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John Baez <johnb@ucr.edu>

pullback bundles

4 messages

JAMES DOLAN <james.dolan1@students.mq.edu.au> Fri, Mar 24, 2023 at 2:34 AM
To: john.baez@ucr.edu

a minor follow-up comment to thursday's discussion:

roughly speaking our main focus recently has been on z/3-torsors over the rational field, but in the discussion on thursday in studying such a torsor t we used as an auxiliary device a z/6-torsor t' obtained by "algebraically joining t with the eisenstein field" aka "algebraic pushforward of t from the rational field to the eisenstein field".

(hmm, i realize now that i'm blurring together the resulting z/3-torsor that lives over the eisenstein field with the resulting z/6-torsor that lives over the rational field but let's try not to let this bother us too much since they're probably more or less just different ways of talking about the same thing ....)

this is a good situation in which to apply the philosophy of "contravariant geometric interpretation of algebra"; thus that algebraic pushforward corresponds to a geometric pullback, aka "pulling a bundle back to a cover or covering", thus exhibiting the bundle as locally trivial (or at least somewhat more tractable) wrt that covering, which may enable useful applications of some kind of gauge-fixing.

at this point i have the feeling that i should be muttering something about "descent" except that i still get a headache from trying to understand what that word really means in contexts like this ....

anyway, one rough theme going on here seems to be that the way the eisenstein field acts as a "splitting field" for the group z/3 seems to be telling us to algebraically push forward / geometrically pull back z/3-torsors to live over the eisenstein field where we can describe them in a more conceptual way, and that from the tannakian viewpoint this is telling us to syntactically analyze "the theory of a z/3-torsor" in ways involving stuff that lives over the eisenstein field (such as "eisenstein line objects").

(for the moment i'm mostly going to try to suppress the urge to wonder how any of this might relate to "the splitting principle" whatever that should mean ....)

(maybe there's a philosophy that says that "when it comes to z/3-torsors you might as well work over the eisenstein field because even if you didn't start there that's where you're going to end up", but i still have the feeling that it's worthwhile to develop a subtler approach where the eisenstein aspect is dealt with "formally" over a non-eisenstein base .... i'm not articulating this idea very clearly yet but it feels like i'm advocating some sort of "gentzen-style" ("natural deduction") approach over some sort of "hilbert-style" approach, in the context of the "total 2-rig" belief-doctrine ....)

anyway, i hope you're getting some idea of where i'm going with all this but on the other hand i'm also hoping that you're wondering with me about where it's all really going: what are we doing here? are we giving a truly geometric underpinning of algebraic number theory or are we instead developing a more refined conceptual apparatus that has the conceptual apparatus of "geometry" as a mere coarsening? or what??

along these lines, i find myself wondering for example about the extent to which we can apply stuff such as the idea that "z/2-torsors are classified by square-free quantities" to the case of over a continuous or smooth manifold ....

but also, in the other direction, it's good that there's such a wealth of concrete examples in the form of number fields, hard data against which to test abstract conceptual ideas. again, i hope that you can see how the working out of these concrete examples connects to the tannakian viewpoint, and how the tannakian viewpoint may help in conceptual unification of a lot of disparate classical ideas about reciprocity theorems and so forth; and that you can have almost as much fun as i have in working out the concrete examples, oblivious to the fact that a lot of other people have presumably already worked out most of this stuff.

John Baez <john.baez@ucr.edu> Fri, Mar 24, 2023 at 10:23 AM
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>

Hi -

> at this point i have the feeling that i should be muttering something about "descent" except that i still get a headache from trying to understand what that word really means in contexts like this ....

Me too, but maybe sometime I should explain in a modern way how Albert-Brauer-Hasse-Noether classified central simple algebras over a field k that 'split' (become isomorphic to a matrix algebra) when you tensor then with some field extension. This nowadays goes by the name of 'Galois descent'. The result is this: there's a bijection between

isomorphism classes of central simple algebras over k that become isomorphic to n×n matrix algebras when tensored with K,

and

elements of H^1(G,PGL_n(K)), where G, the Galois group of K over k, acts on the projective general linear group PGL_n(K) in the obvious way.

But the fun part is understanding this using groupoid theory, or geometrically as 'pulling back' a nontrivial bundle of n x n matrix algebras over some space to a covering space on which it becomes trivializable.

The groupoid theory fact is this:

Theorem. Suppose G is a group acting strictly on some skeletal groupoid, let a be an object of that groupoid, and let A=Aut(a) be its automorphism group. Then the set of isomorphism classes of homotopy fixed points of the action of G having a as their underlying object is in bijection with H^1(G,A).

Anyway, I'm not explaining it now, just writing some stuff to help me remember it, so I can explain it sometime.

> (for the moment i'm mostly going to try to suppress the urge to wonder how any of this might relate to "the splitting principle" whatever that should mean ....)

The splitting principle idea is indeed similar: trying to pull back a nontrivial vector bundle up to some space where it splits as a sum of (not necessarily trivial!) line bundles.

> anyway, i hope you're getting some idea of where i'm going with all this but on the other hand i'm also hoping that you're wondering with me about where it's all really going: what are we doing here? are we giving a truly geometric underpinning of algebraic number theory or are we instead developing a more refined conceptual apparatus that has the conceptual apparatus of "geometry" as a mere coarsening? or what??

It seems the 'etale topology' is some sort of refined version of geometry that people try to use around here - not ordinary geometry/topology, but something that's supposed to be similar. But you're doing similar things in a way that avoids the more traditional approach to sheaves and stacks.

> again, i hope that you can see how the working out of these concrete examples connects to the tannakian viewpoint, and how the tannakian viewpoint may help in conceptual unification of a lot of disparate classical ideas about reciprocity theorems and so forth; and that you can have almost as much fun as i have in working out the concrete examples, oblivious to the fact that a lot of other people have presumably already worked out most of this stuff.

It's definitely fun for me, including the concrete examples, since I have lots of examples up my sleeve when it comes to geometry and topology and some kinds of algebra, but not much when it comes to Galois theory or the splitting of primes.

Best,
jb

JAMES DOLAN <james.dolan1@students.mq.edu.au>
To: baez@math.ucr.edu

Fri, Mar 24, 2023 at 11:45 AM

i'm getting some really interesting results from my mathematica programs now. i've been saying that those z/6-torsors over the rationals extending the eisenstein z/2-torsor must correspond to cube roots of some specific eisenstein integers, and i knew that if that _was_n't true then my understanding of all this stuff had to be pretty badly screwed up; but i still didn't really believe it for sure (except in that one "trivial" case of the cube root of a primitive cube root of unity itself). but i now have a mathematica program that using probabilistic tricks seems to actually explicitly find those eisenstein integers. the next step in my plan is to stare at those specific eisenstein integers to see if i can find a pattern to them, to explain why the splitting fields of their cube roots are abelian ....

again this is all probably equivalent to standard stuff, but it seems like a good validation of (this version of) the tannakian viewpoint ....

(so far i'm getting a vague sense of what you're saying about "galois descent" and it seems like it should fit with what we've been discussing. maybe i'm hoping that they're just one cohomology degree apart or something ....)

....
[Quoted text hidden]

John Baez <john.baez@ucr.edu>
Reply-To: baez@math.ucr.edu
To: JAMES DOLAN <james.dolan1@students.mq.edu.au>
Cc: baez@math.ucr.edu

Fri, Mar 24, 2023 at 12:10 PM

Hi -

I'm glad things are working, though again doing things with Mathematica is more helpful to you (since you're doing it) than me (since I'm not).

> (so far i'm getting a vague sense of what you're saying about "galois descent" and it seems like it should fit with what we've been discussing. maybe i'm hoping that they're just one cohomology degree apart or something ....)

We both know how H^2(G,A) describes some ways of glomming A onto G to get a new group.

We've had a lot of fun thinking about how H^3(G,A) describes ways of glomming A on top of G to get a 2-group with G as the pi_0 and A as the pi_1. And so on for H^4(G,A), H^5(G,A), ...

This is usually called Postnikov tower philosophy of group cohomology, but I call it the "layer-cake philosophy" of group cohomology, where we use a cocycle to build a cake with G on the bottom and A on top.

The case H^2 is a bit degenerate since the top is at the same height as the bottom, so we're smushing the top of the cake right into the bottom.

This story is even more degenerate for H^1(G,A) and H^0(G,A), since then the top of the layer cake is below the bottom! And the H^1 case, at least, seems important for "Galois descent". So I like to think about these cases using a different story:

When G is acting on A, H^0(G,A) is the group of fixed points of this action.

H^1(G,A) is the group of homotopy fixed points of G acting on BA, i.e. the groupoid with one object and A as morphisms.

So:

An element of H^0(G,A) is an element of A with the property of being fixed by G.

An element of H^1(G,A) is an object of BA with the *structure* of being fixed (up to coherent isomorphism) by G. There's just one object of BA, so all the fun comes from the structure of being fixed!

The last paragraph underlies "Galois descent". Anyway, I don't know if this was helpful to you, but it was helpful to me because I now see how to extend this other story about cohomology up to higher H^n, and why it then becomes the same as first story.

Best,
jb