{q}_i + eA_i)\dot{q}^i - (\tfrac{1}{2} m \dot{q}_i \dot{q}^i + eA_i \dot{q}^i) \\ &= \tfrac{1}{2} m \dot{q}_i \dot{q}^i \end{aligned}$$

Comments. This is vaguely like how a nonrelativistic particle in a potential V has

$$H = p_i \dot{q}^i - L = 2K - (K - V) = K + V,$$

but now the "potential' $V = eA_i \dot{q}^i$ is *linear in velocity*, so now

$$H = p_i \dot{q}^i - L = (2K - V) - (K - V) = K.$$

As claimed H is not zero, and the fact that it's conserved says $\|\dot{q}(t)\|$ is constant as a function of t, so the particle's path is parameterized by proper time up to rescaling of t. That is, we're getting "conservation of speed" rather than some more familiar "conservation of energy". The reason is that this Hamiltonian comes from the symmetry

$$q_s(t) = q(t + s)$$

instead of spacetime translation symmetry

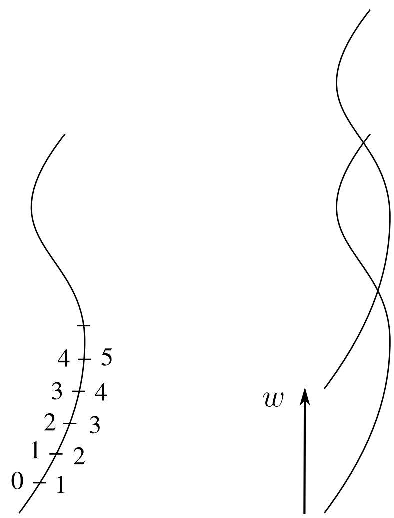
$$q_s(t) = q(t) + s\,w, \quad w \in \mathbb{R}^{n+1}$$

the difference is illustrated schematically in Fig. 3.4.

Our Lagrangian

$$L(q, \dot{q}) = \tfrac{1}{2} m \|\dot{q}\|^2 + A_i(q)\dot{q}^i$$

has time translation symmetry iff A is translation invariant (but it's highly unlikely a given system of interest will have $A(q) = A(q + sw)$). In general then there's no conserved "energy" for our particle corresponding to translations in time.



$q_s(t) = q(t+s)$ $\qquad$ $q_s(t) = q(t) + sw$

Figure 3.4: Proper time rescaling vs spacetime translation.

3.8 The General Relativistic Particle

In GR spacetime, Q, is an $(n+1)$-dimensional Lorentzian manifold, namely a smooth $(n+1)$-dimensional manifold with a *Lorentzian metric* g. We define the metric as follows.

1. For each $x \in Q$, we have a bilinear map

$$\begin{aligned} g(x) : T_xQ \times T_xQ &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto g(x)(v, w) \end{aligned}$$

   or we could write $g(v, w)$ for short.

2. With respect to some basis of T_xQ we have

$$\begin{aligned} g(v, w) &= g_{ij} v^i w^j \\ g_{ij} &= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} \end{aligned}$$

   Of course we can write $g(v, w) = g_{ij} v^i w^j$ in any basis, but for different bases g_{ij} will have a different form.

3. $g(x)$ varies smoothly with x.

3.8.1 Free Particle Lagrangian in GR

The Lagrangian for a free point particle in the spacetime Q is

$$\begin{aligned} L(q,\dot{q}) &= m\sqrt{g(q)(\dot{q},\dot{q})} \\ &= m\sqrt{g_{ij}\dot{q}^i\dot{q}^j} \end{aligned}$$

just like in special relativity but with η_{ij} replaced by g_{ij}. Alternatively we could just as well use

$$\begin{aligned} L(q,\dot{q}) &= \tfrac{1}{2}mg(q)(\dot{q},\dot{q}) \\ &= \tfrac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j \end{aligned}$$

The big difference between these two Lagrangians is that now spacetime translation symmetry (and rotation, and boost symmetry) is gone! So there is no conserved energy-momentum (nor angular momentum, nor velocity of center of energy) anymore!

Let's find the equations of motion. Suppose then Q is a Lorentzian manifold with metric g and $L : TQ \to \mathbb{R}$ is the Lagrangian of a free particle,

$$L(q,\dot{q}) = \tfrac{1}{2}mg_{ij}\dot{q}^i\dot{q}^j$$

We find equations of motion from the Euler-Lagrange equations, which in this case start from

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = mg_{ij}\dot{q}^j$$

The velocity $\dot{q}$ here is a tangent vector, the momentum p is a cotangent vector, and we need the metric to relate them, via

$$\begin{aligned} g : T_q\mathcal{M} \times T_q\mathcal{M} &\longrightarrow \mathbb{R} \\ (v,w) &\longmapsto g(v,w) \end{aligned}$$

which gives

$$\begin{aligned} T_q\mathcal{M} &\longrightarrow T_q^*\mathcal{M} \\ v &\longmapsto g(v,-). \end{aligned}$$

In coordinates this would say that the tangent vector v^i gets mapped to the cotangent vector $g_{ij}v^j$. This is lurking behind the passage from $\dot{q}^i$ to the momentum $mg_{ij}v^j$.

Getting back to the E-L equations,

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}^i} = mg_{ij}\dot{q}^j \\ F_i &= \frac{\partial L}{\partial q^i} = \frac{\partial}{\partial q^i}\left(\tfrac{1}{2}mg_{jk}(q)\dot{q}^j\dot{q}^k\right) \\ &= \tfrac{1}{2}m\partial_i g_{jk}\dot{q}^j\dot{q}^k, \quad (\text{where } \partial_i = \frac{\partial}{\partial q^i}). \end{aligned}$$

So the Euler-Lagrange equations say

$$\frac{d}{dt} m g_{ij}\dot{q}^j = \tfrac{1}{2} m \partial_i g_{jk}\dot{q}^j\dot{q}^k.$$

The mass factors away, so the motion is independent of the mass! Essentially we have a geodesic equation.

We can rewrite this *geodesic equation* as follows

$$\begin{aligned} \frac{d}{dt} g_{ij}\dot{q}^j &= \tfrac{1}{2}\partial_i g_{jk}\dot{q}^j\dot{q}^k \\ \therefore \quad \partial_k g_{ij}\dot{q}^k\dot{q}^j + g_{ij}\ddot{q}^j &= \tfrac{1}{2}\partial_i g_{jk}\dot{q}^j\dot{q}^k \\ \therefore \quad g_{ij}\ddot{q}^j &= \left(\tfrac{1}{2}\partial_i g_{jk} - \partial_k g_{ij}\right)\dot{q}^j\dot{q}^k \\ &= \tfrac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} - \partial_j g_{ki}\right)\dot{q}^j\dot{q}^k \end{aligned}$$

where the last line follows by symmetry of the metric, $g_{ik} = g_{ki}$. Now let,

$$\Gamma_{ijk} = -\left(\partial_i g_{jk} - \partial_k g_{ij} - \partial_j g_{ki}\right)$$

the minus sign being just a convention (so that we agree with everyone else). This defines what we call the Christoffel symbols Γ^i_{jk}. Then

$$\begin{aligned} \ddot{q}_i = g_{ij}\ddot{q}^j &= -\Gamma_{ijk}\dot{q}^j\dot{q}^k \\ \therefore \quad \ddot{q}^i &= -\Gamma^i_{jk}\dot{q}^j\dot{q}^k. \end{aligned}$$

So we see that $\ddot{q}$ can be computed in terms of $\dot{q}$ and the Christoffel symbols Γ^i_{jk}, which is really a particular type of *connection* that a Lorentzian manifold has (the Levi-Civita connection), a *connection* is just the rule for *parallel transporting* tangent vectors around the manifold.

Parallel transport is just the simplest way to compare vectors at different points in the manifold. This allows us to define, among other things, a *covariant derivative*.

3.8.2 Charged particle in EM Field in GR

We can now apply what we've learned in consideration of a charged particle, of charge e, in an electromagnetic field with potential A, in our Lorentzian manifold. The Lagrangian would be

$$L = \tfrac{1}{2} m g_{jk}\dot{q}^j\dot{q}^k + eA_i\dot{q}^i$$

which again was conjured up be replacing the flat space metric η_{ij} by the metric for GR g_{ij}. Not surprisingly, the Euler-Lagrange equations then yield the following equations of motion,

$$m\ddot{q}_i = -m\Gamma_{ijk}\dot{q}^j\dot{q}^k + e\mathsf{F}_{ij}\dot{q}^j.$$

If you want to know more about Lagrangians for general relativity we recommend the paper by Peldan [Pel94], and also the "black book" of Misner, Thorne & Wheeler [WTM71].

3.9 The Principle of Least Action and Geodesics

(Week 4, April 18, 20, 22.)

3.9.1 Jacobi and Least Time vs Least Action

We've mentioned that Fermat's principle of least time in optics is analogous to the principle of least action in particle mechanics. This analogy is strange, since in the principle of least action we fix the time interval $q : [0, 1] \to Q$. Also, if one imagines a force on a particle resulting from a potential gradient at an interface as analogous to light refraction then you also get a screw-up in the analogy (Fig. 3.5).

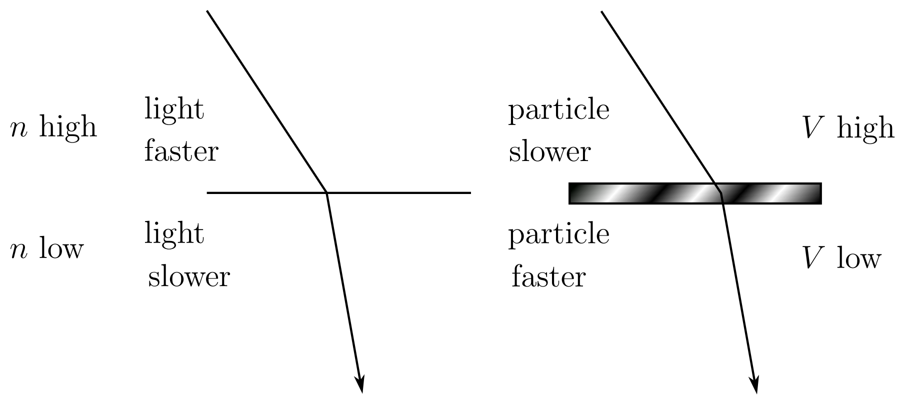


Figure 3.5: Least time versus least action.

Nevertheless, Jacobi was able to reinterpret the mechanics of a particle as an optics problem and hence "unify" the two minimization principles. First, let's consider light in a medium with a varying index of refraction n (recall $1/n \propto$ speed of light). Suppose it's in $\mathbb{R}^n$ with its usual Euclidean metric. If the light s trying to minimize the *time*, its trying to minimize the arclength of its path in the metric

$$g_{ij} = n^2 \delta_{ij}$$

that is, the index of refraction $n : \mathbb{R}^n \to (0, \infty)$, times the usual Euclidean metric

$$\delta_{ij} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

This is just like the free particle in general relativity (minimizing it's proper time) except that now g_{ij} is a Riemannian metric

$$\begin{aligned} g(v,w) &= g_{ij}v^i w^j \\ \text{where} \quad g(v,v) &\geq 0 \end{aligned}$$

So we'll use the same Lagrangian:

$$L(q,\dot{q}) = \sqrt{g_{ij}(q)\dot{q}^i\dot{q}^j}$$

and get the same Euler-Lagrange equations:

$$\frac{d^2q^i}{dt^2} + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0 \tag{3.9}$$

if q is parameterized by arclength or more generally

$$\|\dot{q}\| = \sqrt{g_{ij}(q)\dot{q}^i\dot{q}^j} = \text{ constant.}$$

As before the Christoffel symbols Γ are built from the derivatives of the metric g.

Now, what Jacobi did is show how the motion of a particle in a potential could be viewed as a special case of this. Consider a particle of mass m in Euclidean $\mathbb{R}^n$ with potential $V : \mathbb{R}^n \to \mathbb{R}$. It satisfies $F = ma$, i.e.,

$$m\frac{d^2q^i}{dt^2} = -\partial_i V \tag{3.10}$$

How did Jacobi see (3.10) as a special case of (3.9)? He considered a particle of energy E and he chose the index of refraction to be

$$n(q) = \sqrt{\frac{2}{m}\big(E - V(q)\big)}$$

which is just the *speed* of a particle of energy E when the potential energy is $V(q)$, since

$$\sqrt{\frac{2}{m}(E-V)} = \sqrt{\frac{2}{m}\frac{1}{2}m\|\dot{q}\|^2} = \|\dot{q}\|.$$

Note: this is precisely backwards compared to optics, where $n(q)$ is proportional to the *reciprocal* of the speed of light!! But let's see that it works.

$$\begin{aligned} L &= \sqrt{g_{ij}(q)\dot{q}^i\dot{q}^j} \\ &= \sqrt{n^2(q)\dot{q}^i\dot{q}^j} \\ &= \sqrt{2/m(E-V(q))\dot{q}^2} \end{aligned}$$

where $\dot{q}^2 = \dot{q} \cdot \dot{q}$ is just the usual Euclidean dot product, $v \cdot w = \delta_{ij} v^i w^j$. We get the Euler-Lagrange equations,

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}^i} = \sqrt{\frac{2}{m}(E-V)} \cdot \frac{\dot{q}}{\|\dot{q}\|} \\ F_i &= \frac{\partial L}{\partial q^i} = \partial_i \sqrt{\frac{2}{m}(E - V(q))} \cdot \|\dot{q}\| \\ &= \frac{1}{2} \frac{-2/m \partial_i V}{\sqrt{2/m(E - Vq)}} \cdot \|\dot{q}\| \end{aligned}$$

Then $\dot{p} = F$ says,

$$\frac{d}{dt}\sqrt{2/m(E - V(q))} \cdot \frac{\dot{q}_i}{\|\dot{q}\|} = -\frac{1}{m}\partial_i V \frac{\|\dot{q}\|}{\sqrt{2/m(E-V)}}$$

Jacobi noticed that this is just $F = ma$, or $m\ddot{q}_i = -\partial_i V$, that is, provided we reparameterize q so that,

$$\|\dot{q}\| = \sqrt{2/m(E - V(q))}.$$

Recall that our Lagrangian gives reparameterization invariant Euler-Lagrange equations! This is the unification between least time (from optics) and least action (from mechanics) that we sought.

3.9.2 The Ubiquity of Geodesic Motion

We've seen that many classical systems trace out paths that are geodesics, i.e., paths $q : [t_0, t_1] \to Q$ that are critical points of

$$S(q) = \int_{t_0}^{t_1} \sqrt{g_{ij} \dot{q}^i \dot{q}^j}\, dt$$

which is proper time when (Q, g) is a *Lorentzian manifold*, or arclength when (Q, g) is a *Riemannian manifold*. We have

1. The metric at $q \in Q$ is,

   $$\begin{aligned} g(q) : T_q Q \times T_q Q &\to \mathbb{R} \\ (v, w) &\mapsto g(v, w) \end{aligned}$$

   and it is bilinear.

2. w.r.t a basis of $T_q Q$

   $$g(v, w) = \delta_{ij} v^i w^j$$

3. $g(q)$ varies smoothly with $q \in Q$.

An important distinction to keep in mind is that Lorentzian manifolds represent *space-times*, whereas Riemannian manifolds represent that we'd normally consider as just *space*.

We've seen at least three important things.

(1) In the geometric optics approximation, light in $Q = \mathbb{R}^n$ acts like particles tracing out geodesics in the metric
$$g_{ij} = n(q)^2 \delta_{ij}$$
where $n : Q \to (0, \infty)$ is the index of refraction function.

(2) Jacobi saw that a particle in $Q = \mathbb{R}^n$ in some potential $V : Q \to \mathbb{R}$ traces out geodesics in the metric
$$g_{ij} = \frac{2}{m}(E - V)\delta_{ij}$$
if the particle has energy E (where[1] $V < E$).

(3) A free particle in general relativity traces out a geodesic on a Lorentzian manifold (Q, q).

In fact all three of these results can be generalized to cover every problem that we've discussed!

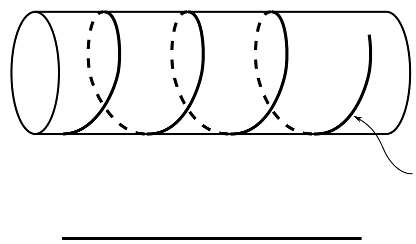
(1′) Light on *any* Riemannian manifold (Q, q) with index of refraction $n : Q \to (0, \infty)$ traces out geodesics in the metric $h = n^2 g$.

(2′) A particle on a Riemannian manifold (Q, q) with potential $V : Q \to \mathbb{R}$ traces out geodesics w.r.t the metric
$$h = \frac{2}{m}(E - V)g$$
if it has energy E. Lots of physical systems can be described this way, e.g., the Atwood machine, a rigid rotating body ($Q = SO(3)$), spinning tops, and others. All of these systems have a Lagrangian which is a quadratic function of position, so they all fit into this framework.

(3′) **Kaluza-Klein Theory.** A particle with charge e on a Lorentzian manifold (Q, q) in an electromagnetic vector potential follows a path with
$$\ddot{q}_i = -\Gamma_{ijk}\dot{q}^j\dot{q}^k + \frac{e}{m}\mathsf{F}_{ij}\dot{q}^j$$
where
$$\mathsf{F}_{ij} = \partial_i A_j - \partial_j A_i$$
but this is actually geodesic motion on the manifold $Q \times U(1)$ where $U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$ is a circle.

[1]The case $V > E$, if they exist, would be classically forbidden regions.

Let's examine this last result a bit further. To get the desired equations for motion on $Q \times U(1)$ we need to given $Q \times U(1)$ a cleverly designed metric built from g and A where the amount of "spiralling"—the velocity in the $U(1)$ direction is e/m. The metric h on



$Q \times U(1)$ is built from g and A in a very simple way. Let's pick coordinates x^i on Q where $i \in \{0, \ldots, n\}$ since we're in $n+1$-dimensional spacetime, and θ is our local coordinate on S^1. The components of h are

$$\begin{aligned} h_{ij} &= g_{ij} + A_i A_j \\ h_{\theta i} &= h_{\theta i} = -A_i \\ h_{\theta\theta} &= 1 \end{aligned}$$

Working out the equations for a geodesic in this metric we get

$$\begin{aligned} \ddot{q}_i &= -\Gamma_{ijk}\dot{q}^j\dot{q}^k + \frac{e}{m}\mathsf{F}_{ij}\dot{q}^j \\ \ddot{q}_\theta &= 0, \\ &\text{if } \dot{q}_\theta = e/m \end{aligned}$$

since F_{ij} is part of the Christoffel symbols for h.

To summarize this section on least time versus least action we can say that every problem that we've discussed in classical mechanics can be regarded as geodesic motion!

Chapter 4

From Lagrangians to Hamiltonians

In the Lagrangian approach we focus on the *position* and *velocity* of a particle, and compute what the particle does starting from the Lagrangian $L(q, \dot{q})$, which is a function

$$L : TQ \longrightarrow \mathbb{R}$$

where the *tangent bundle* is the space of position-velocity pairs. But we're led to consider momentum

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

since the equations of motion tell us how it changes

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q^i}.$$

4.1 The Hamiltonian Approach

In the Hamiltonian approach we focus on *position* and *momentum*, and compute what the particle does starting from the energy

$$H = p_i \dot{q}^i - L(q, \dot{q})$$

reinterpreted as a function of position and momentum, called the *Hamiltonian*

$$H : T^*Q \longrightarrow \mathbb{R}$$

where the *cotangent bundle* is the space of position-momentum pairs. In this approach, position and momentum will satisfy *Hamilton's equations*:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp^i}{dt} = -\frac{\partial H}{\partial q_i}$$

where the latter is the Euler-Lagrange equation

$$\frac{dp^i}{dt} = \frac{\partial L}{\partial q_i}$$

in disguise (it has a minus sign since $H = p\dot{q} - L$).

To obtain this Hamiltonian description of mechanics rigorously we need to study this map

$$\begin{aligned} \lambda : TQ &\longrightarrow T^*Q \\ (q, \dot{q}) &\longmapsto (q, p) \end{aligned}$$

where $q \in Q$, and $\dot{q}$ is any tangent vector in T_qQ (*not* the time derivative of something), and p is a cotangent vector in $T_q^*Q := (T_qQ)^*$, given by

$$\dot{q} \xrightarrow{\lambda} p_i = \frac{\partial L}{\partial \dot{q}^i}$$

So λ is defined using $L : TQ \to \mathbb{R}$. Despite appearances, λ can be defined in a coordinate-free way, as follows (referring to Fig. 4.1). We want to define "$\frac{\partial L}{\partial \dot{q}^i}$" in a coordinate-free

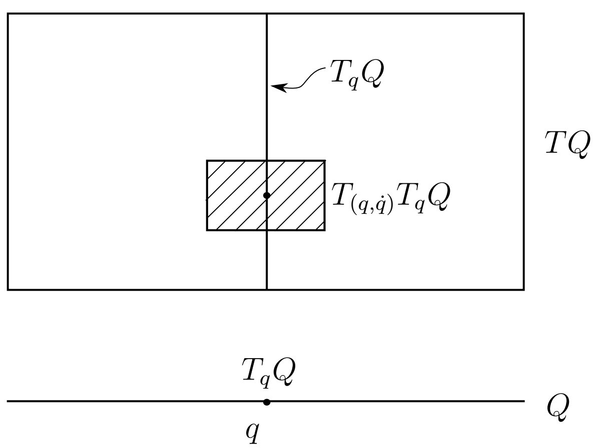


Figure 4.1:

way; it's the "differential of L in the vertical direction"—i.e., the $\dot{q}^i$ directions. We have

$$\begin{aligned} \pi : TQ &\longrightarrow Q \\ (q, \dot{q}) &\longmapsto q \end{aligned}$$

and

$$d\pi : T(TQ) \longrightarrow TQ$$

has kernel[1] consisting of *vertical vectors*:

$$VTQ = \ker d\pi \subseteq TTQ$$

The differential of L at some point $(q, \dot{q}) \in TQ$ is a map from TTQ to $\mathbb{R}$, so we have

$$(dL)_{(q,\dot{q})} \in T^*_{(q,\dot{q})}TQ$$

that is,

$$dL_{(q,\dot{q})} : T_{(q,\dot{q})}TQ \longrightarrow \mathbb{R}.$$

We can restrict this to $VTQ \subseteq TTQ$, getting

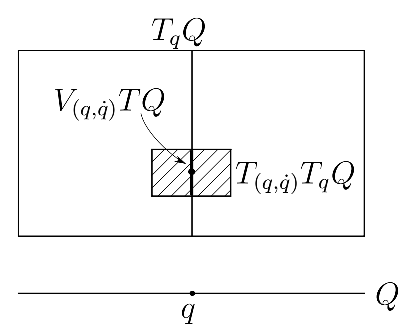
$$f : V_{(q,\dot{q})}TQ \longrightarrow \mathbb{R}.$$

But note

$$V_{(q,\dot{q})}TQ = T(T_qQ)$$

and since T_qQ is a vector space,

$$T_{(q,\dot{q})}T_qQ \cong T_qQ$$



in a canonical way[2]. So f gives a linear map

$$p : T_qQ \longrightarrow \mathbb{R}$$

that is,

$$p \in T^*_qQ$$

this is the momentum!

(Week 6, May 2, 4, 6.)

Given $L : TQ \to T^*Q$, we now know a coordinate-free way of describing the map

$$\begin{aligned} \lambda : TQ &\longrightarrow T^*Q \\ (q, \dot{q}) &\longmapsto (q, p) \end{aligned}$$

given in local coordinates by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}.$$

[1]The *kernel* of a map is the set of all elements in the domain that map to the null element of the range, so $\ker d\pi = \{v \in TTQ : d\pi(v) = 0 \in TQ\}$.

[2]The fiber T_vV at $v \in V$ of vector manifold V has the same dimension as V.

We say L is *regular* if λ is a diffeomorphism from TQ to some open subset $X \subseteq T^*Q$. In this case we can describe what our system is doing equally well by specifying position and velocity,

$$(q, \dot{q}) \in TQ$$

or position and momentum

$$(q, p) = \lambda(q, \dot{q}) \in X.$$

We call X the *phase space* of the system. In practice often $X = T^*Q$, then L is said to be *strongly regular*.

4.2 Regular and Strongly Regular Lagrangians

This section discusses some examples of the above theory.

4.2.1 Example: A Particle in a Riemannian Manifold with Potential $V(q)$

For a particle in a Riemannian manifold (Q, q) in a potential $V : Q \to \mathbb{R}$ has Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} m g_{ij} \dot{q}^i q^j - V(Q)$$

Here

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij} \dot{q}^j$$

so

$$\lambda(q, \dot{q}) = \big(q, mg(\dot{q}, -)\big)$$

so[3] L is strongly regular in this case because

$$\begin{aligned} T_q Q &\longrightarrow T_q^* Q \\ v &\longrightarrow g(v, -) \end{aligned}$$

is 1-1 and onto, i.e., the metric is nondegenerate. Thus λ is a diffeomorphism, which in this case extends to all of T^*Q.

[3] The missing object there "$-$" is of course any tangent vector, not inserted since λ itself is an operator on tangent vectors, not the result of the operation.

4.2.2 Example: General Relativistic Particle in an E-M Potential

For a general relativistic particle with charge e in an electromagnetic vector potential A the Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2} m g_{ij} \dot{q}^i q^j - e A_i \dot{q}^i$$

and thus

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = m g_{ij} \dot{q}^i q^j + e A_i.$$

This L is still strongly regular, but now each map

$$\begin{aligned} \lambda \mid_{T_q Q} : T_q Q &\longrightarrow T_q^* Q \\ \dot{q} &\longmapsto m\, g(\dot{q}, -) + eA(q) \end{aligned}$$

is *affine* rather than linear[4].

4.2.3 Example: Free General Relativistic Particle with Reparameterization Invariance

The free general relativistic particle with reparameterization invariant Lagrangian has,

$$L(q, \dot{q}) = m \sqrt{g_{ij} \dot{q}^i \dot{q}^j}$$

This is terrible from the perspective of regularity properties—it's not differentiable when $g_{ij} \dot{q}^i \dot{q}^j$ vanishes, and undefined when the same is negative. Where it *is* defined

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \frac{m g_{ij} \dot{q}^j}{\|\dot{q}\|}$$

(where $\dot{q}$ is timelike), we can ask about regularity. Alas, the map λ is not 1-1 where defined since multiplying $\dot{q}$ by some number has *no effect* on p! (This is related to the reparameterization invariance—this always happens with reparameterization-invariant Lagrangians.)

4.2.4 Example: A Regular but not Strongly Regular Lagrangian

Here's a Lagrangian that's regular but not strongly regular. Let $Q = \mathbb{R}$ and

$$L(q, \dot{q}) = f(\dot{q})$$

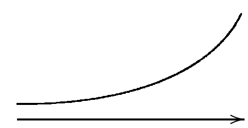
[4] All linear transforms are affine, but affine transformations include translations, which are nonlinear. In affine geometry there is no defined origin. For the example the translation is the "$+eA(q)$" part.

so that

$$p = \frac{\partial L}{\partial \dot q} = f'(\dot q)$$

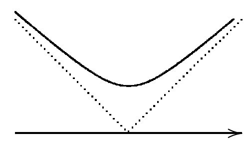
This will be regular but not strongly so if $f' : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism from $\mathbb{R}$ to some proper subset $U \subset \mathbb{R}$. For example, take $f(\dot q) = e^{\dot q}$ so $f' : \mathbb{R} \xrightarrow{\sim} (0, \infty) \subset \mathbb{R}$. So

$$L(q, \dot q) = e^{\dot q}$$



or

$$L(q, \dot q) = \sqrt{1 + \dot q^2}$$



and so forth.

4.3 Hamilton's Equations

Now let's assume L is regular, so

$$\begin{aligned} \lambda : TQ &\xrightarrow{\sim} X \subseteq T^*Q \\ (q, \dot q) &\longmapsto (q, p) \end{aligned}$$

This lets us have the best of both worlds: we can identify TQ with X using λ. This lets us treat q^i, p^i, L, H, etc., all as functions on X (or TQ), thus writing

$$\dot q^i \qquad \text{(function on } TQ\text{)}$$

for the function

$$\dot q^i \circ \lambda^{-1} \qquad \text{(function on } X\text{)}$$

In particular

$$\dot p_i := \frac{\partial L}{\partial q^i} \qquad \text{(Euler-Lagrange eqn.)}$$

which is really a function on TQ, will be treated as a function on X. Now let's calculate:

$$\begin{aligned} dL &= \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial \dot q^i} d\dot q^i \\ &= \dot p_i dq^i + p_i d\dot q^i \end{aligned}$$

while

$$\begin{aligned} dH &= d(p_i\dot{q}^i - L) \\ &= \dot{q}^i dp_i + p_i d\dot{q}^i - dL \\ &= \dot{q}^i dp_i + p_i d\dot{q}^i - (\dot{p}_i dq^i + p_i d\dot{q}^i) \\ &= \dot{q}^i dp_i - \dot{p}_i d\dot{q}^i \end{aligned}$$

so

$$dH = \dot{q}^i dp_i - \dot{p}^i dq_i.$$

Assume the Lagrangian $L : TQ \to \mathbb{R}$ is regular, so

$$\begin{aligned} \lambda : TQ &\xrightarrow{\sim} X \subseteq T^*Q \\ (q,\dot{q}) &\longmapsto (q,p) \end{aligned}$$

is a diffeomorphism. This lets us regard both L and the Hamiltonian $H = p_i\dot{q}^i - L$ as functions on the phase space X, and use $(q^i, \dot{q}^i)$ as local coordinates on X. As we've seen, this gives us

$$\begin{aligned} dL &= \dot{p}_i dq^i + p_i d\dot{q}^i \\ dH &= \dot{q}^i dp_i - \dot{p}_i dq^i. \end{aligned}$$

But we can also work out dH directly, this time using local coordinates (q^i, p_i), to get

$$dH = \frac{\partial H}{\partial p^i} dp_i + \frac{\partial H}{\partial q^i} dq^i.$$

Since dp_i, dq^i form a basis of 1-forms, we conclude:

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$$

These are *Hamilton's Equations.*

4.3.1 Hamilton and Euler-Lagrange

Though $\dot{q}^i$ and $\dot{p}_i$ are just functions of X, when the Euler-Lagrange equations hold for some path $q : [t_0, t_1] \to Q$, they will be the time derivatives of q^i and p_i. So when the Euler-Lagrange equations hold, Hamilton's equations describe the motion of a point $x(t) = \big(q(t), p(t)\big) \in X$. In fact, in this context, Hamilton's equations are just the Euler-Lagrange equations in disguise. The equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

really just lets us recover the velocity $\dot{q}$ as a function of q and p, inverting the formula

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

which gave p as a function of q and $\dot{q}$. So we get a formula for the map

$$\begin{aligned} \lambda^{-1} : X &\longrightarrow TQ \\ (q,p) &\longmapsto (q,\dot{q}). \end{aligned}$$

Given this, the other Hamilton equation

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

is secretly the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \qquad \text{or} \qquad \dot{p} = \frac{\partial L}{\partial q^i}$$

These are the same because

$$\frac{\partial H}{\partial q^i} = \frac{\partial}{\partial q^i}\big(p_i\dot{q}^i - L\big) = -\frac{\partial L}{\partial q^i}.$$

Example: Particle in a Potential $V(q)$

For a particle in $Q = \mathbb{R}^n$ in a potential $V : \mathbb{R}^n \to \mathbb{R}$ the system has Lagrangian

$$L(q,\dot{q}) = \frac{m}{2}\|\dot{q}\|^2 - V(q)$$

which gives

$$\begin{aligned} p &= m\dot{q} \\ \dot{q} &= \frac{p}{m}, \qquad \text{(though really that's } \dot{q} = \frac{g^{ij}p_j}{m}\text{)} \end{aligned}$$

and Hamiltonian

$$\begin{aligned} H(q,p) = p_i\dot{q}^i - L &= \frac{1}{m}\|p\|^2 - \left(\frac{\|p\|^2}{2m} - V\right) \\ &= \frac{1}{2m}\|p\|^2 + V(q). \end{aligned}$$

So Hamilton's equations say

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \quad \Rightarrow \quad \dot{q} = \frac{p}{m} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \quad \Rightarrow \quad \dot{p} = -\boldsymbol{\nabla} V \end{aligned}$$

The first just recovers $\dot{q}$ as a function of p; the second is $F = ma$.

Note on Symplectic Structure

Hamilton's equations push us toward the viewpoint where p and q have equal status as coordinates on the phase space X. Soon, we'll drop the requirement that $X \subseteq T^*Q$ where Q is a configuration space. X will just be a manifold equipped with enough structure to write down Hamilton's equations starting from any $H : X \to \mathbb{R}$.

The coordinate-free description of this structure is the major 20th century contribution to mechanics: a symplectic structure.

This is important. You might have some particles moving on a manifold like S^3, which is not symplectic. So the Hamiltonian mechanics point of view says that the abstract manifold that you are really interested in is something different: it must be a symplectic manifold. That's the phase space X. We'll introduce symplectic geometry more completely in later chapters.

4.3.2 Hamilton's Equations from the Principle of Least Action

Before, we obtained the Euler-Lagrange equations by associating an "action" S with any $q : [t_0, t_1] \to Q$ and setting $\delta S = 0$. Now let's get Hamilton's equations *directly* by assigning an action S to any path $x : [t_0, t_1] \to X$ and setting $\delta S = 0$. Note: we don't impose any relation between p and q, $\dot{q}$! The relation will follow from $\delta S = 0$.

Let P be the space of paths in the phase space X and define the action

$$S : P \longrightarrow \mathbb{R}$$

by

$$S(x) = \int_{t_0}^{t_1} (p_i \dot{q}^i - H) dt$$

where $p_i \dot{q}^i - H = L$. More precisely, write our path x as $x(t) = \big(q(t), p(t)\big)$ and let

$$S(x) = \int_{t_0}^{t_1} \left[p_i(t) \frac{d}{dt} q^i(t) - H\big(q(t), p(t)\big) \right] dt$$

we write $\frac{d}{dt} q^i$ instead of $\dot{q}^i$ to emphasize that we mean the time derivative rather than a coordinate in phase space.

Let's show $\delta S = 0 \Leftrightarrow$Hamilton's equations.

$$\begin{aligned} \delta S &= \delta \int (p_i \dot{q}^i - H) dt \\ &= \int \big(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \delta H \big) dt \end{aligned}$$

then integrating by parts,

$$\begin{aligned}
&= \int \big(\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \delta H\big) dt \\
&= \int \Big(\delta p_i \dot{q}^i - \dot{p}_i \delta q^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i\Big) dt \\
&= \int \left(\delta p_i \Big(\dot{q}^i - \frac{\partial H}{\partial p_i}\Big) + \delta q^i \Big(-\dot{p}_i - \frac{\partial H}{\partial q^i}\Big)\right) dt
\end{aligned}$$

This vanishes $\forall \delta x = (\delta q, \delta p)$ if and only if Hamilton's equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad p_i = -\frac{\partial H}{\partial q^i}$$

hold. Just as we hoped.

We've seen two principles of "least action":

1. For paths in configuration space Q, $\delta S = 0 \Leftrightarrow$Euler-Lagrange equations.
2. For paths in phase space X, $\delta S = 0 \Leftrightarrow$Hamilton's equations.

Additionally, since $X \subseteq T^*Q$, we might consider a third version based on paths in position-velocity space TQ. But when our Lagrangian is regular we have a diffeomorphism $\lambda : TQ \xrightarrow{\sim} X$, so this third principle of least action is just a reformulation of principle 2. However, the *really* interesting principle of least action involves paths in the *extended phase space* where we have an additional coordinate for time: $X \times \mathbb{R}$.

Recall the action

$$\begin{aligned}
S(x) &= \int (p_i \dot{q}^i - H)\, dt \\
&= \int p_i \frac{dq^i}{dt} dt - H\, dt \\
&= \int p_i dq^i - H\, dt
\end{aligned}$$

We can interpet the integrand as a 1-form

$$\beta = p_i dq^i - H\, dt$$

on $X \times \mathbb{R}$, which has coordinates $\{p_i, q^i, t\}$. So any path

$$x : [t_0, t_1] \longrightarrow X$$

gives a path

$$\begin{aligned}
\sigma : [t_0, t_1] &\longrightarrow X \times \mathbb{R} \\
t &\longmapsto (x(t), t)
\end{aligned}$$

and the action becomes the integral of a 1-form over a curve:

$$S(x) = \int p_i \, dq^i - H \, dt = \int_\sigma \beta$$

4.4 Waves versus Particles—The Hamilton-Jacobi Equations

(Week 7, May 9, 11, 13.)

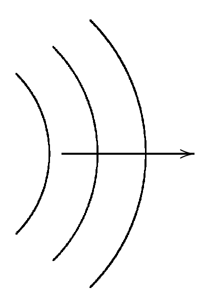
In quantum mechanics we discover that every particle—electrons, photons, neutrinos, etc.—is a wave, and vice versa. Interestingly Newton already had a particle theory of light (his "corpuscules") and various physicists argued against it by pointing out that diffraction is best explained by a wave theory. We've talked about geometrized optics, an approximation in which light consists of particles moving along geodesics. Here we start with a Riemannian manifold (Q, g) as space, but we use the new metric

$$h_{ij} = n^2 g_{ij}$$

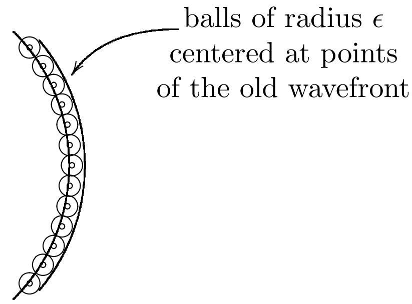
where $n : Q \to (0, \infty)$ is the index of refraction throughout space (generally not a constant).

4.4.1 Wave Equations

Huygens considered this same setup (in simpler language) and considered the motion of a wavefront:



and saw that the wavefront is the envelope of a bunch of little wavelets centered at points along the big wavefront:



In short, the wavefront moves at unit speed in the normal direction with respect to the "optical metric" h. We can think about the distance function

$$d : Q \times Q \longrightarrow [0, \infty)$$

on the Riemannian manifold (Q, h), where

$$d(q_0, q_1) = \inf_{\Upsilon}(\text{arclength})$$

where $\Upsilon = \{\text{paths from } q_0 \text{ to } q_1\}$. (Secretly this $d(q_0, q_1)$ *is* the least action—the infimum of action over all paths from q_0 to q_1.) Using this we get the wavefronts centered at $q_0 \in Q$ as the level sets

$$\{q : d(q_0, q) = c\}$$

or at least for small $c > 0$, as depicted in Fig. 4.2. For larger c the level sets can cease to

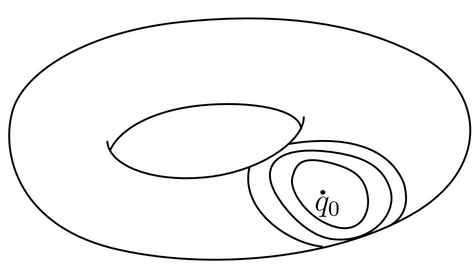


Figure 4.2:

be smooth—we say a *catastrophe* occurs—and then the wavefronts are no longer the level sets. This sort of situation can happen for topological reasons (as when the waves smash into each other in the back of Fig. 4.2) and it can also happen for geometrical reasons

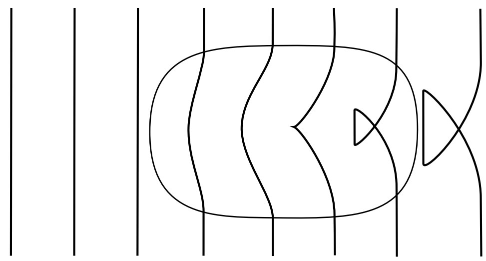


Figure 4.3:

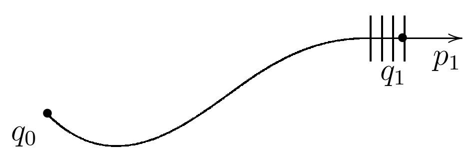
(Fig. 4.3). Assuming no such catastrophes occur, we can approximate the waves of light by a wavefunction:

$$\psi(q) = A(q)e^{ik\,d(q,q_0)}$$

where k is the wavenumber of the light (i.e., its color) and $A : Q \to \mathbb{R}$ describes the amplitude of the wave, which drops off far from q_0. This becomes the *eikonal approximation* in optics[5] once we figure out what A should be.

Hamilton and Jacobi focused on distance $d : Q \times Q \to [0, \infty)$ as a function of *two* variables and called it W=*Hamilton's principal function.* They noticed that:

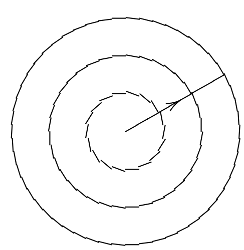
$$\frac{\partial}{\partial q_1^i} W(q_0, q_1) = (p_1)_i$$



where p_1 is a cotangent vector *pointing normal to the wavefronts.*

4.4.2 The Hamilton-Jacobi Equations

We've seen that in optics, particles of light move along geodesics, but wavefronts are level sets of the distance functions:



[5]Eikonal comes form the Greek word for 'image' or 'likeness', in optics the eikonal approximation is the basis for ray tracing methods.

at least while the level sets remain smooth. In the eikonal approximation, light is described by waves

$$\begin{aligned} &\psi : Q \longrightarrow \mathbb{C} \\ &\psi(q_1) = A(q_1) e^{ik\,W(q_0,q_1)} \end{aligned}$$

where (Q, h) is a Riemannian manifold, h is the optical metric, $q_0 \in Q$ is the light source, k is the frequency and

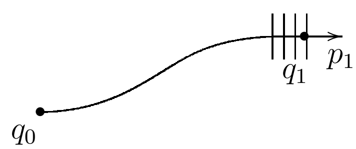
$$W : Q \times Q \longrightarrow [0, \infty)$$

is the distance function on Q, or *Hamilton's principal function*:

$$W(q_0, q_1) = \inf_{q \in \Upsilon} S(q)$$

where Υ is the space of paths from q_0 to q and $S(q)$ is the action of the path q, i.e., its arclength. This is begging to be generalized to other Lagrangian systems! (At least it is retrospectively, with the advantage of our historical perspective.) We also saw that

$$\frac{\partial}{\partial q_1^i} W(q_0, q_1) = (p_1)_i,$$



"points normal to the wavefront"—really the tangent vector

$$p_1^i = h^{ij}(p_1)_j$$

points in this direction. In fact kp_{1i} is the *momentum* of the light passing through q_1. This foreshadows quantum mechanics! After all, in quantum mechanics, the momentum is an operator that acts to differentiating the wavefunction.

Jacobi generalized this to the motion of point particles in a potential $V : Q \to \mathbb{R}$, using the fact that a particle of energy E traces out geodesics in the metric

$$h_{ij} = \frac{2(E - V)}{m} g_{ij}.$$

We've seen this reduces point particle mechanics to optics—but only for particles of fixed energy E. Hamilton went further, and we now can go further still.

Suppose Q is any manifold and $L : TQ \to \mathbb{R}$ is any function (Lagrangian). Define Hamilton's principal function

$$W : Q \times \mathbb{R} \times Q \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$W(q_0, t_0; q_1, t_1) = \inf_{q \in \Upsilon} S(q)$$

where

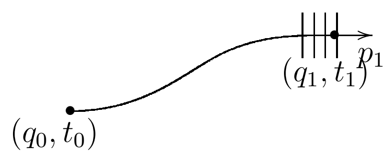
$$\Upsilon = \{q : [t_0, t_1] \to Q,\ q(t_0) = q_0,\ \&\ q(t_1) = q_1\}$$

and

$$S(q) = \int_{t_0}^{t_1} L\Big(q(t), \dot{q}(t)\Big)\, dt$$

Now W is just the *least action* for a path from (q_0, t_0) to (q_1, t_1); it'll be smooth if (q_0, t_0) and (q_1, t_1) are close enough—so let's assume that is true. In fact, we have

$$\frac{\partial}{\partial q_1^i} W(q_0, q_1) = (p_1)_i,$$



where p_1 is the momentum of the particle going from q_0 to q_1, at time t_1, and

$$\begin{aligned} \frac{\partial W}{\partial q_0^i} &= -(p_0)_i, && \text{(-momentum at time } t_0) \\ \frac{\partial W}{\partial t_1} &= -H_1, && \text{(-energy at time } t_1) \\ \frac{\partial W}{\partial t_0} &= H_0, && \text{(+energy at time } t_0) \end{aligned}$$

($H_1 = H_0$ as energy is conserved). These last four equations are the *Hamilton-Jacobi equations.* The mysterious minus sign in front of energy was seen before in the 1-form,

$$\beta = p_i dq^i - H\, dt$$

on the extended phase space $X \times \mathbb{R}$. Maybe the best way to get the Hamilton-Jacobi equations is from this extended phase space formulation. But for now let's see how Hamilton's principal function W and variational principles involving least action also yield the Hamilton-Jacobi equations.

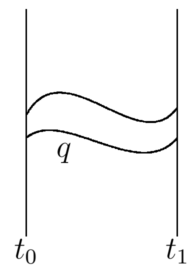
Given (q_0, t_0), (q_1, t_1), let

$$q : [t_0, t_1] \longrightarrow Q$$

be the action-minimizing path from q_0 to q_1. Then

$$W(q_0, t_0;\ q_1, t_1) = S(q)$$

Now consider varying q_0 and q_1 a bit



and thus vary the action-minimizing path, getting a variation δq which does not vanish at t_0 and t_1. We get

$$\begin{aligned}
\delta W &= \delta S \\
&= \delta \int_{t_0}^{t_1} L(q, \dot{q})\, dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \delta q^i - \dot{p}_i \delta q^i \right) dt + p_i \delta q^i \Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} - \dot{p}_i \right) \delta q^i \, dt
\end{aligned}$$

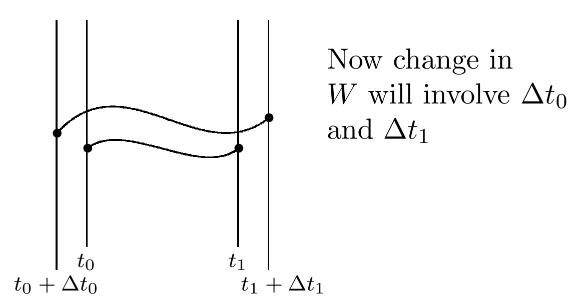
the term in parentheses is zero because q minimizes the action *and* the Euler-Lagrange equations hold. So we δq^i have

$$\delta W = p_{1i} \delta q_1^i - p_{0i} \delta q_0^i$$

and so

$$\frac{\partial W}{\partial q_1^i} = p_{1i}, \quad \text{and} \quad \frac{\partial W}{\partial q_0^i} = -p_{0i}$$

These are two of the four Hamilton-Jacobi equations! To get the other two, we need to vary t_0 and t_1:



(you can imagine $\Delta t_0 < 0$ in this figure if you like).

We want to derive the Hamilton-Jacobi equations describing the derivatives of Hamilton's principal function

$$W(q_0, t_0;\, q_1, t_1) = \inf_{q \in \Upsilon} S(q)$$

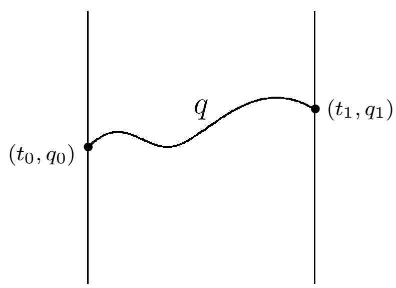
where Υ is the space of paths $q : [t_0, t_1] \to Q$ with $q(t_0) = q$, $q(t_1) = q_1$ and

$$S(q) = \int_{t_0}^{t_1} L(q, \dot{q})\, dt$$

where the Lagrangian $L : TQ \to \mathbb{R}$ will now be assumed *regular*, so that

$$\begin{aligned} \lambda TQ &\longrightarrow X \subseteq T^*Q \\ (q, \dot{q}) &\longmapsto (q, p) \end{aligned}$$

is a diffeomorphism. We need to ensure that (q_0, t_0) is close enough to (q_i, t_1) that there is a unique $q \in \Upsilon$ that minimizes the action S, and assume that this q depends smoothly on $U = (q_0, t_0; q_1, t_1) \in (Q \times \mathbb{R})^2$. We'll think of q as a function of U:



$$\begin{aligned} (Q \times \mathbb{R})^2 &\to \Upsilon \\ u &\mapsto q \end{aligned}$$

defined only when (q_0, t_0) and (q_1, t_1) are sufficiently close.

Then Hamilton's principal function is

$$\begin{aligned} W(u) \;:=\; W(q_0, t_0; q_1, t_1) &= S(q) \\ &= \int_{t_0}^{t_1} L(q, \dot{q})\, dt \\ &= \int_{t_0}^{t_1} \Big(p\dot{q} - H(q, p) \Big)\, dt \\ &= \int_{t_0}^{t_1} p\, dq - H\, dt \\ &= \int_C \beta \end{aligned}$$

where $\beta = pdq - H(q, p)dt$ is a 1-form on the extended phase space $X \times \mathbb{R}$, and C is a curve in the extended phase space:

$$C(t) = \Big(q(t), p(t), t \Big) \in X \times \mathbb{R}.$$

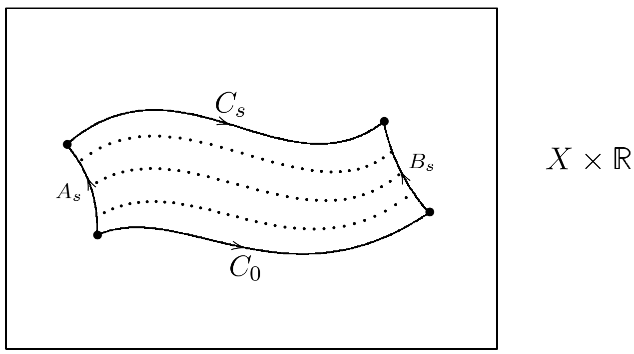
Note that C depends on the curve $q \in \Upsilon$, which in turn depends upon $u = (q_0, t_0; q_1, t_1) \in (Q \times \mathbb{R})^2$. We are after the derivatives of W that appear in the Hamilton-Jacobi relations, so let's differentiate

$$W(u) = \int_C \beta$$

with respect to u and get the Hamilton-Jacobi equations from β. Let u_s be a 1-parameter family of points in $(Q \times \mathbb{R})^2$ and work out

$$\frac{d}{ds} W(u_s) = \frac{d}{ds} \int_{C_s} \beta$$

where C_s depends on u_s as above



Let's compare

$$\int_{C_s} \beta \quad \text{and} \quad \int_{A_s + C_s + B_s} = \int_{A_s} \beta + \int_{C_s} \beta + \int_{B_s} \beta$$

Since C_0 minimizes the action among paths with the given end-points, and the curve $A_s + C_s + B_s$ has the same end-points, we get

$$\frac{d}{ds} \int_{A_s + C_s + B_s} \beta = 0$$

(although $A_s + C_s + B_s$ is not smooth, we can approximate it by a path that is smooth). So

$$\frac{d}{ds} \int_{C_s} \beta = \frac{d}{ds} \int_{B_s} \beta - \frac{d}{ds} \int_{A_s} \beta \quad \text{at } s = 0.$$

Note

$$\begin{aligned} \frac{d}{ds} \int_{A_s} \beta &= \frac{d}{ds} \int \beta(A_r')\, dr \\ &= \beta(A_0') \end{aligned}$$

where $A_0' = v$ is the tangent vector of A_s at $s = 0$. Similarly,

$$\frac{d}{ds} \int_{B_s} \beta = \beta(w)$$

where $w = B_0'$. So,

$$\frac{d}{ds}W(u_s) = \beta(w) - \beta(v)$$

where w keeps track of the change of (q_1, p_1, t_1) as we move C_s and v keeps track of (q_0, p_0, t_0). Now since $\beta = p^i dq_i - H dt$, we get

$$\frac{\partial W}{\partial q_1^i} = p_1^i$$
$$\frac{\partial W}{\partial t_1} = -H$$

and similarly

$$\frac{\partial W}{\partial q_0^i} = -p_0^i$$
$$\frac{\partial W}{\partial t_0} = H$$

So, if we define a wavefunction:

$$\psi(q_0, t_0; q_1, t_1) = e^{iW(q_0,t_0;q_1,t_1)/\hbar}$$

then we get

$$\frac{\partial \psi}{\partial t_1} = -\frac{i}{\hbar} H_1 \psi$$
$$\frac{\partial \psi}{\partial q_1^i} = \frac{i}{\hbar} p_1 \psi$$

At the time of Hamilton and Jacobi's research this would have been new... but nowadays it is thoroughly familiar from *quantum mechanics*!

Bibliography

[Pel94] Peter Peldan. Actions for gravity, with generalizations: A review. *Classical and Quantum Gravity*, 11:1087, 1994.

[WTM71] J. A. Wheeler, K. S. Thorne, and C. W. Misner. *Gravitation*. W. H. Freeman, New York, 1971.