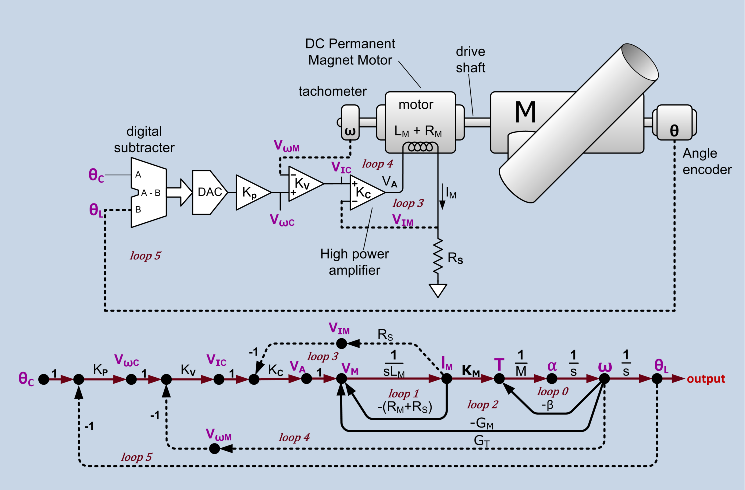
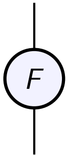
Categories in Control

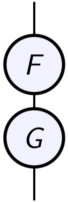
John Baez and Jason Erbele



Categories are great for describing processes. A process with input x and output y is a *morphism* $F\colon x \to y$, and we can draw it like this:

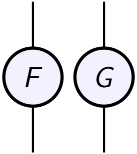


We can do one process after another if the output of the first equals the input of the second:



Here we are composing morphisms $F\colon x \to y$ and $G\colon y \to z$ to get a morphism $GF\colon x \to z$.

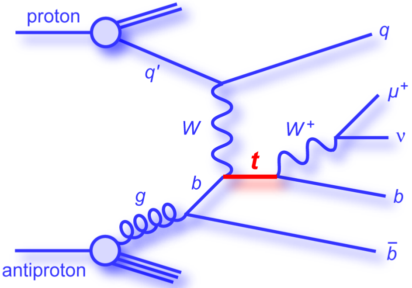
In a monoidal category, we can also do processes 'in parallel':



Here we are 'tensoring' $F\colon x \to y$ and $G\colon x' \to y'$ to get a morphism $F \otimes G\colon x \otimes x' \to y \otimes y'$.

The category with vector spaces as objects and linear maps as morphisms becomes a monoidal category with the usual $\otimes$.

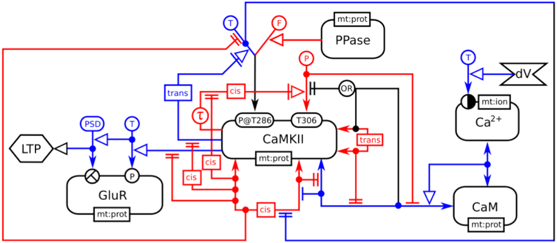
In quantum field theory, 'Feynman diagrams' are pictures of morphisms in this monoidal category:



But why should quantum field theorists have all the fun? This is the century of biology, and ecology.

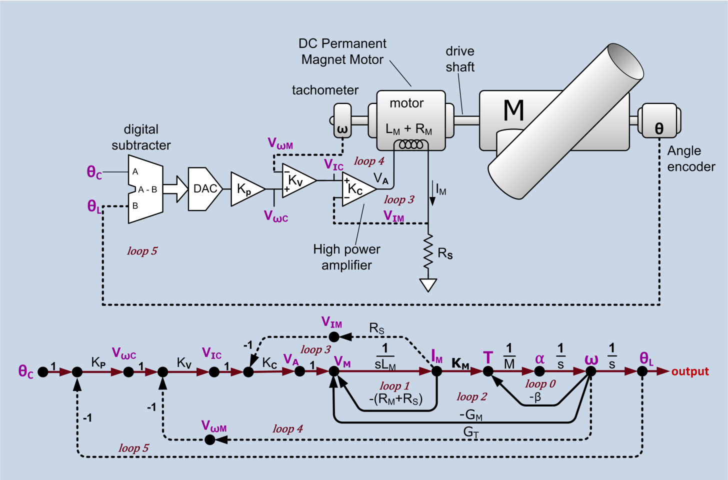
Now is our chance to understand the biosphere, and stop destroying it! We should be using category theory — and everything else — to do these things.

Biologists use diagrams to describe the complex processes they find in life. They use at least three different diagram languages, as formalized in Systems Biology Graphical Notation.



We should try to understand these diagrams using all the tools of modern mathematics!

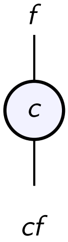
But let's start with something easier: engineering. Engineers use 'signal-flow graphs' to describe processes where signals flow through a system and interact:



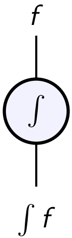
Think of a signal as a smooth real-valued function of time:

$$f : \mathbb{R} \to \mathbb{R}$$

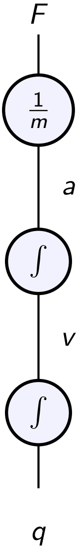
We can multiply a signal by a constant and get a new signal:



We can integrate a signal:



Here is what happens when you push on a mass m with a time-dependent force F:



Integration introduces an ambiguity: the constant of integration. But electrical engineers often use Laplace transforms to write signals as linear combinations of exponentials

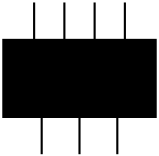
$$f(t) = e^{-st} \quad \text{for some } s > 0$$

Then they define

$$(\textstyle\int f)(t) = \frac{e^{-st}}{s}$$

This lets us think of integration as a special case of scalar multiplication! We extend our field of scalars from $\mathbb{R}$ to $\mathbb{R}(s)$, the field of rational real functions in one variable s.

Let us be general and work with an arbitrary field k. For us, any signal-flow graph with m input edges and n output edges



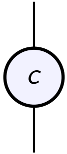
will stand for a linear map

$$F \colon k^m \to k^n$$

In other words: signal-flow graphs are pictures of morphisms in $\mathrm{FinVect}_k$, the category of finite-dimensional vector spaces over k... *where we make this into a monoidal category using $\oplus$, not $\otimes$.*

We build these pictures from a few simple 'generators'.

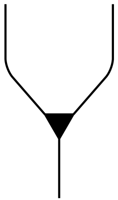
First, we have scalar multiplication:



This is a notation for the linear map

$$\begin{array}{ccc} k & \to & k \\ f & \mapsto & cf \end{array}$$

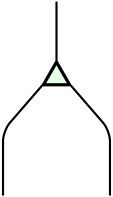
Second, we can add two signals:



This is a notation for

$$+ \colon k \oplus k \to k$$

Third, we can 'duplicate' a signal:



This is a notation for the diagonal map

$$\begin{array}{rccc} \Delta\colon & k & \to & k \oplus k \\ & f & \mapsto & (f, f) \end{array}$$

Fourth, we can 'delete' a signal:

This is a notation for the linear map

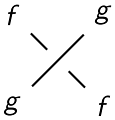
$$\begin{array}{rcl} k & \to & \{0\} \\ f & \mapsto & 0 \end{array}$$

Fifth, we have the zero signal:

This is a notation for the linear map

$$\begin{array}{ccc} \{0\} & \to & k \\ 0 & \mapsto & 0 \end{array}$$

Furthermore, $(\mathrm{FinVect}_k, \oplus)$ is a *symmetric* monoidal category. This means we have a ‘braiding’: a way to switch two signals:

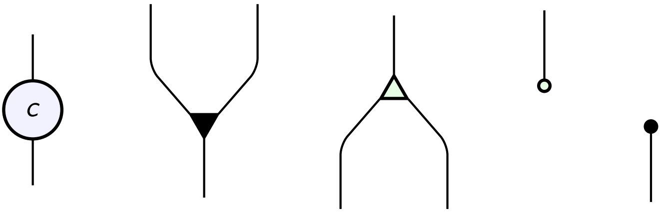


This is a notation for the linear map

$$\begin{array}{rcl} k \oplus k & \to & k \oplus k \\ (f, g) & \mapsto & (g, f) \end{array}$$

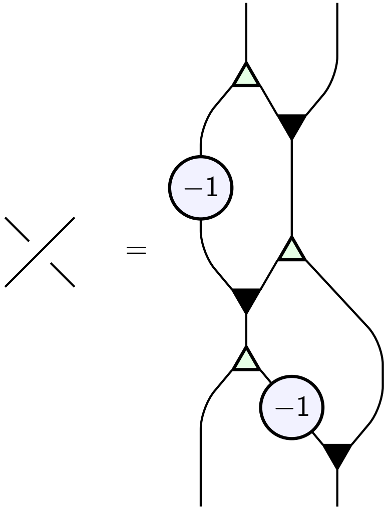
In a symmetric monoidal category, the braiding must obey a few axioms. I won’t list them here, since they are easy to find.

From these ‘generators’:



together with the braiding, we can build complicated signal flow diagrams. In fact, we can describe *any* linear map $F\colon k^m \to k^n$ this way!

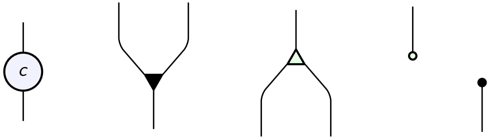
But these generators obey some unexpected relations:



Luckily, we can derive *all* the relations from some very nice ones!

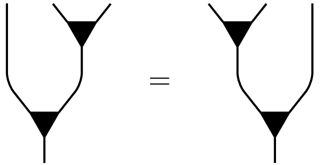
Theorem (Jason Erbele)

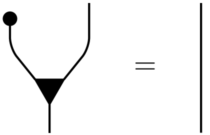
$FinVect_k$ is equivalent to the symmetric monoidal category generated by the object k and these morphisms:

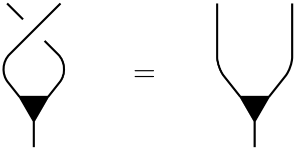


where $c \in k$, with the following relations.

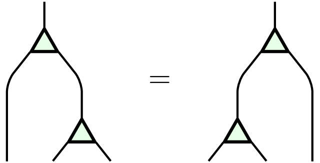
Addition and zero make k into a commutative monoid:

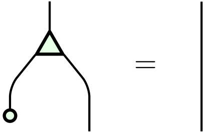


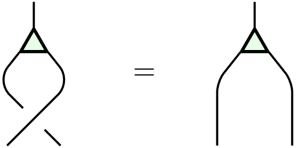




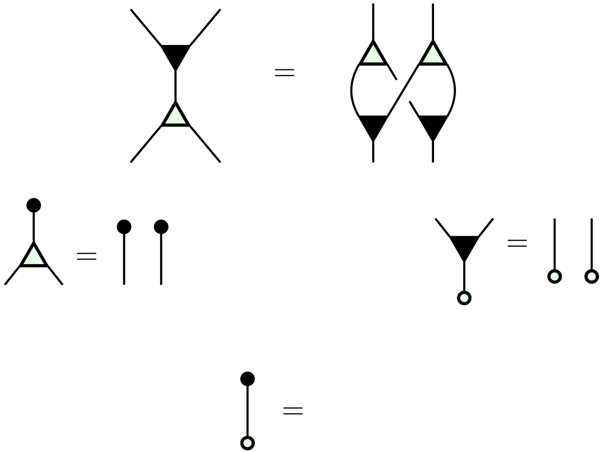
Duplication and deletion make k into a cocommutative comonoid:



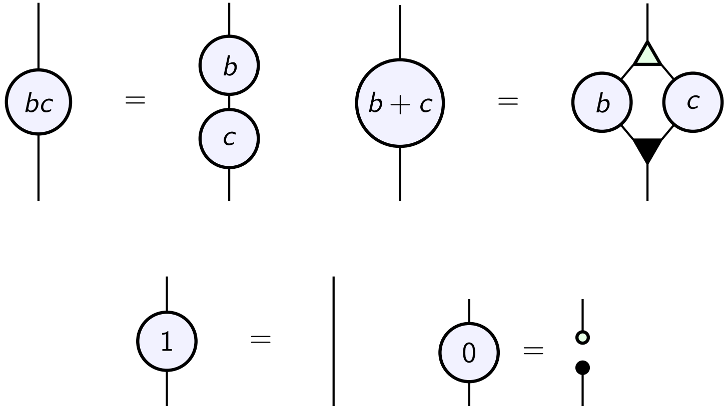




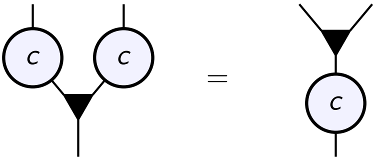
The monoid and comonoid operations are compatible, as in a bialgebra:

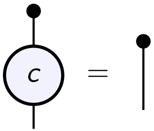


The ring structure of k can be recovered from the generators:

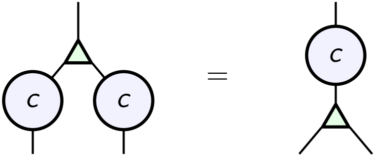


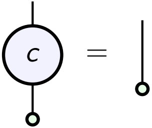
Scalar multiplication is linear (compatible with addition and zero):





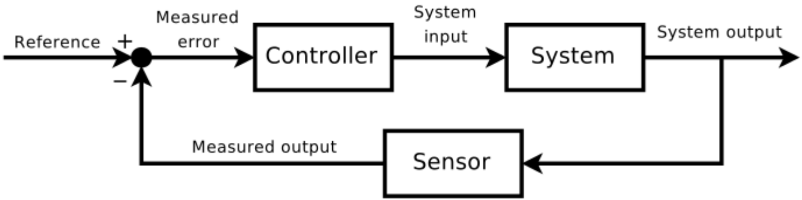
Scalar multiplication is ‘colinear’ (compatible with duplication and deletion):





Those are all the relations we need!

However, control theory also needs more general signal-flow graphs, which have ‘feedback loops’:



This is the most important concept in control theory: letting the output of a system affect its input.

To allow feedback loops we need morphisms more general than linear maps. We need linear relations!

A **linear relation** $F \colon U \rightsquigarrow V$ from a vector space U to a vector space V is a linear subspace $F \subseteq U \oplus V$.

We can compose linear relations $F \colon U \rightsquigarrow V$ and $G \colon V \rightsquigarrow W$ and get a linear relation $G \circ F \colon U \rightsquigarrow W$:

$$G \circ F = \{(u, w) : \exists v \in V \;\; (u, v) \in F \text{ and } (v, w) \in G\}.$$

A linear map $\phi \colon U \to V$ gives a linear relation $F \colon U \rightsquigarrow V$, namely the graph of that map:

$$F = \{(u, \phi(u)) : u \in U\}$$

Composing linear maps becomes a special case of composing linear relations.

There is a symmetric monoidal category FinRel_k with finite-dimensional vector spaces over the field k as objects and linear relations as morphisms. This has $\mathrm{FinVect}_k$ as a subcategory.

Fully general signal-flow diagrams are pictures of morphisms in FinRel_k, typically with $k = \mathbb{R}(s)$.

Jason Erbele showed that besides the previous generators of $\mathrm{FinVect}_k$, we only need two more morphisms to generate all the morphisms in FinRel_k: the 'cup' and 'cap'.

These linear relations say that when a signal goes around a bend in a wire, the signal coming out equals the signal going in!

More formally, the cup is the linear relation

$$\cup \colon k \oplus k \rightsquigarrow \{0\}$$

that is, the subspace

$$\cup \;\subseteq\; k \oplus k \oplus \{0\}$$

given by:

$$\cup = \{(f, f, 0) : f \in k\}$$

Similarly, the cap

$$\cap \colon \{0\} \rightsquigarrow k \oplus k$$

is the subspace

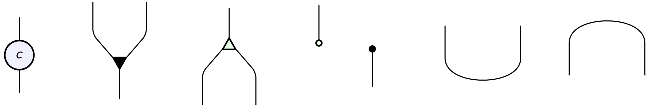
$$\cap \;\subseteq\; \{0\} \oplus k \oplus k$$

given by:

$$\cap = \{(0, f, f) : f \in k\}$$

Theorem (Jason Erbele)

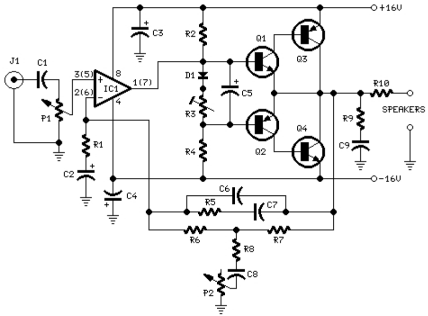
FinRel_k is equivalent to the symmetric monoidal category generated by the object k and these morphisms:



where $c \in k$, and an explicit list of relations.

Instead of listing the relations, let me just sketch what comes next!

I have only talked about *linear* control theory. There is also a nonlinear version. In both the linear and nonlinear case there's a general issue: engineers want to build devices that actually *implement* a given signal-flow graph. One way is to use electrical circuits. These are described using 'circuit diagrams':



At least in the linear case, there is a category Circ whose morphisms are circuit diagrams. Thanks to work in progress by Brendan Fong, we know there is a *functor* from this category to FinRel_k:

$$Z\colon \mathrm{Circ} \to \mathrm{FinRel}_k$$

This functor says, for any linear circuit, how the voltages and currents on the input wires are related to those on the ouput wires.

However, we do not get arbitrary linear relations this way. The space of voltages and currents on n wires,

$$k^n \oplus k^n$$

is naturally a *symplectic* vector space. And the kind of linear relation

$$F \colon k^m \oplus k^m \rightsquigarrow k^n \oplus k^n$$

we get from a linear circuit is naturally a *Lagrangian* relation. That is,

$$F \subseteq (k^m \oplus k^m) \oplus (k^n \oplus k^n)$$

is a **Lagrangian subspace**: a maximal subspace on which the symplectic 2-form vanishes! These are important in classical mechanics.

So, we can see the beginnings of an interesting relation between:

- control theory
- electrical engineering
- category theory
- symplectic geometry

This should become even more interesting when we study nonlinear systems. And as we move from the networks important in *human-engineered* systems to those important in *biology* and *ecology*, the mathematics should become even more rich!

For more, see what the Azimuth Project is doing on network theory:

www.azimuthproject.org