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class fields ....

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so i just want to give an extremely sketchy outline here of the motivation for the false conjecture that i was trying to discuss yesterday; i'm hoping that this might help in showing how we should plan to tie together the seemingly disparate themes that were coming up in the discussion.

so there are 3 interesting relevant facts here:

1. ideal class groups are used in classifying unramified abelian extensions of number fields. (analogous things happen in the ramified case as well.)

2. nonzero ideals (or more accurately fractional ideals) in number fields are invertible modules aka "line objects" aka torsors of the algebraic abelian group gl(1). (this also means that we can think of ideals of order n in the ideal class group as giving torsors of the algebraic subgroup of the nth roots of unity in gl(1).)

3. abelian extensions of a number field can be thought of as torsors of the finite abelian galois group of the extension.

thus there are 3 fundamental concepts here (ideals in number fields, torsors of finite abelian groups, and abelian extensions of number fields) and each of the 3 facts above ties together 2 of those concepts, creating a tight 3-point circle of concepts which of course we hope to see as homotopically trivial in the world of concepts.

and in fact there's an obvious way to try to unify the 3 facts together-- roughly speaking by seeing an abelian extension e of a number field as being directly "built out of" the ideals in the subgroup s of the ideal class group, where e and s correspond to each other in the "covariant galois correspondence".

(by "covariant galois correspondence" here i mean that instead of the usual contravariant galois correspondence between subfields and subgroups, we can compose it with another contravariant correspondence (between subgroups of the finite abelian galois group and subgroups of its fourier dual which is another finite abelian group unnaturally isomorphic to the original) to get a covariant correspondence instead.)

but this idea fails! or at least, the most naive versions of the idea have all failed for me, so far; but i still suspect there's something important to be learned here .... and it has a lot to do with torsors and line objects and 2-homomorphisms from stacky 2-rigs into unstacky ones and so forth ....

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