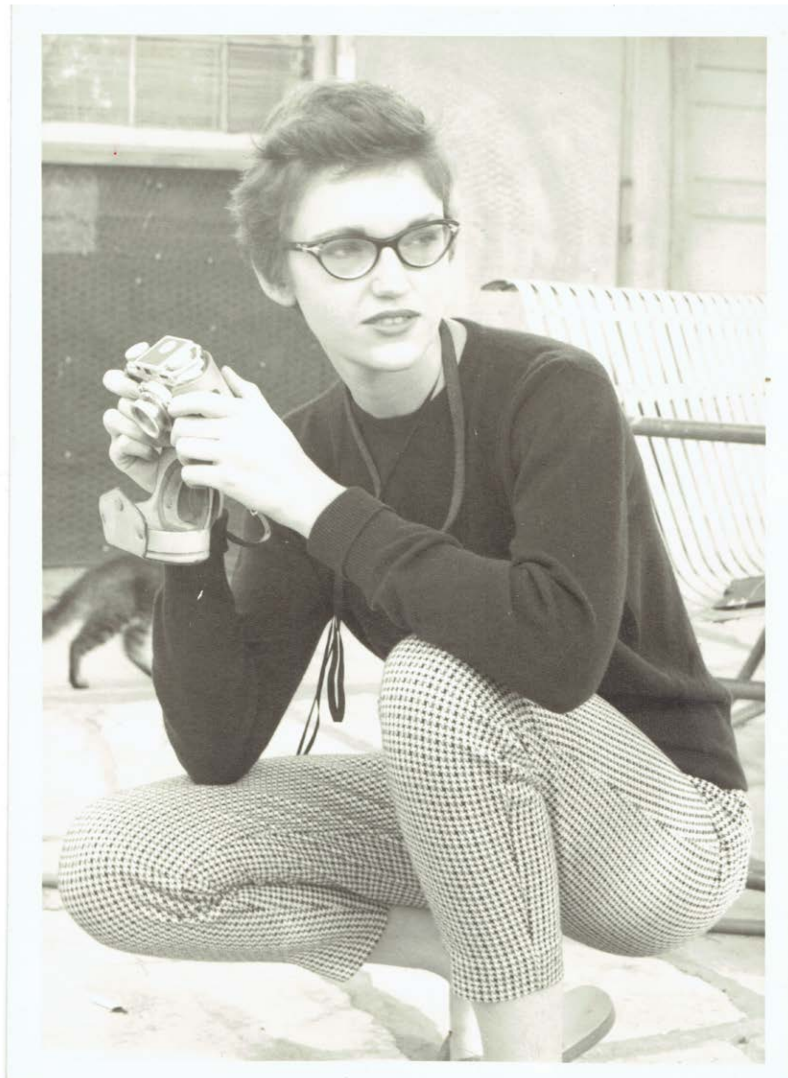
[**For my December 2017 diary, go here.**]

Diary — January 2018

John Baez

January 1, 2018

Phyllis Baez
November 14, 1930 - March 15, 2017



Last year was a bad year for me. Not just because the US has fallen into the hands of a buffoonish, lying, incompetent, self-absorbed would-be dictator. Not just because our puppet Congress voted to hand 1.5 trillion dollars to their wealthy masters. Not just because the world is teetering on the brink of a nuclear war with North Korea. Not just because the Tories have trapped themselves into ripping the UK out of the EU. Not just because climate change is breeding disasters — Harvey flooding Texas, Irma destroying Puerto Rico, half that island still without electric power, California burning, etc. — while the incompetent buffoon tries to prop up the dying coal industry and his henchmen erase the words "climate change" from government websites. All this is our common woe.

Last year my mother died on March 15th. In the photo above, taken before I was born, she looks rather feline and

quite different, but she was always her own person — unconventional, indomitable.

The daughter of farmers in Grand Rapids, Michigan, she went to college and studied art. Then she taught it at a local high school — just long enough to save up money for a trip around the world. A friend of hers who planned to travel with her got pregnant and bailed out at the last minute. So she went alone, taking a steamer to Japan. She fell in love with their esthetics, and it stuck with her the rest of her life. Then she went to Angkor Wat, India, and so on. She had pizza for the first time in Italy: this was another age.

When she returned, Michigan was not exciting enough for her, so she moved to San Francisco. She taught art and hung out with Zen monks. My dad met her at party there, on the fringe of the beatnik scene of the late 1950s. He brought over some jazz records, conveniently forgot them, came back the next day to pick them up... and in a year they were married.

When I was in high school, she had us buy 5 acres in the woods and build a house there in the style of Frank Lloyd Wright. Later she decorated it with art, ceramics and textiles. Later she spent her time making jewelry. She was always focused on visual esthetics, with an intense perfectionism.

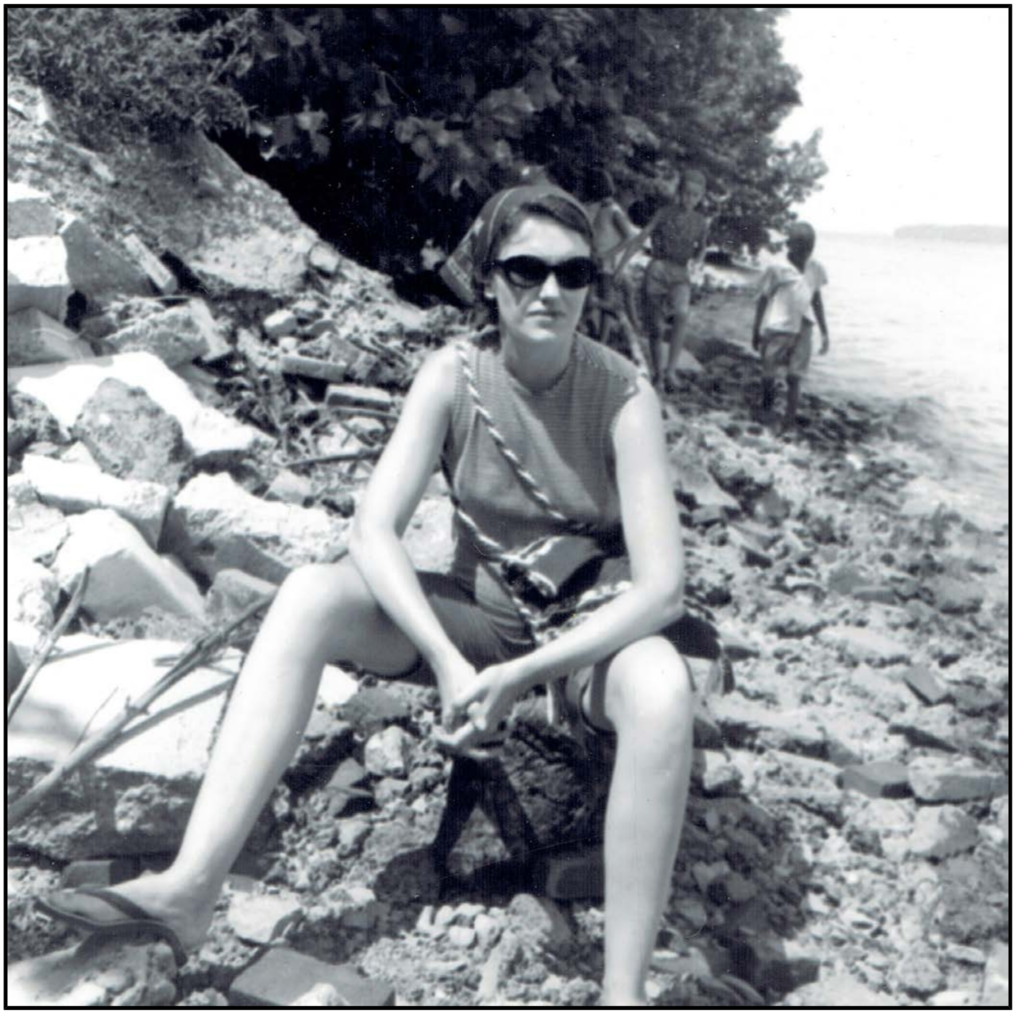
My knowledge of math and physics goes back to my dad and uncle, but my knowledge of music, art and religion is mainly due to her. Sure, my dad liked jazz and taught me how to play the recorder — but she liked Miles Davis, the Modern Jazz Quartet, Bartok and Ravi Shankar. I read my dad's books on logical positivism — but also her books on Zen and Taoism.

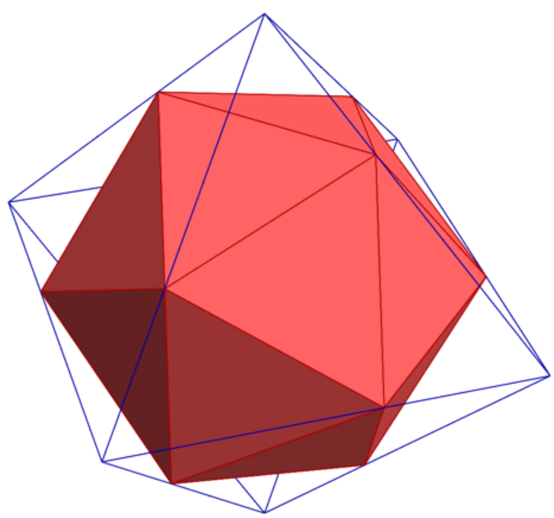
More importantly, she taught me to always seek out what was really interesting. It's a waste of time to pay attention to mediocre crap just because other people do. This has always seemed obvious to me. But I'm slowly realizing why. I was very lucky to have parents who lived that way.

I'm not very good at dealing with negative emotions: I tend to suppress them, let them fester. I heard the news of my mother's death while I was meeting my grad students, teaching them category theory — a phone call from my sister. I froze inside and soldiered on. As I spent this year dealing with my mother's estate, I found myself becoming not sad but angry. Every day, waking up to see the evil clown president dragging our country down, too busy reading contracts and talking to lawyers to think about math.... a world of pointless bullshit. I need to apologize to my wife and everyone else who suffered through my bitter moods.

Things are getting a bit better now — if not in the world at large, at least in my tiny corner. The legal work on my mom's estate is winding down. Her house is sold. I'm getting back the joy in life that powers me forward.

My mother died at 87, probably very happy that she didn't have to move out of her house into an old age home, as she'd been planning, very reluctantly, to do. She lived a great life, and she always said that dying would be okay. No condolences required.





New Year's Resolution: ***do all my procrastination right now — don't keep putting it off.***

I figure if I completely waste the first day of the year, I'll be ready to work ceaselessly starting tomorrow.

So, after lying in bed browsing the web for a few hours and listening to the radio, I got up and had pancakes for breakfast. Then I read old magazines that had been accumulating by my bed. Then I spent a few hours taking photos and putting them in my online diary.

The one above, created by Greg Egan, shows a shape morphing between a cuboctahedron and a regular icosahedron. When we bisect the edges of a regular tetrahedron, we get 12 points that form the vertices of a cuboctahedron. But when we divide these edges in the proportion of the golden ratio, we get 12 points that form the corners of a regular icosahedron!

It's fun to watch this, so I spent half an hour doing that. Note that as the icosahedron becomes a cuboctahedron, some pairs of the 20 triangular faces flatten out to form squares.

I've almost run out of ways to procrastinate — I guess I'm not very good at this. Pretty soon I'll have to start working! I have a bunch of papers to finish off.

Hey, I know — I can waste some more time by listing them! I've heard that listing your goals is a good way to postpone actually achieving them.

- *Network models*, with John Foley, Joseph Moeller and Blake Pollard. This is about using categories and operads to design systems of mobile, interacting agents. John works for Metron Scientific Solutions, and they're helping the Coast Guard create a tool to design search and rescue missions. But I'm over at the theoretical end of this project. Our paper is done, and we just need to take a few last looks at it, make sure it's okay, and submit it for publication.

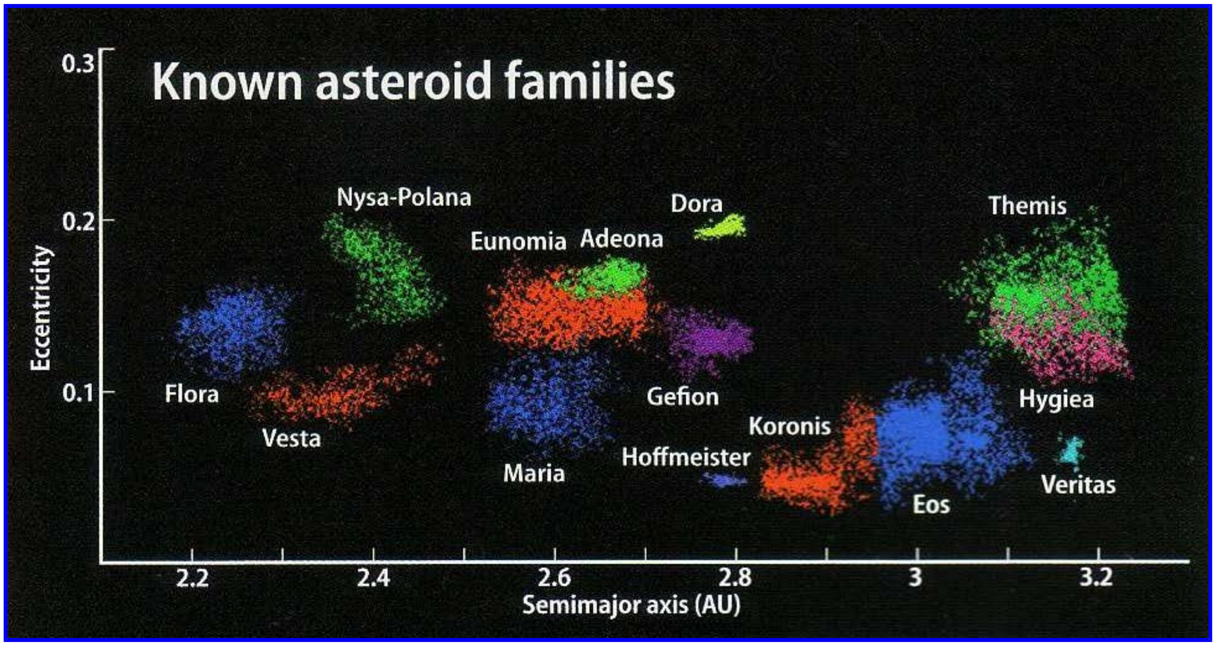
- *Struggles with the continuum*. This is about ways in which assuming space is a continuum causes trouble in our favorite theories of physics — infinities and stuff like that. I need to edit the introduction, because it talks a lot about the foundations of mathematics, while the paper itself is all about physics. It somehow fools the reader into thinking the paper is going in a different direction than it actually does. I've been putting this off, because I'm not quite sure how to handle it.

- *Coarse-graining open Markov processes* with Kenny Courser. This studies a double category where the horizontal arrows are open Markov processes and the squares are 'coarse-grainings': ways of simplifying an open Markov process. The paper is almost done, we just need to insert some extra figures and make sure the notation is optimal.

- *Magnitude homology versus persistent homology* with Nina Otter. Persistent homology is really fashionable in 'topological data analysis', where you do things like try to count the number of holes in a cloud of data. Magnitude homology, on the other hand, was created by category theorists because of its beautiful abstract properties. They should be related and they are. We explain how they are. But we need to finish writing this paper!

So those are the main things I'm not doing yet. Starting tomorrow I'll get to work.

January 4, 2018



Many asteroids belong to families. These formed when bigger asteroids collided and broke into pieces. Asteroids in a family have similar orbits. But finding these families takes work!

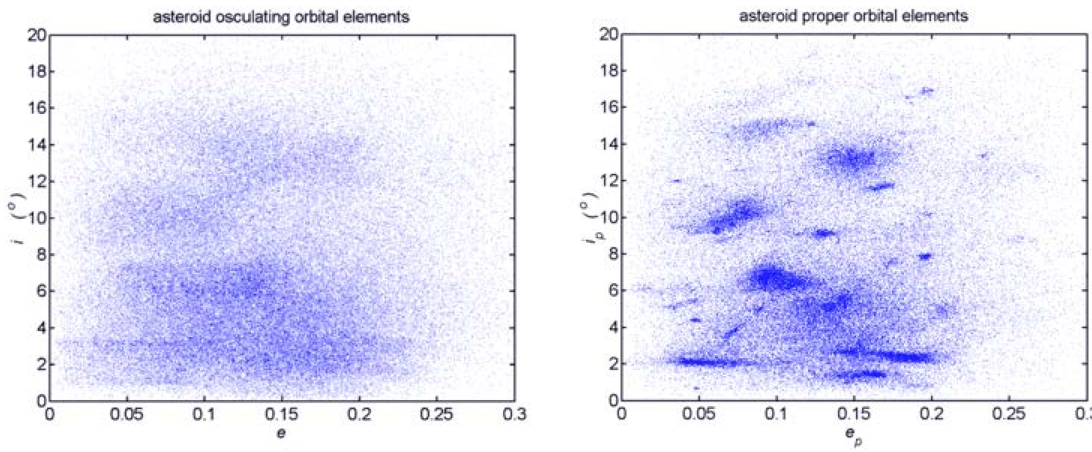
This picture makes it look easy. But it's hard. In reality, the different families don't come in different colors. There are lots of asteroids that don't belong to families, not shown on this chart. And there's another even more important reason!

We can describe an orbit using some numbers. An orbit is an ellipse. The **semimajor axis** is half the length of this ellipse. The **eccentricity** says how stretched-out the ellipse is: it's 0 for a circle. The **inclination** is an angle that describes the orbit's tilt. Most planets have inclination almost zero, because they move close to a plane called the **ecliptic**.

So, you might try to find asteroid families by taking lots of asteroids and making a chart of their semimajor axis, eccentricity and inclination. Would that work? This picture makes it seem so!

But it wouldn't work. You'd get random junk — no obvious families.

To find the families, you have to *correct for the fact that orbits keep changing!* Look at this picture:



At left you see a chart of the eccentricity and inclination of lots of asteroids. Random junk! At right you see a corrected chart. Now you see asteroid families!

How does this 'correction' business work? It was invented by the Japanese astronomer Kiyotsugu Hirayama in 1918. He noticed that asteroid orbits change in a roughly periodic way over thousands of years due to the gravitational pull of the planets... but one can create ideal unchanging orbits by correcting for this fact.

Hirayama didn't have a computer, so he had to do these computations by hand, approximately! He succeeded in finding several families of asteroids this way: the Koronis, Eos, and Themis families, and later the Flora and Maria families.

By now we can do much better, and find many more families... and more asteroids in each family. For example, the **Eos family** was formed about 1.1 billion years ago between Jupiter and Mars. Hirayama found 19 asteroids in this family. The biggest, which gives the family its name, is **Eos**, named after the Greek goddess of the dawn. It's almost 100 kilometers across! But now we know almost 300 members of this family.

Here's what Wikipedia says about this business of correcting orbits:

> The **proper orbital elements** of an orbit are constants of motion of an object in space that remain practically unchanged over an astronomically long timescale. The term is usually used to describe the three quantities:_
>
> - proper semimajor axis (a_p),
> - proper eccentricity (e_p), and
> - proper inclination (i_p).
>
> The proper elements can be contrasted with the **osculating Keplerian orbital elements** observed at a particular time or epoch, such as the semi-major axis, eccentricity, and inclination. Those osculating elements change in a quasi-periodic and (in principle) predictable manner due to such effects as perturbations from planets or other bodies, and precession (e.g. perihelion precession). In the Solar System, such changes usually occur on timescales of thousands of years, while proper elements are meant to be practically constant over at least tens of millions of years.

I suspect the math here is quite interesting. Unfortunately Wikipedia doesn't go into details.

For a table of asteroid families, go here:

- Wikipedia, Asteroid family.

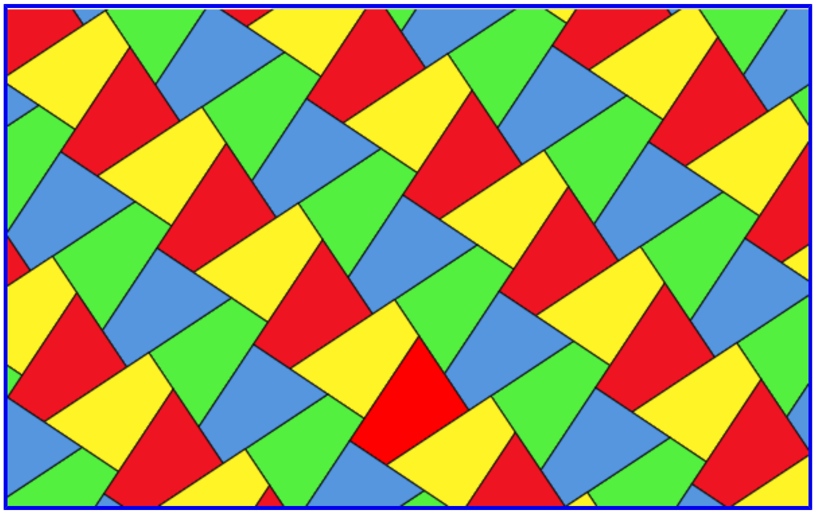
The picture at the top of this entry is from here:

- Joel Duff, Rewinding the clock: an asteroid family history, *Naturalis Historia*, May 14, 2014.

In the discussion on my G+ post, Greg Roelofs took a look at this picture and wondered what distinguished the Eunomia family from the Adeona family: they overlap in this chart. It turns out the Eunomia family consists of bright S-type asteroids, which are 'stony': mainly made of iron and magnesium silicates. The Adeona family consists of dark C-type asteroids, which are 'carbonaceous'.

He also asked about the overlapping Hygeia family and Themis family. It turns out these can be distinguished by their inclination, but not so much by their semimajor axis and eccentricity.

January 11, 2018



Al Grant has a great interactive page of tilings that move on 'hinges'. Check it out:

- Al Grant, Hinged tesselations (part 1), February 1, 2014.

January 12, 2018

Flying foxes and the tao of bats

This is a flying fox. What could be cooler than *flying foxes?*

How about this: *megabats.*

'Megabats' sound like science-fictional monsters with 12-foot wing spans, swooping down in ruthless packs. But no, it's a standard scientific term. Bats come in two main kinds: the insect-eating **microbats** and the fruit-eating **megabats**, which include flying foxes.

At least that was the story until recently.

The fruit-eaters are generally larger than the insectivores, so the order of bats, **Chiroptera**, was subdivided into the **Macrochiroptera** or megabats and **Microchiroptera** or microbats. But this subdivision was recently found to be somewhat incorrect, so now the two suborders or bats are called — get ready for this! — **Yinpterochiroptera** and **Yangopterochiroptera**.

That's right: yin and yang, the Taoist terms for cold and hot, or dark and light, or negative and positive!

Puzzle. Who started using these terms for bats, and why?

In China, bats are considered lucky. At several Taoist monasteries I've visited, I've seen carvings of bats on the walls. But that could be a red herring.

Anyway, I'll use the old names for now. Here's how to tell the difference between megabats and microbats:

1. In general megabats are larger and eat fruit, while microbats are smaller and eat insects. However, there are exceptions. For example, vampire bats, which drink blood, are descended from fruit-eating bats that developed sharp teeth that could puncture thick-skinned fruit... but these particular fruit-eating bats are microbats!

2. Microbats use echolocation, whereas megabats generally don't. It seems they have *lost* this trait because they're looking for plants rather than insects, and they use more energy flying than the microbats. Again there's an exception: the Egyptian fruit bat is a macrobat that uses echolocation — but it doesn't use the same method: scientists think that instead of using its larynx, it clicks using its nasal passages and the back of its tongue.

3. Microbats lack a claw on the second finger of their forelimb.

4. Megabats never have tails, but some species of microbats do.

5. Microbats have ears with a special organ that seems to be crucial in echolocation, and their ears are larger than megabats'.

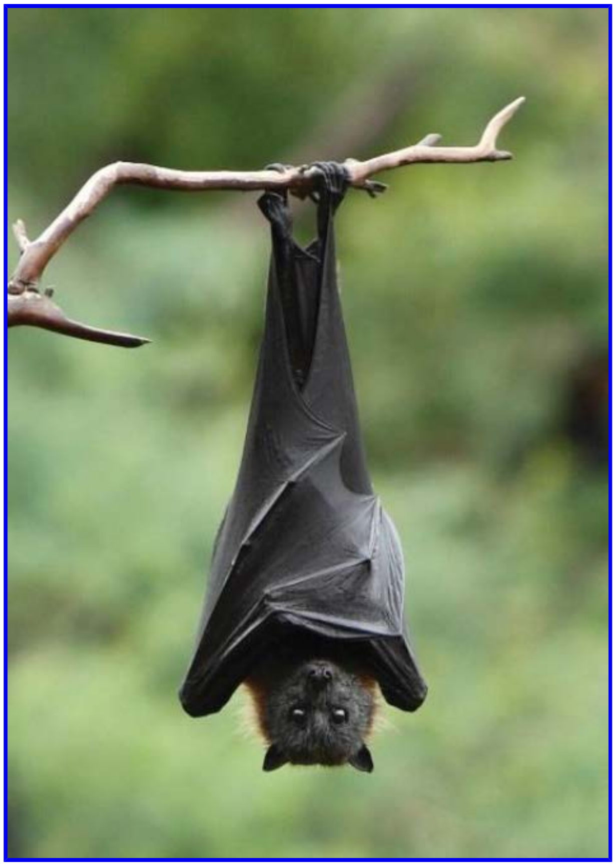
6. Megabats have larger eyes.

This is based on various Wikipedia articles, especially the articles on megabats and microbats.

The picture was taken by Michael Cleary, who wrote:

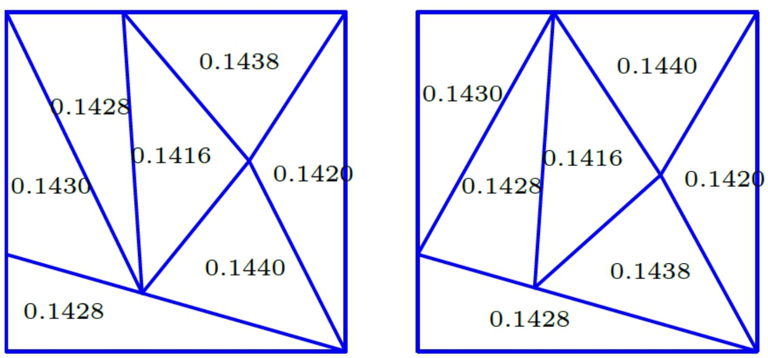
> A Grey Headed Flying Fox having dipped coming straight at me, you can see the reflection in its eyes. Hope people don't mind me putting a few more flying Fox photos on, we are getting to the hot part of summer where they drink during the day.

For more, read the conversation on my G+ post, where Irina T. gave a nice answer to the puzzle.



January 13, 2018

Chopping a square into 7 triangles with almost the same area



It's impossible to divide a square into an odd number of triangles with the same area. This amazing result was proved by Paul Monsky in 1970, but I only heard about it today, from David Eppstein.

So it's impossible. But you can still try your best! These are the two best tries with 7 triangles and at most 8 vertices. These were discovered last year by Jean-Philippe Labbé, Güter Rote, and Günter Ziegler. They used a lot of clever math and programming to do this.

What do they mean by "best"? They mean that the standard deviation of the area of the triangles is as small as possible. For these two it's 0.0008051.

They couldn't check all the ways that used more than 8 vertices — it took too much time!

This was just the start of what Labbé, Rote and Ziegler did. They used a lot of clever math and programming to do this. David Eppstein summarized the rest:

> According to Monsky's theorem, it is impossible to divide a square into an odd number of triangles, all the same area as each other. But if it can't be done exactly, how close to equal-area can you get? The answer is, at least superpolynomially close. The construction uses the Thue–Morse sequence to balance out the differences in area.

'Superpolynomially close' means that for any number p, with n triangles you can get the difference in areas to be less than $1/n^p$ if n is big enough.

Puzzle. Show that if n is even, you can divide a square into n triangles all having the same area.

This puzzle should be quite easy for most of you out there. How many seconds did it take you?

Here's the actual paper:

- Jean-Philippe Labbé, Güter Rote, and Günter Ziegler, Area difference bounds for dissections of a square into an odd number of triangles.

It's full of cool techniques... a great example of how even 'useless' and 'unimportant' problems in math can inspire interesting thoughts.

Here's the sketch of Monsky's proof, from Wikipedia:

1. Take the square to be the unit square with vertices at (0,0), (0,1), (1,0) and (1,1). If there is a dissection into n triangles of equal area then the area of each triangle is 1/n.
2. Color each point in the square with one of three colors, depending on the 2-adic valuation of its coordinates.
3. Show that a straight line can contain points of only two colors.
4. Use Sperner's Lemma to show that every triangulation of the square into triangles meeting edge-to-edge must contain at least one triangle whose vertices have three different colors.
5. Conclude from the straight-line property that a tricolored triangle must also exist in every dissection of the square into triangles, not necessarily meeting edge-to-edge.
6. Use Cartesian geometry to show that the 2-adic valuation of the area of a triangle whose vertices have three different colors is greater than 1. So every dissection of the square into triangles must contain at least one triangle whose area has a 2-adic valuation greater than 1.
7. If n is odd then the 2-adic valuation of 1/n is 1, so it is impossible to dissect the square into triangles all of which have area 1/n.

Thinking about this this will force you to understand the p-adic valuation and and Sperner's Lemma, both of which are more important and interesting than this particular theorem! The Thue-Morse sequence is also lots of fun.

For more, including some answers to the puzzle, read the discussion on my G+ post.

For my March 2018 diary, go here.

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For my January 2018 diary, go here.

Diary — March 2018

John Baez

March 15, 2018

When comets meet white dwarfs

A white dwarf is a star that's run out of fuel and is slowly cooling down... but still very hot thanks to the energy it got from gravity crushing it down. White dwarfs should have atmospheres that are almost pure hydrogen and helium, since heavier elements quickly sink down further. But about a quarter have noticeable amounts of heavier elements in their atmosphere. How did those get there?

Alexander Stephan and other scientists at UCLA argue that they come from comets! More precisely, large icy objects like those in the Kuiper belt of our Solar System, beyond the orbit of Pluto.

But it takes work to explain how so many of these objects hit white dwarfs.

The theory is that these white dwarfs are in binary star systems. When the star that becomes the white dwarf begins to die it emits a lot of gas and loses mass - we know that's how it works. So, the Kuiper belt objects orbiting it start to move further out. There are lots of these things. So, some will interact gravitationally with the other star in binary system and get thrown this way and that... and eventually some will hit the white dwarf!

The scientists did detailed computer simulations to check that this could account for what we see. Even more exciting: sometimes Neptune-like planets will hit the white dwarf! And indeed we see some white dwarfs that have a lot more heavy elements in their atmosphere.

> By running large Monte Carlo simulations, Stephan and collaborators demonstrate that this scenario can successfully produce accretion of both Neptune-like planets and Kuiper-belt-analog objects. Their simulation results indicate that ~1% of all white dwarfs should accrete Neptune-like planets, and ~7.5% of all white dwarfs should accrete Kuiper-belt-analog objects.

> While these fractions are broadly consistent with observations, it's hard to say with certainty whether this model is correct, as observations are scant. Only ~200 polluted white dwarfs have been observed, and of these, only ~15 have had detailed abundance measurements made. Next steps for understanding white-dwarf pollution certainly must include gathering more observations of polluted white dwarfs and establishing the statistics of what is polluting them.

Also, 7.5% is a lot less than 25%.

I got this from here:

- Susanna Kohler, Throwing icebergs at white dwarfs, *AAS Nova*, July 28, 2017.

For my July 2018 diary, go here.

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[For my January 2018 diary, go here.](#)

Diary — April 2018

John Baez

April 11, 2018

Two weeks ago I started teaching an online course based on this free book:

- Brendan Fong and David Spivak, *Seven Sketches in Compositionality: An Invitation to Applied Category Theory*.

It's eating up the time I used to spend on Google+ or this diary. But I'm happy with that, because over 250 people have registered, and a bunch of them are very energetic. It's exciting!

Four days I week I write short 'lectures' on the book. If you take the course you can read those lectures, read the book, try the exercises in the book and the puzzles I create, and discuss everything with me and the other students! The best part of the course, in my opinion, is the conversations. People are starting to dream up projects to work on together.

If this sounds interesting, go here and register in the box at upper left.

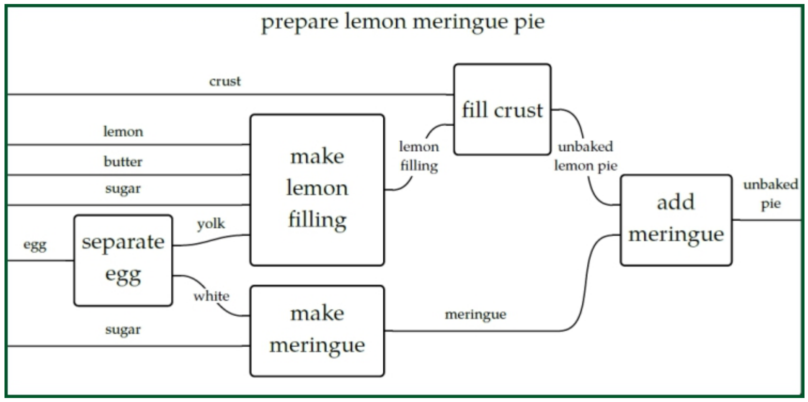
Use your full real name as your username, with no spaces. I will get back to you, so use a working email address. You can move through the course at your own pace: all the discussions can go on indefinitely.

Brendan Fong was my grad student; now he's doing a postdoc at MIT with David Spivak. They're at the cutting edge of applied category theory, and I'm having a lot of fun working through their book by teaching it. Here is the preface to their book, just so you can get an idea of what it's like:

> **Preface**
>
> Category theory is becoming a central hub for all of pure mathematics. It is unmatched in its ability to organize and layer abstractions, to find commonalities between structures of all sorts, and to facilitate communication between different mathematical communities. But it has also been branching out into science, informatics, and industry. We believe that it has the potential to be a major cohesive force in the world, building rigorous bridges between disparate worlds, both theoretical and practical. The motto at MIT is *mens et manus*, Latin for mind and hand. We believe that category theorbyand pure math in generablhas stayed in the realm of mind for too long; it is ripe to be brought to hand.
>
> **Purpose and audience**
>
> The purpose of this book is to offer a self-contained tour of applied category theory. It is an invitation to discover advanced topics in category theory through concrete real-world examples. Rather than try to give a comprehensive treatment of these topics — which include adjoint functors, enriched categories, proarrow equipments, toposes, and much more. We want to give readers some insight into how it feels to work with these structures as well as some ideas about how they might show up in practice.
>
> The audience for this book is quite diverse: anyone who finds the above description intriguing. This could include a motivated high school student who hasn't een calculus yet but has loved reading a weird book on mathematical logic they found at the library. Or a machine learning researcher who wants to understand what vector spaces, design theory, and dynamical systems could possibly have in common. Or a pure mathematician who wants to imagine what sorts of applications their work might have. Or a recently-retired programmer who's always had an eerie feeling that category theory is what they've been looking for to tie it

all together, but who's found the usual books on the subject impenetrable.

For example, we find it something of a travesty that in 2018 there seems to be no introductory material available on monoidal categories. Even beautiful modern introductions to category theory, e.g. by Riehl or Leinster, do not include anything on this rather central topic. The basic idea is certainly not too abstract; modern human intuition seems to include a pre-theoretical understanding of monoidal categories that is just waiting to be formalized. Is there anyone who wouldn't correctly understand the basic idea being communicated in the diagram below?



Many applied category theory topics seem to take monoidal categories as their jumping off point. So one aim of this book is to provide a reference — even if unconventional — for this important topic.

We hope this book inspires both new visions and new questions. We intend it to be self-contained in the sense that it is approachable with minimal prerequisites, but not in the sense that the complete story is told here. On the contrary, we hope that readers use this as an invitation to further reading, to orient themselves in what is becoming a large literature, and to discover new applications for themselves.

This book is, unashamedly, our take on the subject. While the abstract structures we explore are important to any category theorist, the specific topics have simply been chosen to our personal taste. Our examples are ones that we find simple but powerful, concrete but representative, entertaining but in a way that feels important and expansive at the same time. We hope our readers will enjoy themselves and learn a lot in the process.

How to read this book

The basic idea of category theory — which threads through every chapter — is that if one pays careful attention to structures and coherence, the resulting systems will be extremely reliable and interoperable. For example, a category involves several structures: a collection of objects, a collection of morphisms relating objects, and a formula for combining any chain of morphisms into a morphism. But these structures need to cohere or work together in a simple commonsense way: a chain of chains is a chain, so combining a chain of chains should be the same as combining the chain. That's it!

We will see structures and coherence come up in pretty much every definition we give: "here are some things and here are how they fit together." We ask the reader to be on the lookout for structures and coherence as they read the book, and to realize that as we layer abstraction on abstraction, it is the coherence that makes everything function like a well-oiled machine.

Each chapter in this book is motivated by a real-world topic, such as electrical circuits, control theory, cascade failures, information integration, and hybrid systems. These motivations lead us into and through various sorts of category-theoretic concepts.

We generally have one motivating idea and one category-theoretic purpose per chapter, and this forms the title of the chapter, e.g. Chapter 4 is "Collaborative design: profunctors, categorification, and monoidal categories." In many math books, the difficulty is roughly a monotonically-increasing function of the page number. In this book, this occurs in each chapter, but not so much in the book as a whole. The chapters start out fairly easy and progress in difficulty.

The upshot is that if you find the end of a chapter very difficult, hope is certainly not lost: you can start on the next one and make good progress. This format lends itself to giving you a first taste now, but also leaving open the opportunity for you to come back at a later date and get more deeply into it. But by all means, if you have the gumption to work through each chapter to its end, we very much encourage that!

We include many exercises throughout the text. Usually these exercises are fairly straightforward; the only thing they demand is that the reader's mind changes state from passive to active, rereads the previous paragraphs with intent, and puts the pieces together. A reader becomes a student when they work the exercises; until then they are more of a tourist, riding on a bus and listening off and on to the tour guide. Hey, there's nothing wrong with that, but we do encourage you to get off the bus and make contact with the natives as often as you can.

For my August 2018 diary, go here.

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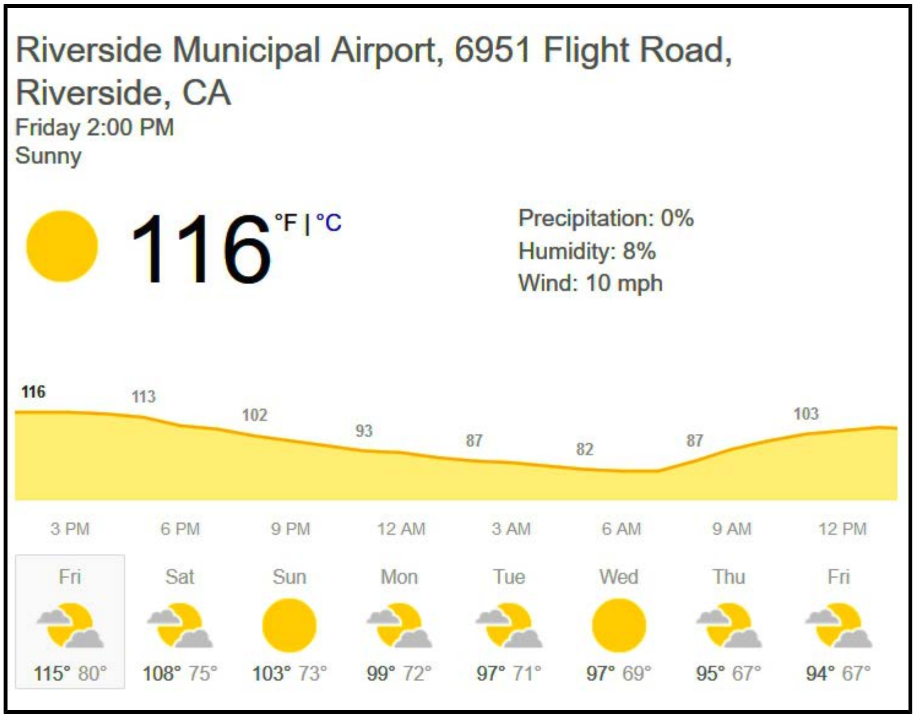
For my March 2018 diary, go here.

Diary — July 2018

John Baez

July 6, 2018

I haven't been keeping up my diary — too busy! But today there's some real new: it's the hottest day ever recorded where I live, in Riverside California!



It's almost 47 Celsius. Tons of flies are buzzing around our front porch. Unusual: I think they're trying to keep cool. And as soon as I started watering a plant, a hummingbird landed on it and started sipping water from a leaf, undeterred by the spray!

Yesterday, Scott Pruitt resigned his position as head of the Environmental Protection Agency. Global warming keeps cranking up the temperature, but this corrupt bastard never gave a damn. The new interim guy is a coal industry lobbyist. Trump comments: "the future of the EPA is very bright!" We gotta get rid of him.

For my August 2018 diary, go here.

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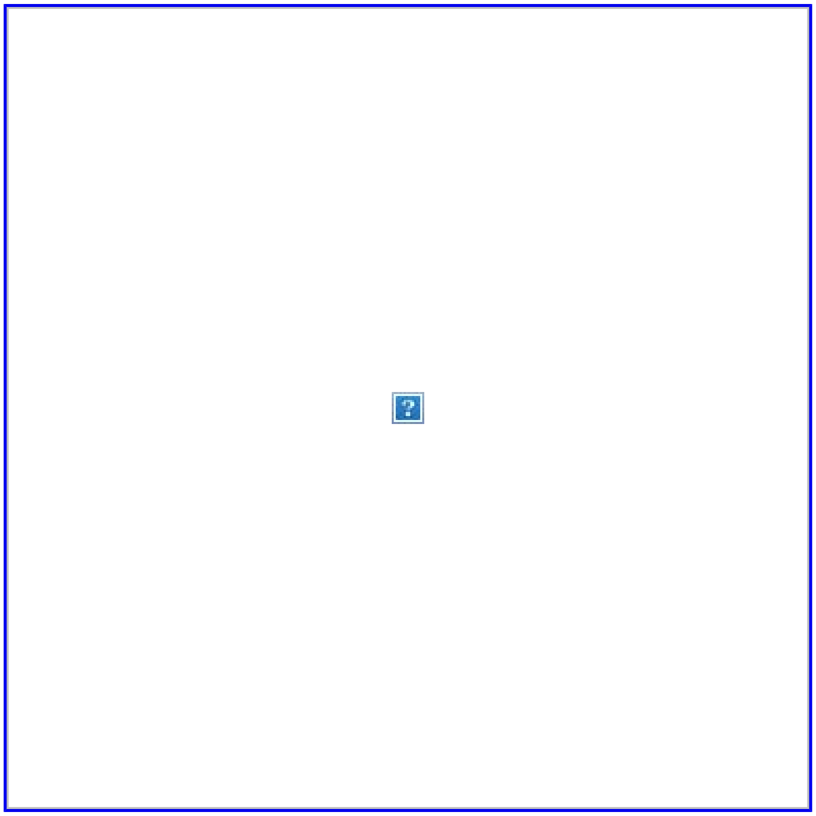
Diary — August 2018

John Baez

August 1, 2018



Caucher Birkar was one of four Fields medalists this year:

- Kevin Hartnett, An innovator who brings order to an infinitude of equations, *Quanta*, August 1, 2018.

Math is such a big subject that someone can win a Fields Medal and I'll read a nice news article about it and think "Wow, who is this — and what's this weird stuff they did?"

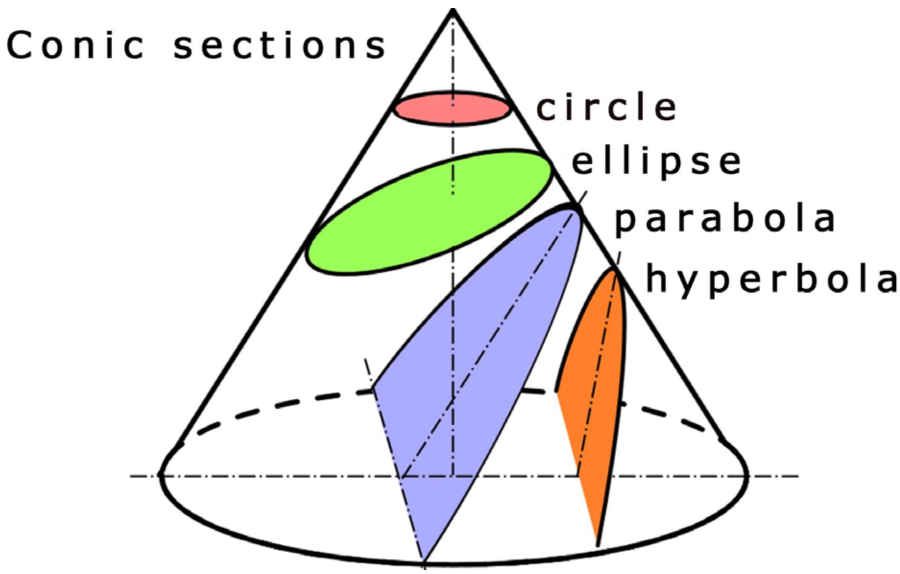
So I read a bit about Birkar's work....

He's studying 'algebraic varieties', which are roughly curves, surfaces and higher-dimensional shapes described by polynomial equations. It's an old subject. In the 1800s people realized it's easier to study solutions with *complex* numbers, not just real numbers.

For example, above you can see the real points in an algebraic variety called Kummer's quartic surface — but there's a lot more going on with the complex solutions, and ultimately they make everything *simpler*, not more 'complex'.

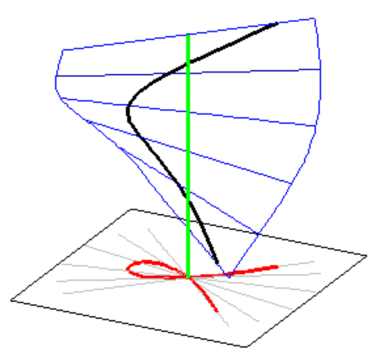
It's also easier to study algebraic varieties if you add 'points at infinity'. For example, then *all* pairs of lines on the plane intersect — perhaps at infinity, like railroad tracks in perspective. This idea is called projective geometry.

Combining these ideas, 'complex projective space' is a nice framework for studying algebraic varieties. In this framework a circle, ellipse, parabola and hyperbola are all just different views of the same curve!



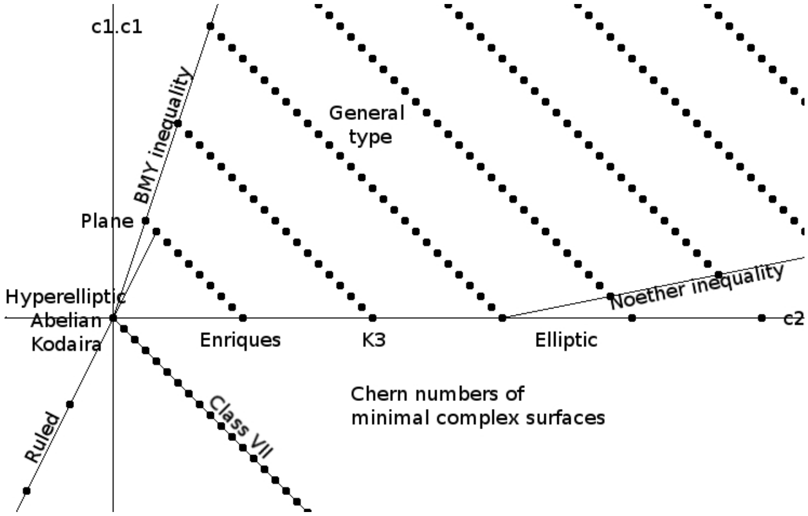
Nice and simple. So we can become more ambitious! We can try to classify *all* algebraic varieties. This sounds insane: classifying all possible curves, surfaces, and higher-dimensional things that you can describe with polynomials? Weird things like the Escudero nonic surface here, drawn here by Abdelaziz Merzouk?

So we have to be clever. First, we have to view algebraic varieties through very blurry glasses, so tons of them count as 'the same'. Not just circles and hyperbolas! We have to use 'birational equivalence', so the red and black curves here count as the same:



Very roughly speaking, two algebraic varieties are birationally equivalent if they're the same except on a very small set; this lets us eliminate singularities like the point where the red curve crossses itself.

Classifying algebraic varieties up to birational equivalence gets harder as their dimension goes up. Curves are *all* birationally equivalent. Surfaces come in 10 families, thanks to the Enriques–Kodaira classification.



But in higher dimensions things get harder. Starting around 1980, the 'minimal model program' sought to find the 'simplest possible' algebraic variety that's birationally equivalent to whichever one you name:

- Braian Lehmann, A snapshot of the minimal model program.

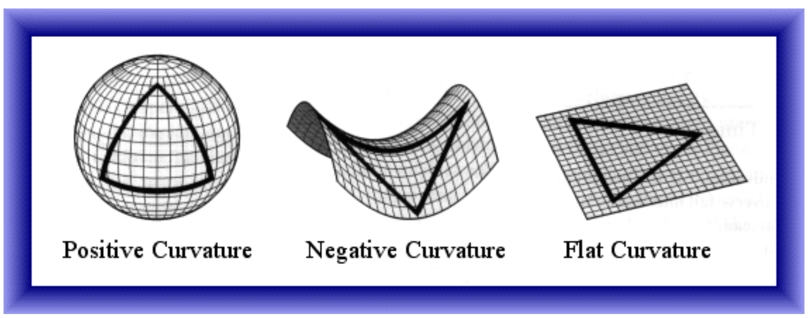
It succeeded in 3 dimensions, thanks largely to the work of Mori:

- J´nos, Koll´r, Minimal models of algebraic threefolds: Mori's program, *Séminaire Bourbaki* **31** (1988-1989), Talk no. 712, 302–326.

Caucher Birkar is pursuing the minimal model program in higher dimensions. The *Quanta* article explains what he's done about as well as possible w/o getting ultra-technical — so read that!

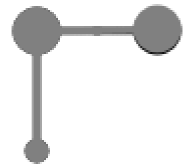
He made progress on 'Fano varieties'. But what are those?

A smooth complex variety is Fano iff it admits a Kähler metric of positive Ricci curvature. Calabi-Yau varieties, famous from string theory, are 'Ricci flat', and most varieties are negatively curved. This rough 3-fold classification shows up all over math!



So very roughly, Caucher Birkar is contributing to a grand program to classify all shapes described by polynomials... with a focus on 'positively curved' ones. But the technical details are way over my head... and he's also done other things.

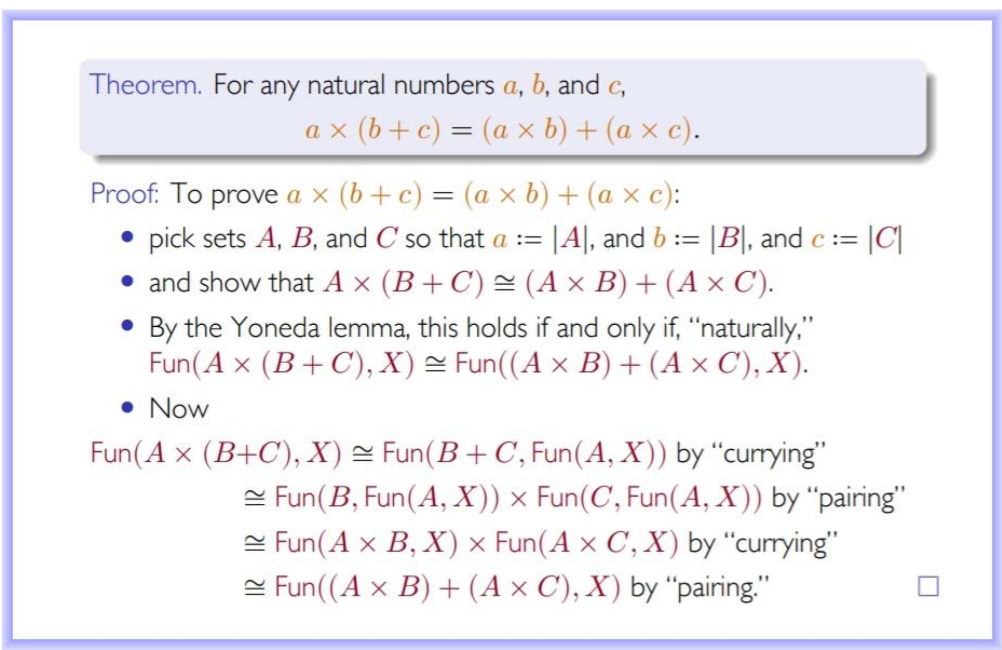
August 4, 2018



Here Jacopo Bertolotti shows us chaos. Starting with very similar positions and velocities, two copies of a frictionless double pendulum drift apart... and soon they do completely different things! The Lyapunov exponent measures how fast this happens.

If you want to learn some cool physics, check out Bertolotti's Physics Factlets on Twitter.

August 5, 2018



The basic laws of arithmetic, like $a \times (b + c) = a \times b + a \times c$, are secretly laws of set theory. But they apply not only to sets, but to many other structures!

Emily Riehl explained this in Eugenia Cheng's recent "Categories for All" session at a meeting of the Mathematical Association of America. Check out her slides:

- Emily Riehl, Categorifying cardinal arithmetic, MAA MathFest, August 4, 2018

To hear her discuss this argument, and math in general, listen to her episode of *My Favorite Theorem*:

- Evelyn Lamb, Emily Riehl's favorite theorem, *Scientific American*, May 24, 2018.

Her proof that $a \times (b + c) = a \times b + a \times c$ involves three facts about the category of sets:

- It has **binary coproducts** (the disjoint union $A + B$ of two sets).
- It has **binary products** (the cartesian product $A \times B$ of two sets).
- It is **Cartesian closed** (there's a set $\mathrm{Fun}(A, B)$ of functions from A to B).

The defining property of the coproduct $A + B$ is that a function from $A + B$ to any set X is 'the same as' a function from A to X together with a function from B to X.

The defining property of the product $A \times B$ is that a function from any set X to $A + B$ is the same as a function from X to A together with a function from X to B.

The defining property of $\mathrm{Fun}(A, B)$ is that a function from any set X to $\mathrm{Fun}(A, B)$ is the same as a function from $A \times X$ to B. This change of viewpoint is called **currying**, and Riehl uses it twice in her proof.

She calls the defining property of $+$ 'pairing', and she uses that twice too. I would call this **copairing**, but she wisely decided that sounds too technical.

Her proof actually never uses the defining property of $\times$, which I would call **pairing**.

Her argument works for any category that has the necessary properties! For example, the category of graphs. But it's nice to see that lurking inside grade-school arithmetic there is an introduction to category theory.

August 6, 2018

Nobody knows if there are infinitely many twin primes: primes that are 2 apart. But Viggo Brun proved the sum of the reciprocals of the twin primes converges. Of course, if there were finitely many twin primes, this sum would be rational. However, in theory, it could still be rational even if there are infinitely many twin primes!

Now someone has conjectured a formula for this sum, which is called Brun's constant, B_2.

BRUN'S CONSTANT:
the sum of reciprocals of the twin primes

$$B_2 = \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots$$
$$\approx 1.902160583104$$

Nobody knows if it's irrational; it was proved finite by Viggo Brun in 1919. Computing Brun's constant turned up a bug in Pentium's floating-point arithmetic in 1994.
The above estimate used all twin primes up to 10^{16}.

The formula involves Catalan's constant G: the alternating sum of reciprocals of odd squares. Nobody knows if this constant is irrational, though I bet it is.

CATALAN'S CONSTANT:
the alternating sum of reciprocals of the odd squares

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} + \cdots$$

$$\approx 0.915965594177219015054603514932384110774\ldots$$

Nobody knows it it's irrational.

The formula also involves the [Ramanujan–Soldner constant]() μ. If you integrate the reciprocal of $\ln(x)$ from 0 to this constant, you get zero.

THE RAMANUJAN–SOLDNER CONSTANT:
the only positive number μ with

$$\int_0^\mu \frac{dx}{\ln x} = 0$$

$$\mu \approx 1.45136923488338105028396848589202744949303228\ldots$$

Marcin Lesniak recently conjectured a formula relating Brun's constant to Catalan's constant and the Ramanujan–Soldner constant.

Marcin Lesniak's conjecture for Brun's constant:

$$B_2 = \frac{(8 + 40(\mu - G))}{(16 - (\mu - G))}$$

$$\approx 1.902160583102973079982261491757436 1\ldots$$

My guess: it's false. The discrepancy in the 12th decimal place is not necessarily a problem, since the [current best estimate of Brun's constant]() uses an extrapolation: it's not necessarily a lower bound. But the formula feels "too good to be true": we don't have any reason to expect such a formula, since twin primes are even more hard to understand than plain old primes. For example, they don't seem to be connected to [Dirichlet series]() the way plain old primes are.

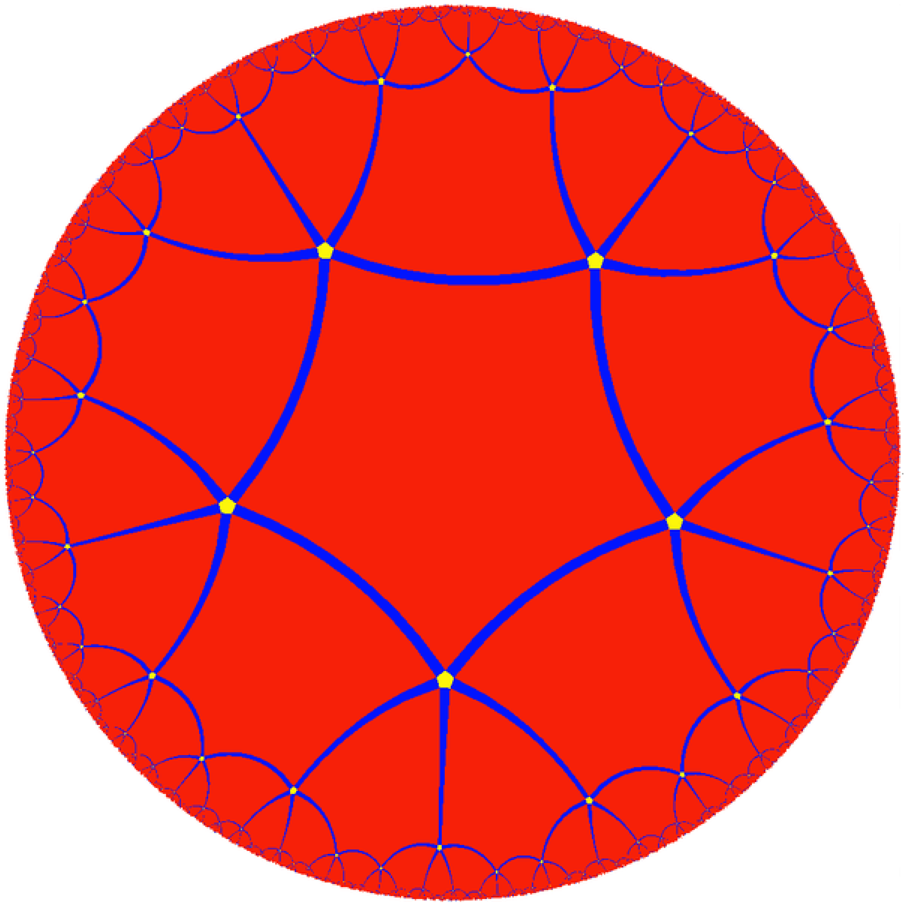
August 11, 2018

Yesterday we arrived in Singapore. We've been visiting here for 9 years now! Everything is instantly familiar, like waking up out of a dream. Someone at the local market asked us if we come here when it's cold where we live. We said no, we come here to escape the heat! (It's been over 100° F almost every day for the last week in Riverside.)

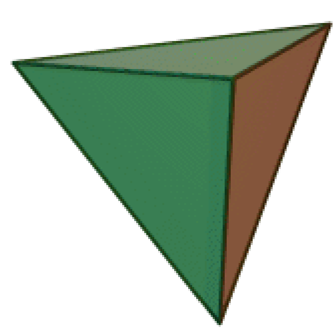
In case you're wondering about the funny new format of some posts, it's that I've moved from Google+ to [Twitter](). I found that people like reading series of tweets where I explain math and physics. So I'm copying some of these to my diary.

August 12, 2018

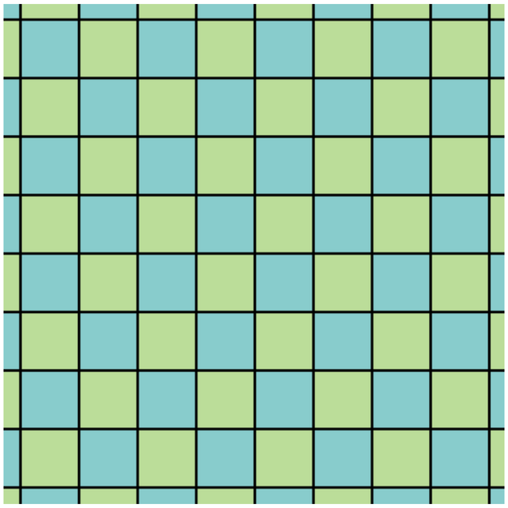
The number 5 is more exotic than all the natural numbers that come before. So, take the hyperbolic plane and tile it with pentagons, 5 meeting at each vertex. This is the {5,5} tiling. Very fiveish! We should be able to have fun with this.



The {3,3} tiling of the sphere gives the tetrahedron, with 3 triangles meeting at each vertex:



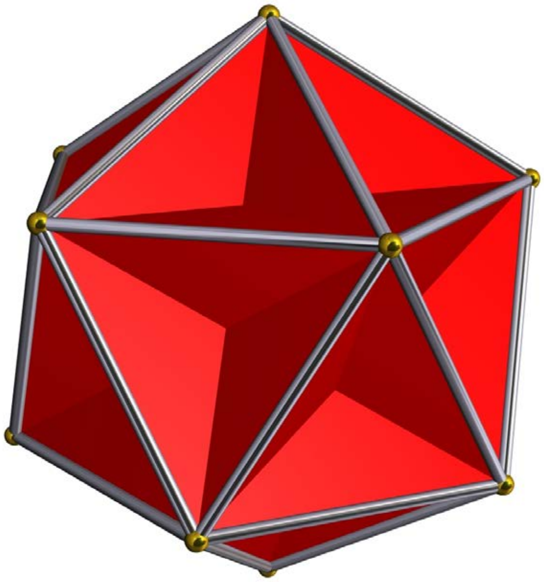
The {4,4} tiling of the plane has 4 squares meeting at each vertex:



I could talk all day about these, but we're going straight for the jugular and doing {5,5}.

The {5,5} tiling has a big symmetry group. If we mod out by some of these symmetries we get a surface tiled by just 12 pentagons, with 5 meeting at each vertex.

Amazingly, we can almost embed this surface in 3d space while preserving most of its symmetries. The result is the great dodecahedron, drawn here using Robert Webb's Stella software:



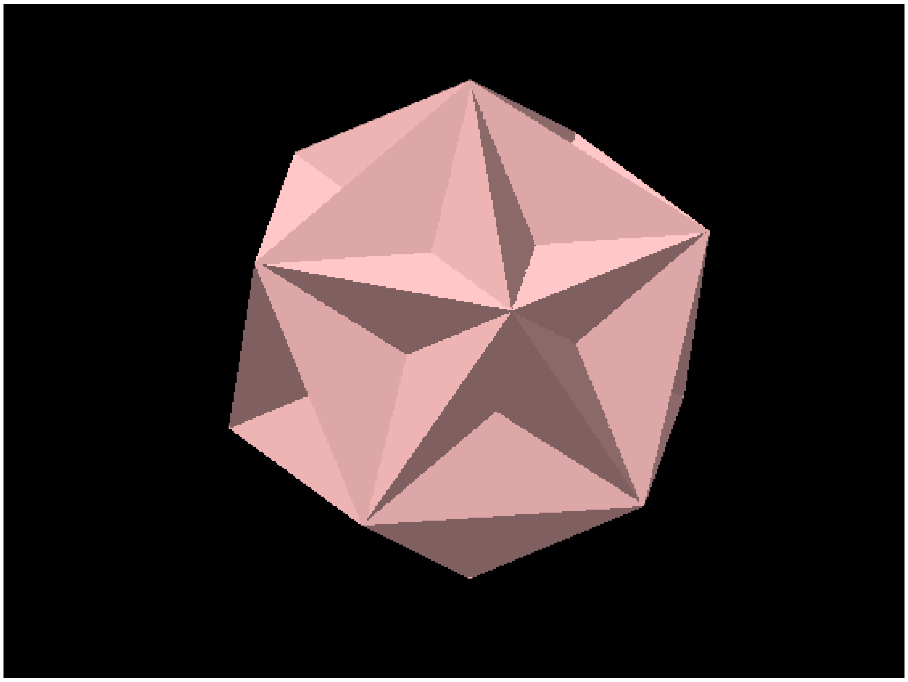
It's not embedded, because the pentagons cross each other, but it's immersed in a piecewise-linear way.

The great dodecahedron has 12 vertices, 30 edges and 12 faces, which are pentagons that cross through each other. Now

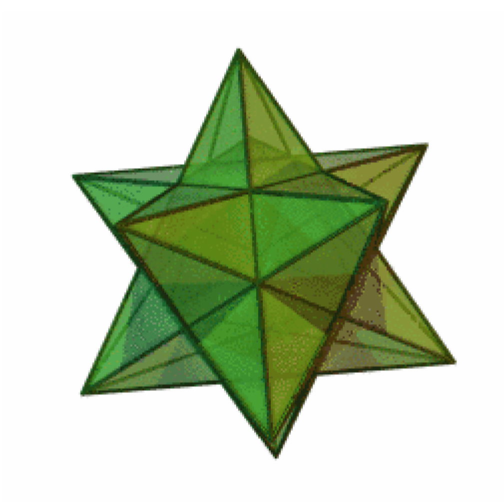
$$12 - 30 + 12 = -6 = 2 - 2g$$

with $g = 4$, so it's really a surface of genus 4. That is, a 4-holed torus!

This picture is by Brokk:



The dual of the great dodecahedron is the small stellated dodecahedron:



However, since the {5,5} tiling is self-dual, the small stellated dodecahedron gives a Riemann surface isomorphic to that coming from the great dodecahedron! It's just mapped into 3d space in a different (and more confusing) way. It has 12 pentagrams as faces, with 5 meeting at each vertex. Each pentagram is just a pentagon mapped into 3d space in a funny way with a `branch point' at the center.

The great mathematician Felix Klein showed that the Riemann surface coming from the small stellated dodecahedron can also be described by 3 equations in 5 complex variables:

$$z_1 + z_2 + z_3 + z_4 + z_5 = 0$$

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0$$

$$z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 = 0$$

They're homogeneous so they describe a complex curve in CP^4. It's a 4-holed torus. I prefer to understand it using the great dodecahedron: this aproach is equivalent, but simpler. What we're doing is giving the 4-holed torus the same conformal structure as if we give the interior of each pentagon in the great dodecahedron the metric it gets from its immersion in R^3. Angles are only distorted at the pentagon's vertices (not their edges).

There are many possible conformal structures on a 4-holed torus, but the one we get this way is the most symmetrical of all! Its symmetry group is S_5 the group of permutations of the 5 variables $z_1, \ldots, z_5$.

See how fiveish it all is?

This amazing complex curve of genus 4 is also the set of ordered 5-tuples of complex numbers $(z_1, \ldots, z_5)$, modulo rescaling, that are roots of some quintic of the form $z^5 + pz + q = 0$, where p and q are arbitrary complex numbers. You can see a sketch of the proof, and references, in my blog article on this stuff:

- Small stellated dodecahedron, *Visual Insight*, June 15, 2016.

August 13, 2018

What's this weird tree of fractions in that strange book Kepler wrote in 1619, *Harmonices Mundi*? It may be connected to a rather deep piece of mathematics!


HARMONICIS LIB. III. 27

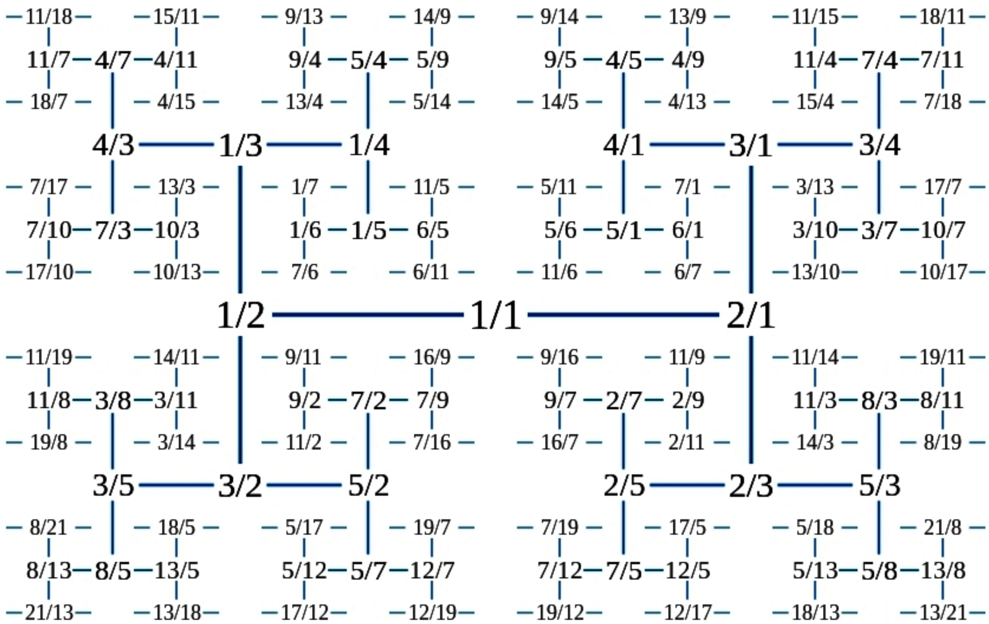
CAp. II.

prà pro numeratore, & unitas infrà pro denominatore: deinde uterq; numerus ſeorſim ponitur ſuprà, & ſumma utriuſq; utrinq; infrà; ut ex una qualibet fractione duo oriantur rami, eouſq; dum ex ſumma fiat numerus index figuræ indemonſtrabilis.

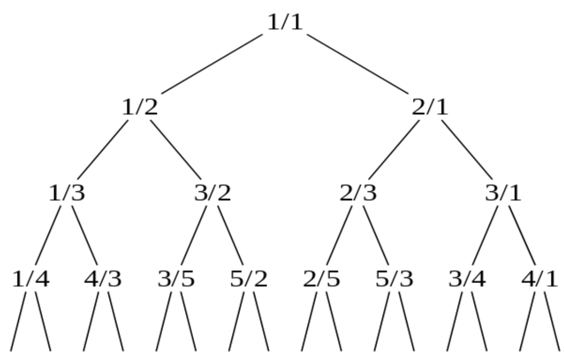
Has ſectiones chordæ ſeptem primùm auditu indice inveni, totidem nempe, quot ſunt Harmoniæ non majores unâ Diapaſon: poſtea cauſas & ſectionum ſingularum, & univerſarum Numeri, non ſine labore ex penitiſſimis Geometriæ fontibus erui. Legat curioſus lector, quæ de his ſectionibus ante annos 22. ſcripſi in Myſterio Coſmographico, Capite XII. & perpendat, quomodo fuerim illo loco hallucinatus ſuper cauſis ſectionum & Harmoniarum; perperam niſus

Cauſæ ſeptenarij hujus ſucceſſivè inventæ.

It's vaguely similar to the Calkin–Wilf tree:

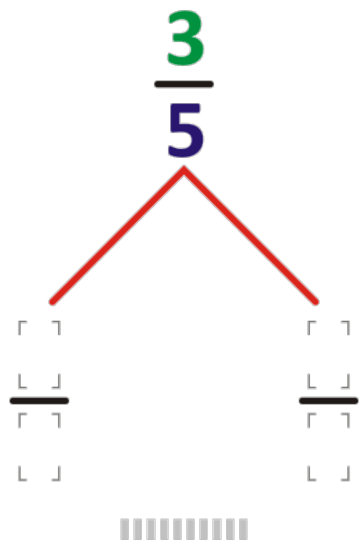


This tree contains all the positive rational numbers — and each shows up just once! Can you spot the Fibonacci numbers? Can you see the pattern that's being used to get all the numbers in this tree?

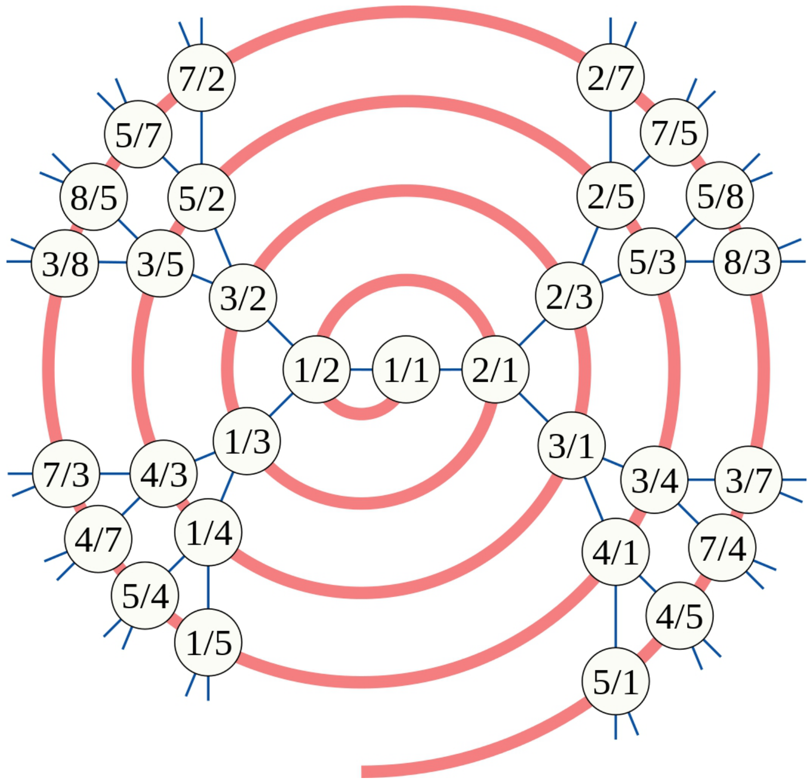


The Calkin–Wilf tree starts with 1/1.

If a/b is any fraction in the tree, then $a/(a+b)$ is below and to the left. $(a+b)/b$ is below and to the right. Amazingly, each positive rational shows up just once, in lowest terms!



You can see why Fibonacci numbers show up! But the Calkin–Wilf tree also gives a nice way to *list* all positive rational numbers....



If you follow this spiral through the Calkin–Wilf tree, you'll meet each positive rational number exactly once, so you can list them all:

$$1/1,\ 1/2,\ 2/1,\ 1/3,\ 3/2,\ 2/3,\ 3/1,\ 1/4,\ 4/3,\ 3/5,\ 5/2,\ 2/5,\ 5/3,\ 3/4, \dots$$

Amazingly, there's also recursive formula for this sequence! If you can't guess that recursive formula, you can look it up on Wikipedia. That's where I got all this great information... and all these nice pictures:

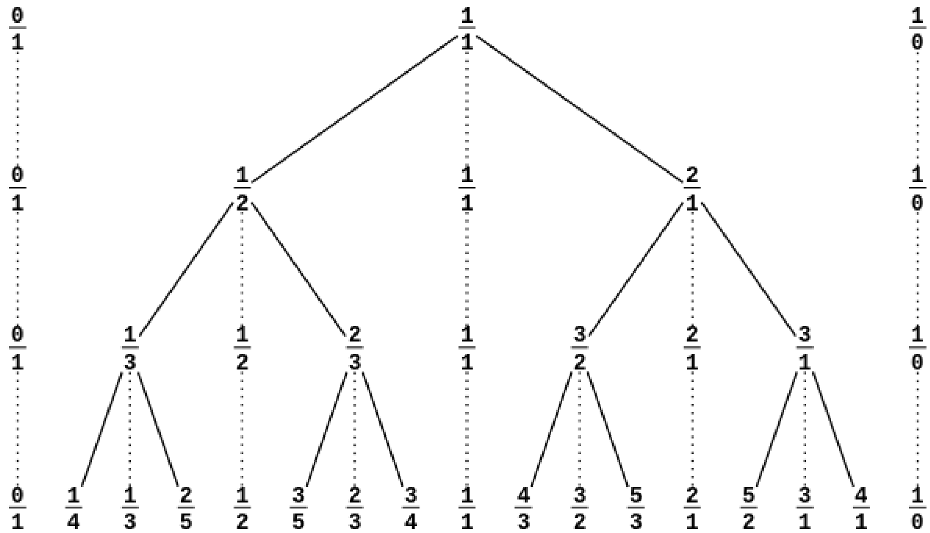
- Wikipedia, Calkin-Wilf tree.

Here is Calkin and Wilf's paper:

- Neil Calkin and Herbert S. Wilf, Recounting the rationals, *The American Mathematical Monthly* **107** (2000), 360

—363.

Also read about the [Stern-Brocot tree](). It has the same numbers in each row, but arranged in increasing order:



But how closely is all this connected to what Kepler was doing?


HARMONICIS LIB. III. 27

prà pro numeratore, & unitas infrà pro denominatore: deinde uterq; numerus seorsim ponitur suprà, & summa utriusq; utrinq; infrà; ut ex una qualibet fractione duo oriantur rami, eousq; dum ex summa fiat numerus index figuræ indemonstrabilis.

CAp. II.

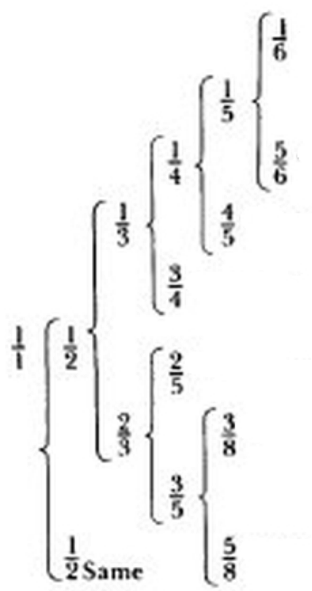
Has sectiones chordæ septem primùm auditu indice inveni, totidem nempe, quot sunt Harmoniæ non majores unâ Diapason: postea causas & sectionum singularum, & universarum Numeri, non sine labore ex penitissimis Geometriæ fontibus erui. Legat curiosus lector, quæ de his sectionibꝰ ante annos 22. scripsi in Mysterio Cosmographico, Capite XII. & perpendat, quomodo fuerim illo loco hallucinatus super causis sectionum & Harmoniarum; perperam nisus

Causæ septenarij hujus successivè inventæ.

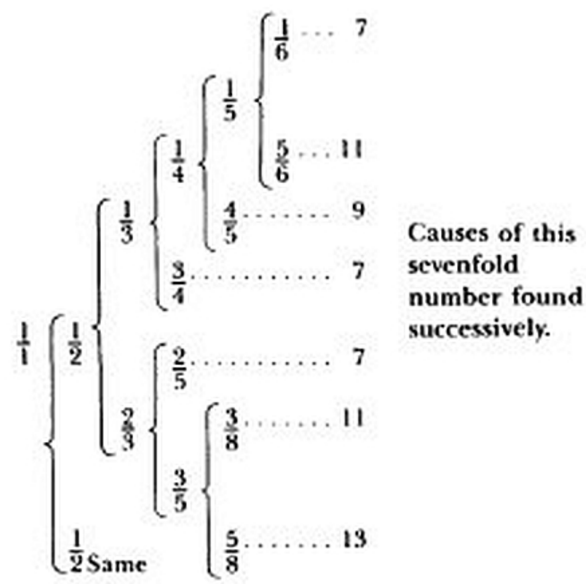
2 Idem.

Here's what he wrote here (translated from the Latin):

> I. The harmonic divisions of a single string are seven in number, not more.
>
> II. The expansion of the numbers which are characteristic of divisions occurs in the following manner. To being with, the whole is expressed in the form of a fraction, that is to say with unity above as numerator, and unity below for denominator. Then each number separately is put as numerator, and the sum of the two is put as denominator in each case. Hence from any given fraction two branches arise, until from the sum occurs the number which indicates an unconstructible figure.



I believe 'unconstructible figure' alludes to a regular polygon that can't be constructed by ruler and compass: note that the 3,4,5,6 and 8-sided polygons can be so constructed, but not the 7, 9, 11, or 13-sided ones, which would come next. He shows these as well in his chart:



I believe he was trying to find the ratios of frequencies that appear in a a major scale, because of the remark that "the harmonic divisions of a single string are seven in number", and also the remarks that follow the above passage:

> I found these seven divisions of the string first with hearing as a guide, in other words the same number as there are harmonies not greater than a single diapason. After that I dug out the causes both of the individual divisions and of the number of the total, not without toil, from the deepest fountains of geometry. Let the diligent reader read what I wrote about these divisions 22 years ago in *The Secret of the Universe*, Chapter XII.

So, he didn't want the tree to go on forever, even though it *could*. He wanted to cut it off to avoid ratios involving 7, 9, 11, 13 or more complicated numbers. And he did this by invoking the nonconstructibility of these regular polygons — a fact that was apparently known even back then, though not *proved* until later.

August 16, 2018

You can use quaternions to understand electrons!

Or: you can use electrons to understand quaternions!

Hamilton invented the formula for quaternion multiplication long before people knew quantum mechanics. But it describes what happens when you rotate an electron.

Usually people describe the spin of an electron using a pair of complex numbers... or in other words, 4 real numbers. But we can package the same information in a single quaternion:

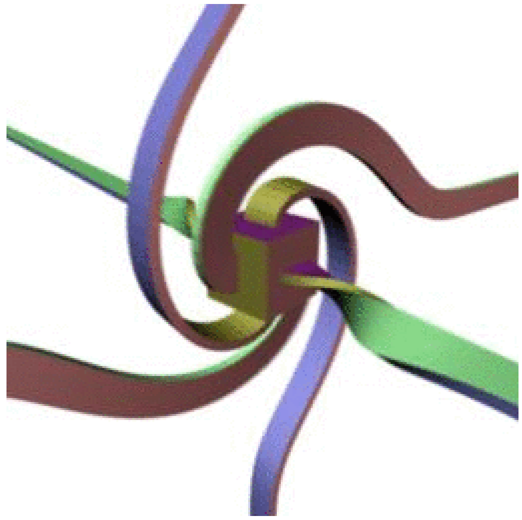
$$a + bi + cj + dk$$

where a, b, c, d are real numbers and i, j, k are square roots of -1.

When you rotate the electron 180° around the x axis, you multiply its quaternion by i.

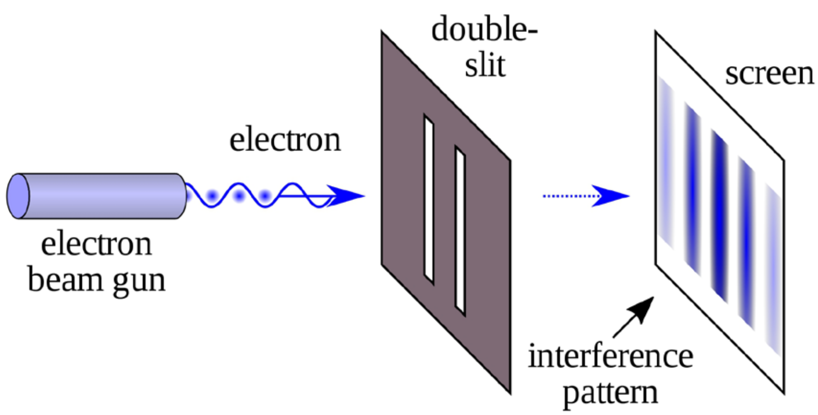
So when you rotate it 360°, it doesn't come back to where it was! Instead, since $i^2 = -1$, it gets multiplied by -1.

You have to rotate it 720° to get it back to where it was, like this contraption here:



But wait — what does it even *mean* to say the electron gets multiplied by -1 if we rotate it 360°?

Shoot an electron through a double slit. It goes through *both* slits, and it's probability of hitting any point on the screen behind the slit forms an interference pattern. Make a machine that rotates the electron 360 degrees when it goes through the left slit. This changes the pattern: where the waves added up, they now subtract.



With work you can show experimentally that rotating an electron 360° around any axis multiplies it by -1.

So, if we say that rotating it 180° around the $x, y,$ or z axes multiplies it by $i, j,$ or k respectively, then we conclude

$$i^2 = j^2 = k^2 = -1.$$

It's also true that rotating an electron 180° around the x axis, then the y)*axis*, *andthenthe*\(z axis multiplies it by -1. This gives

$$ijk = -1$$

In fact we get the whole quaternion multiplication table this way!

So: quaternions are built into physics.

The basis elements i, j, and k commute with the real quaternion 1, that is

$$i \cdot 1 = 1 \cdot i = i, \qquad j \cdot 1 = 1 \cdot j = j, \qquad k \cdot 1 = 1 \cdot k = k.$$

The other products of basis elements are defined by

$$\begin{aligned} i^2 &= j^2 = k^2 = -1, \\ ij &= k, \qquad ji = -k, \\ jk &= i, \qquad kj = -i, \\ ki &= j, \qquad ik = -j. \end{aligned}$$

These multiplication formulas are equivalent to

$$i^2 = j^2 = k^2 = ijk = -1\,.$$

In fact, the equality $ijk = -1$ results from

$$(ij)k = k^2 = -1.$$

August 24, 2018

Primes that give repeating decimals of maximum possible period:

$$\frac{1}{7} = 0.\underbrace{142857}_{6 \text{ digits}}142857\ldots$$

$$\frac{1}{17} = 0.\underbrace{0588235294117647}_{16 \text{ digits}}0588235294117647\ldots$$

and infinitely many more: $7, 17, 19, 23, 29, 47, 59, 61, 97, \ldots.$

ARTIN'S CONJECTURE:
the fraction of primes with this property is

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p(p-1)}\right) = 0.3739558136\ldots$$

Proved by Christopher Hooley in 1967 *assuming* the Generalized Riemann Hypothesis is true.

Number theory is a great source of problems that are easy to state, hard to solve.

For experts, though, the fun comes from big ideas — often connected to other branches of math!

Artin's conjecture sounds cute when you say it's about repeating decimals:

- Wikipedia, Full reptend prime

but it has connections to very deep math:

- Wikipedia, Generalized Riemann hypothesis.

Read more about it here:

- Wikipedia, Artin's conjecture on primitive roots.

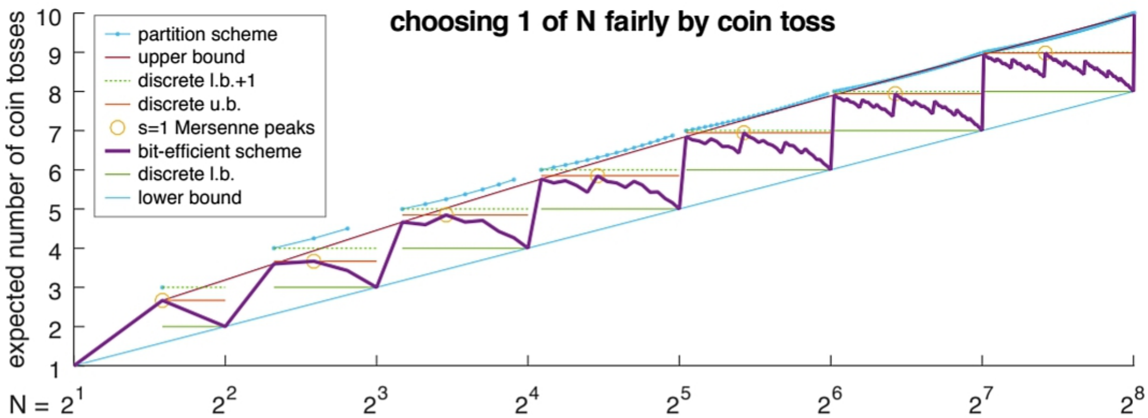
August 27, 2018

What's the fewest flips of a coin you need, on average, to randomly choose a number from 1 to *N*? That's the topic of this paper:

- Matthew Brand, [Choosing 1 of *N* with and without lucky numbers].

Let's assume the coin is fair and — the hard part — each number 1, 2, …, *N* has an equal chance of being chosen.

The answer is the purple curve here:



It's easy to pick a random number between 1 and $N = 2^n$ with n coin flips. But when *N* is not a power of 2, it gets tricky: fractal patterns start to show up! It's all about number theory.

For my September 2018 diary, go here.

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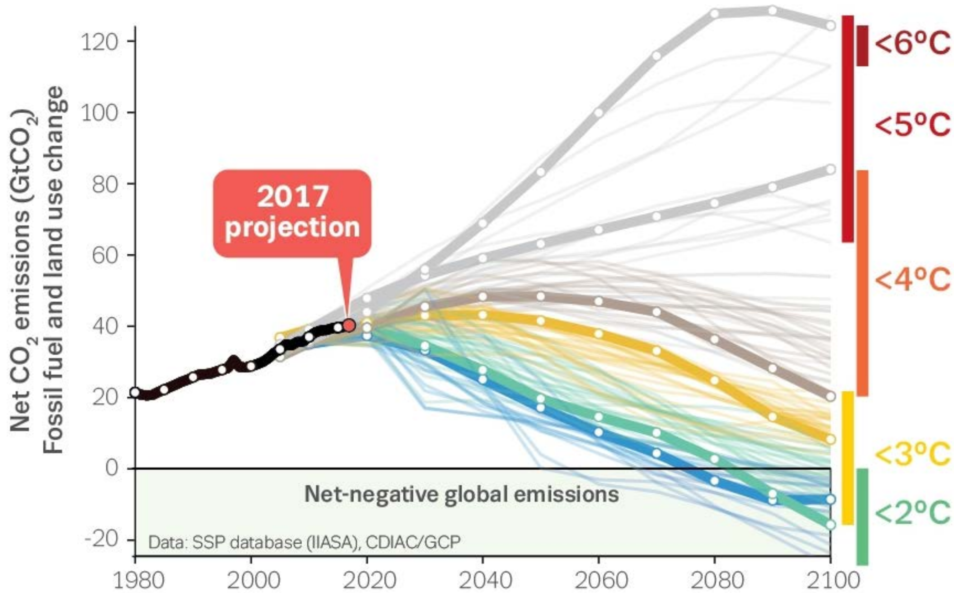
[For my August 2018 diary, go here.](#)

Diary — September 2018

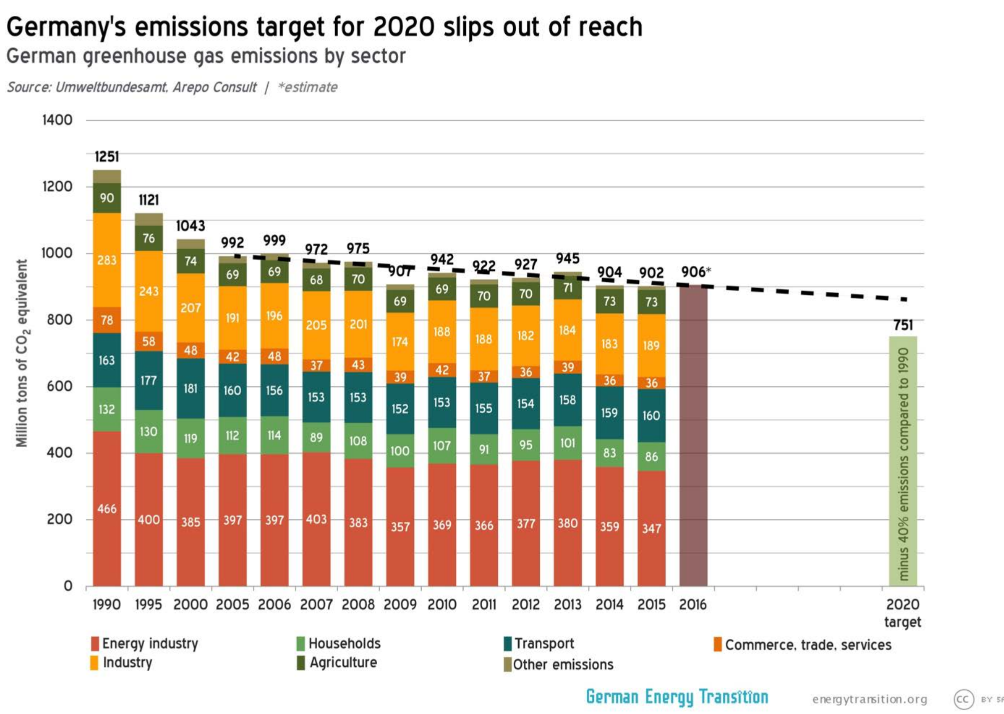
John Baez

September 14, 2018

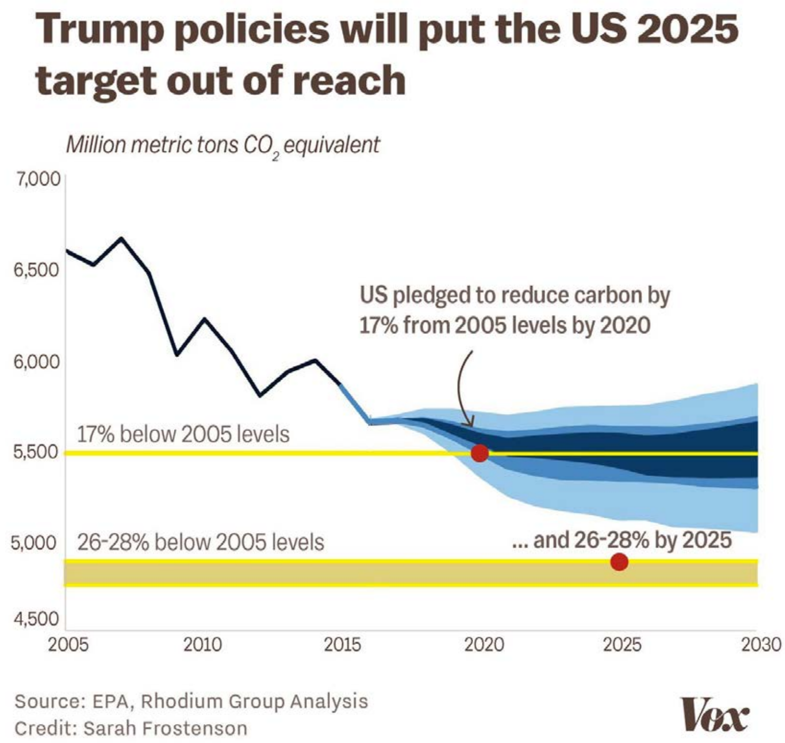
Click the figures for more information.



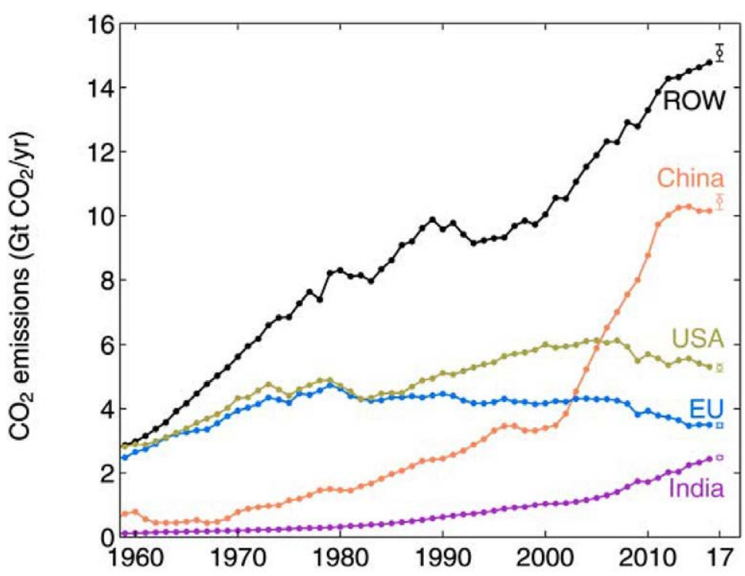
The weather disasters we're seeing will keep getting worse unless we cut carbon emissions until they're *negative*. That's hard. But how much is the Paris Agreement helping?



 ns by 40% from 1990 to 2030. They'd gone down 23% by 2016 — but last year they went *up*. The EU's energy chief wants to do more: cut by 45%. But Merkel is pushing back. Germany is already going to miss its 2020 target.



The US pledged to cut carbon emissions 17% from 2005 to 2020, and 26% by 2025. Trump pulled us out of the agreement. Many in the US are soldiering on, but right now it looks like we'll miss these goals.

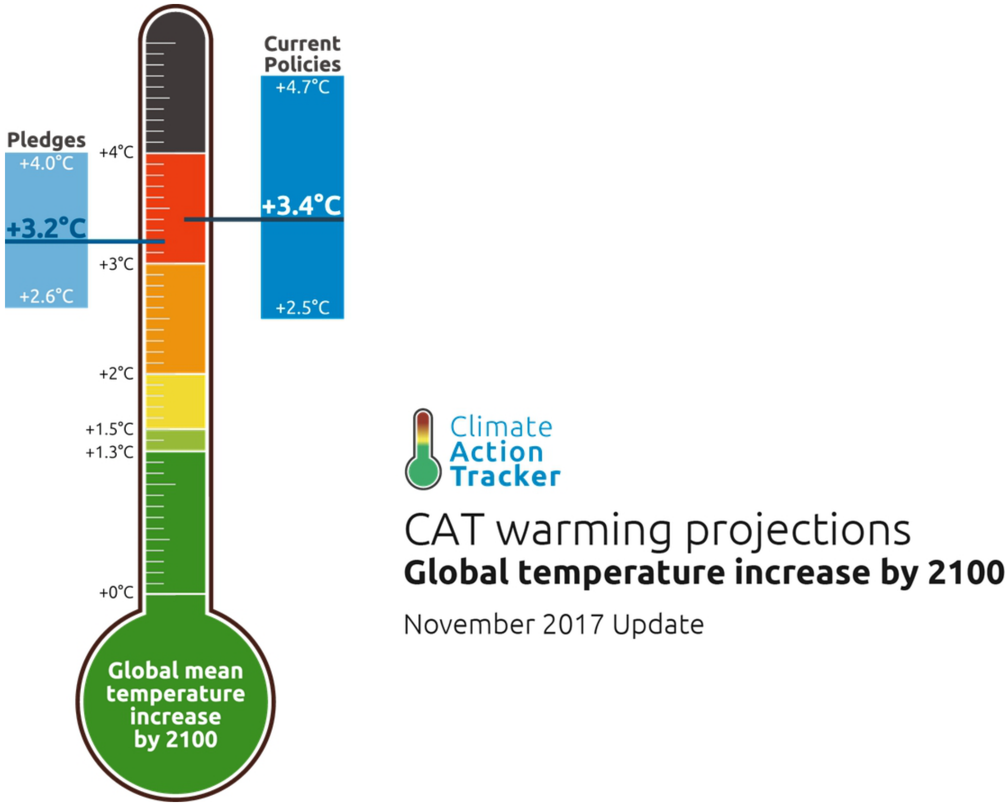


China's carbon emissions shot up starting in 2000. It leveled off after 2010, even going down a bit — but it went up 3.5% last year. India and "ROW" (rest of world) are also going up.

China pledged only to reach peak carbon emissions by 2030. See what other countries pledged — and how they're doing:

- Climate Action Tracker.

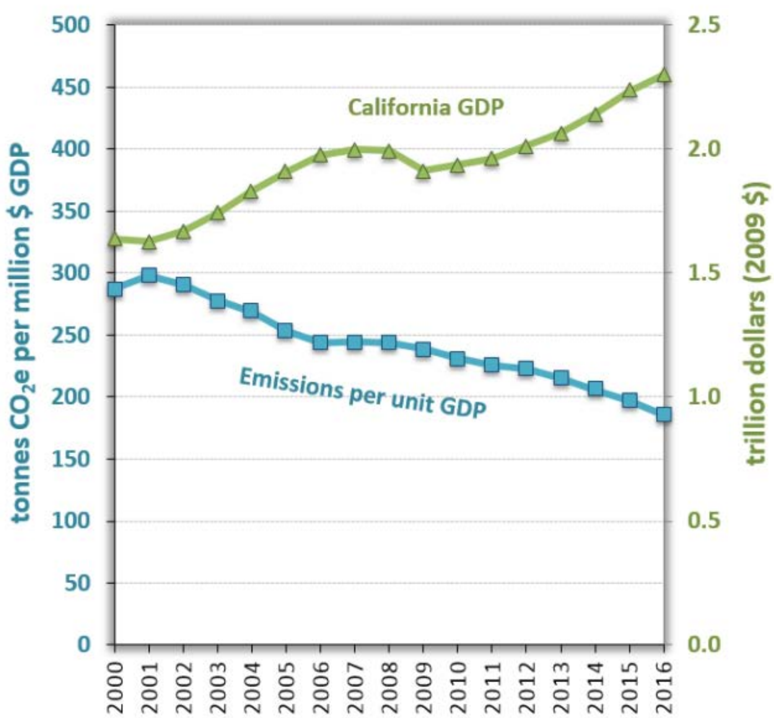
Short version: right now we're heading for 2.5-4.7°C warming by 2100. Between bad and disaster.



But some good news...

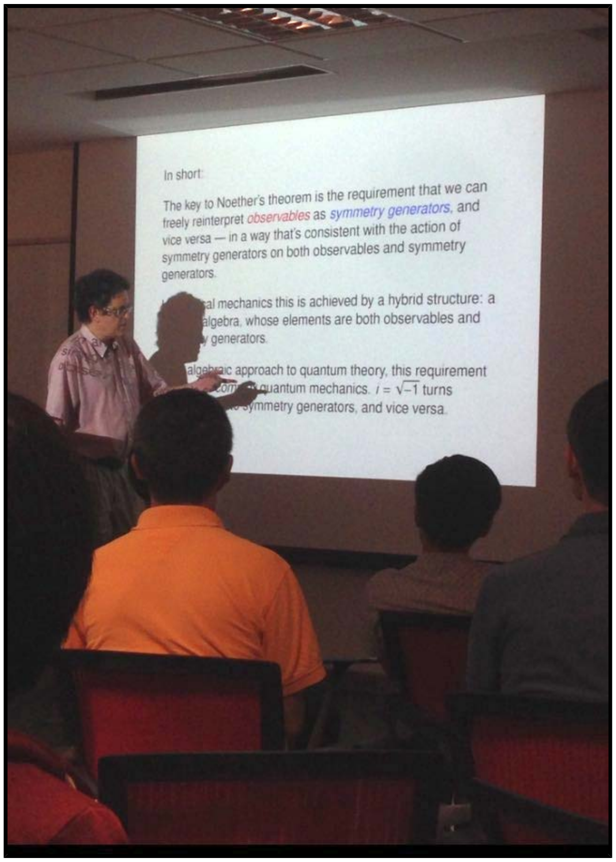
California just passed a law saying we'll switch to completely carbon-free electricity production by 2045! Very hard, but look at this chart:

Figure 1c. Carbon Intensity of California's Economy



California's carbon emissions per GDP has dropped almost 50% from 2000 to 2016! And today Governor Brown announced: "with climate science still under attack... we'll launch our own damn satellite" to track carbon emissions.

September 14, 2018



Lisa and I have been in Singapore since August 10th, and we're leaving soon. As usual, I've been visiting the Centre for Quantum Technologies. Today I gave a talk on Noether's theorem relating symmetries and conserved quantities. It's the 100th anniversary of her paper on this! But we still haven't gotten to the bottom of it.

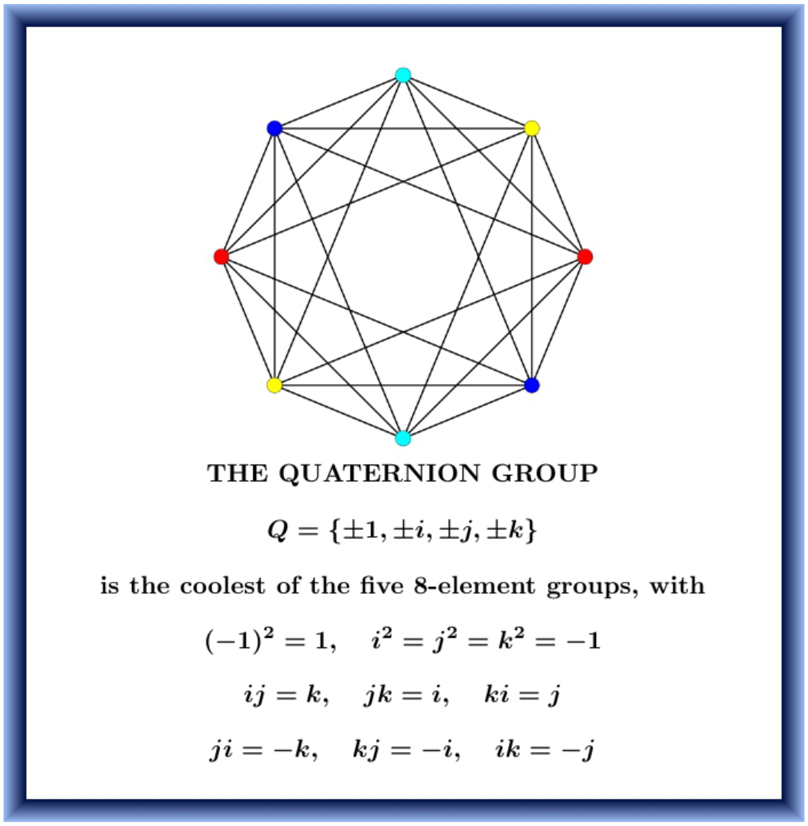
Noether showed that in a theory of physics obeying the principle of least action, any 1-parameter family of transformations preserving the action gives a conserved quantity. But Noether's theorem takes different guises in other approaches to physics, and my talk focuses on the *algebraic* approach using Poisson brackets or commutators. I argue that this explains the role of complex numbers in quantum theory!

I'll also give this talk at the Department of Applied Mathematics and Theoretical Physics (DAMTP) at Cambridge, and in a conference celebrating the anniversary of Noether's theorem in London. You can see my slides here:

- John Baez, Noether's theorem.

  Abstract. In her paper of 1918, Noether's theorem relating symmetries and conserved quantities was formulated in term of Lagrangian mechanics. But if we want to make the essence of this relation seem as self-evident as possible, we can turn to a formulation in term of Poisson brackets, which generalizes easily to quantum mechanics using commutators. This approach also gives a version of Noether's theorem for Markov processes. The key question then becomes: when, and why, do observables generate one-parameter groups of transformations? This question sheds light on why complex numbers show up in quantum mechanics.

September 16, 2018



The quaternion group has 8 elements. Two of them, ± 1, commute with everything. The rest commute with 4 elements each: for example i commutes with $\pm 1, \pm i$.

So, one quarter of the elements commute with every element. Three quarters of the elements commute with half. Thus the probability that two randomly chosen elements in this group commute is

$$\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{2} = \frac{5}{8}$$

But what's the *largest* this probability can be, for *any* noncommutative group?

It's 5/8. And I explain why here:

- John Baez, The 5/8 theorem, *Azimuth*, September 16, 2018.

September 18, 2018

Sylow's theorems are key to understanding the structure of finite groups. Take a finite group G, a prime p, and let p^k be the biggest power of p that divides the number of elements of G. Then G has a subgroup of size p^k, called a **Sylow *p*-subgroup**. And more good stuff is true, as summarized by Robert A. Wilson in his book *Finite Simple Groups*:

Sylow's theorems

If G is a finite group of order $p^k n$, where p is prime and n is not divisible by p, then the *Sylow theorems* state that

(i) G has subgroups of order p^k;
(ii) these *Sylow p-subgroups* are all conjugate; and
(iii) the number s_p of Sylow p-subgroups satisfies $s_p \equiv 1 \bmod p$. (Note also that, by the orbit–stabiliser theorem, s_p is a divisor of n).

But what the hell are Sylow's theorems good for? As an undergrad in love with physics, I never really got it. Now on YouTube you can see examples that show what you can do with them — nice!

You can show there are no simple groups of order 30:

You can show that any group of order 77 is cyclic:

Okay, but how do you prove them?

You can also see proofs of Sylow's theorems on YouTube, like this, the first of a 3-part series.

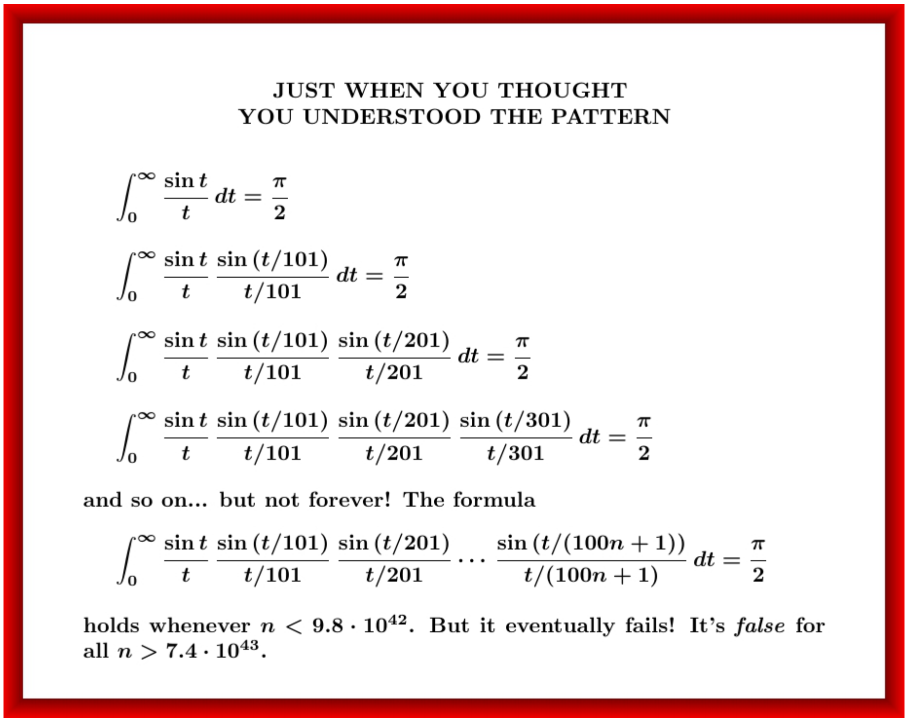
But I'm often too impatient for videos. I'd rather see really short proofs, then think about them in my spare time: washing the dishes, lying in bed, etc.

So here's a really short proof of all 3 Sylow theorems from Robert A. Wilson's book *Finite Simple Groups*. This proof takes lots of work to unravel. But I love it: all the ideas are here.

> To prove the first statement, let G act by right multiplication on all subsets of G of size p^k: since the number of these subsets is not divisible by p, there is a stabiliser of order divisible by p^k, and therefore equal to p^k. To prove the second statement, and also to prove that any p-subgroup is contained in a Sylow p-subgroup, let any p-subgroup Q act on the right cosets Pg of any Sylow p-subgroup P by right multiplication: since the number of cosets is not divisible by p, there is an orbit $\{Pg\}$ of length 1, so $PgQ = Pg$ and gQg^{-1} lies inside P. To prove the third statement, let a Sylow p-subgroup P act by conjugation on the set of all the other Sylow p-subgroups: the orbits have length divisible by p, for otherwise P and Q are distinct Sylow p-subgroups of $N_G(Q)$, which is a contradiction.

Here $N_G(Q)$ is the normalizer of Q in G.

September 20, 2018



JUST WHEN YOU THOUGHT
YOU UNDERSTOOD THE PATTERN

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t} \frac{\sin(t/101)}{t/101}\, dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t} \frac{\sin(t/101)}{t/101} \frac{\sin(t/201)}{t/201}\, dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t} \frac{\sin(t/101)}{t/101} \frac{\sin(t/201)}{t/201} \frac{\sin(t/301)}{t/301}\, dt = \frac{\pi}{2}$$

and so on... but not forever! The formula

$$\int_0^\infty \frac{\sin t}{t} \frac{\sin(t/101)}{t/101} \frac{\sin(t/201)}{t/201} \cdots \frac{\sin(t/(100n+1))}{t/(100n+1)}\, dt = \frac{\pi}{2}$$

holds whenever $n < 9.8 \cdot 10^{42}$. But it eventually fails! It's *false* for all $n > 7.4 \cdot 10^{43}$.

Sometimes you check just a few examples and decide something is always true. But sometimes even 9.8×10^{42} examples are not enough!

Greg Egan and I came up with this shocker on Twitter after he explained some related integrals by the Borwein brothers. To see what's really going on, visit my blog:

- John Baez, Patterns that eventually fail, *Azimuth*, September 20, 2018.

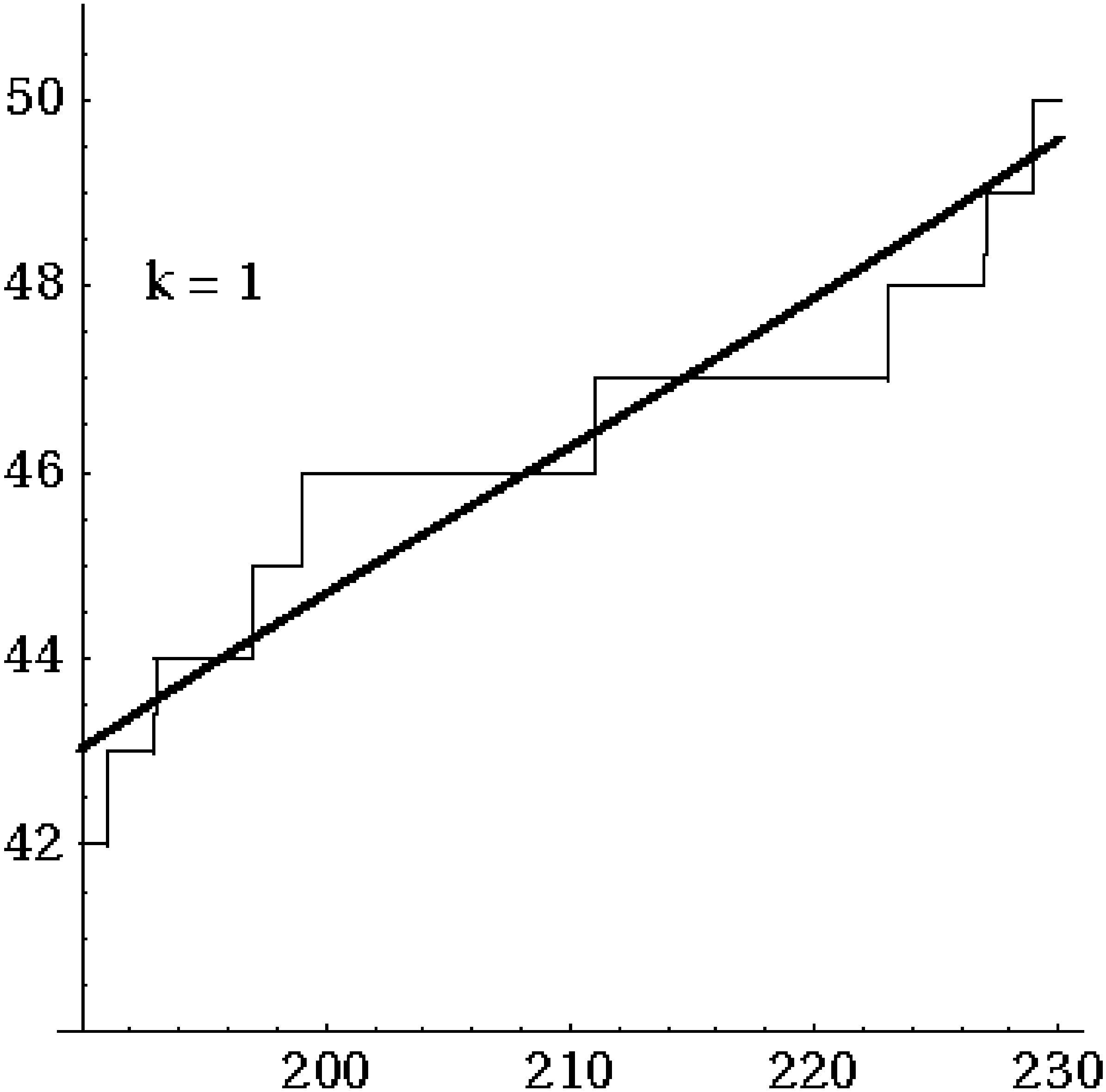
In the comments on my blog, you'll see that Greg figured out the exact value of n for which the identity first fails! It's

$$n = 15,341,178,777,673,149,429,167,740,440,969,249,338,310,889$$

which is about 1.5×10^{43}. When I saw this I breathed a sigh of relief, because it meant my estimates were right.

September 23, 2018

Tomorrow Atiyah will talk about his claimed proof of the Riemann Hypothesis. It's all about "the music of the primes". Here the function that counts primes $< n$ is being approximated by waves whose frequencies come from zeroes of the Riemann zeta function:

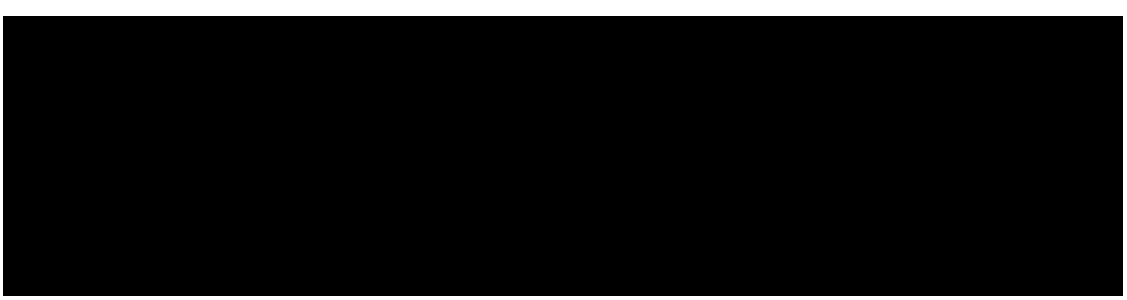


The Riemann zeta function is given by this simple formula when the complex number s has $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Then the sum converges! But we can "analytically continue" the Riemann zeta function to define it for other values of s, and that's where the fun starts.

The Riemann zeta function is zero for some numbers with $0 < \mathrm{Re}(s) < 1$. These are called the "nontrivial zeros" of his zeta function. Riemann computed a few and hypothesized they all have $\mathrm{Re}(s) = 1/2$.



Riemann found a formula for the number of primes $< n$ as a sum over the nontrivial zeros of the zeta function. My first tweet shows the sum over the first k nontrivial zeros. So, if the Riemann Hypothesis is true, we'll get a better understanding of primes!

For example, in 2014, Adrian Dudek proved that the Riemann hypothesis implies that that for all $n \geq 2$ there is a prime p with

$$n - \frac{4}{\pi}\sqrt{x}\ln x < p \leq n$$

So far people have checked, using a computer, that the first 10,000,000,000,000 nontrivial zeros of the Riemann zeta function have $Re(s) = 1/2$. This might seem like damned good evidence for the Riemann Hypothesis. But maybe not!

JUST WHEN YOU THOUGHT
YOU UNDERSTOOD THE PATTERN

The number 6 is the most common gap between consecutive primes if you only look at primes less than about

$$\mathbf{174{,}270{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000.}$$

But then that pattern stops!!!

For me, the best evidence for the Riemann Hypothesis is that it's part of a much bigger story! Mathematicians like Weil, Grothendieck and Deligne proved similar results for related functions. Much remains mysterious, though.

I bet that Atiyah's claimed proof, if and when he writes it up, will not convince experts. In 2017 he claimed to have a 12-page proof of the Feit-Thompson theorem, which usually takes 255 pages:

- Magnus Linklater, Mathematician, 88, hopes to prove himself again with new solution, *The Times*, 12 August 2017.

He showed it to experts, and... silence.

In 2016 Atiyah put a paper on the arXiv claiming to have solved a famous problem in differential geometry. The argument was full of big holes:

- MathOverflow, What is the current understanding regarding complex structures on the 6-sphere?

So, I'm not holding my breath this time.

September 22, 2018

Here is Atiyah's lecture on the Riemann Hypothesis:

Here, apparently, is his paper:

- Michael Atiyah, [The Riemann hypothesis](#).

It refers extensively to this much longer paper, where he attempts to compute the fine structure constant:

- Michael Atiyah, [The fine structure constant](#).

The formula he gives for the fine structure constant does not give the correct answer.

Here's what *Science* says about Atiyah:

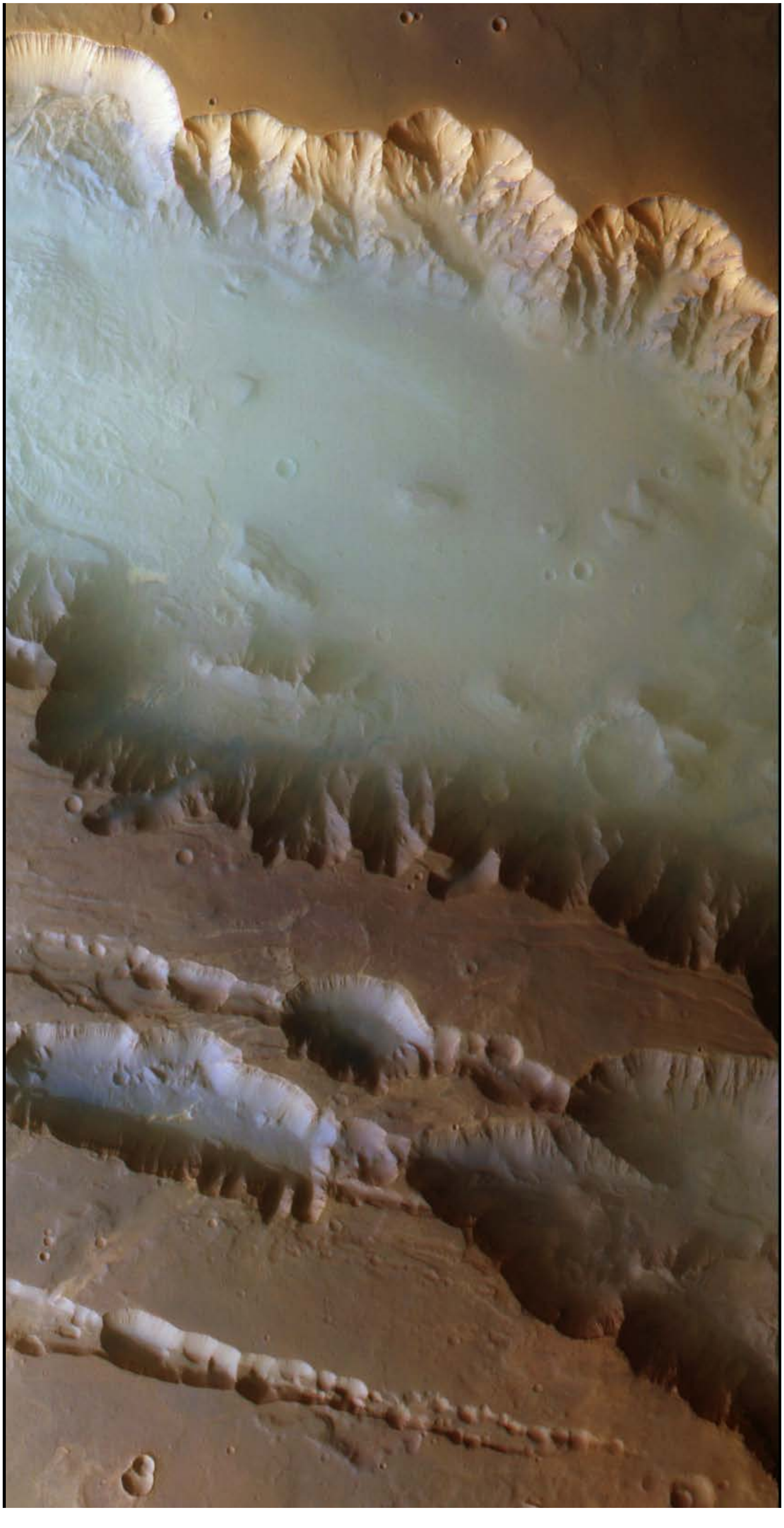
- Frankie Schembri, Skepticism surrounds renowned mathematicianâ€™s attempted proof of 160-year-old hypothesis, *Science*, September 24, 2018.

Only a few mathematicians were willing to be quoted, and I was one. I was also quoted later here:

- David Freeman, Retired mathematician rocks math world with claim that he's solved $1 million problem, *NBC Mach*, September 27, 2018.

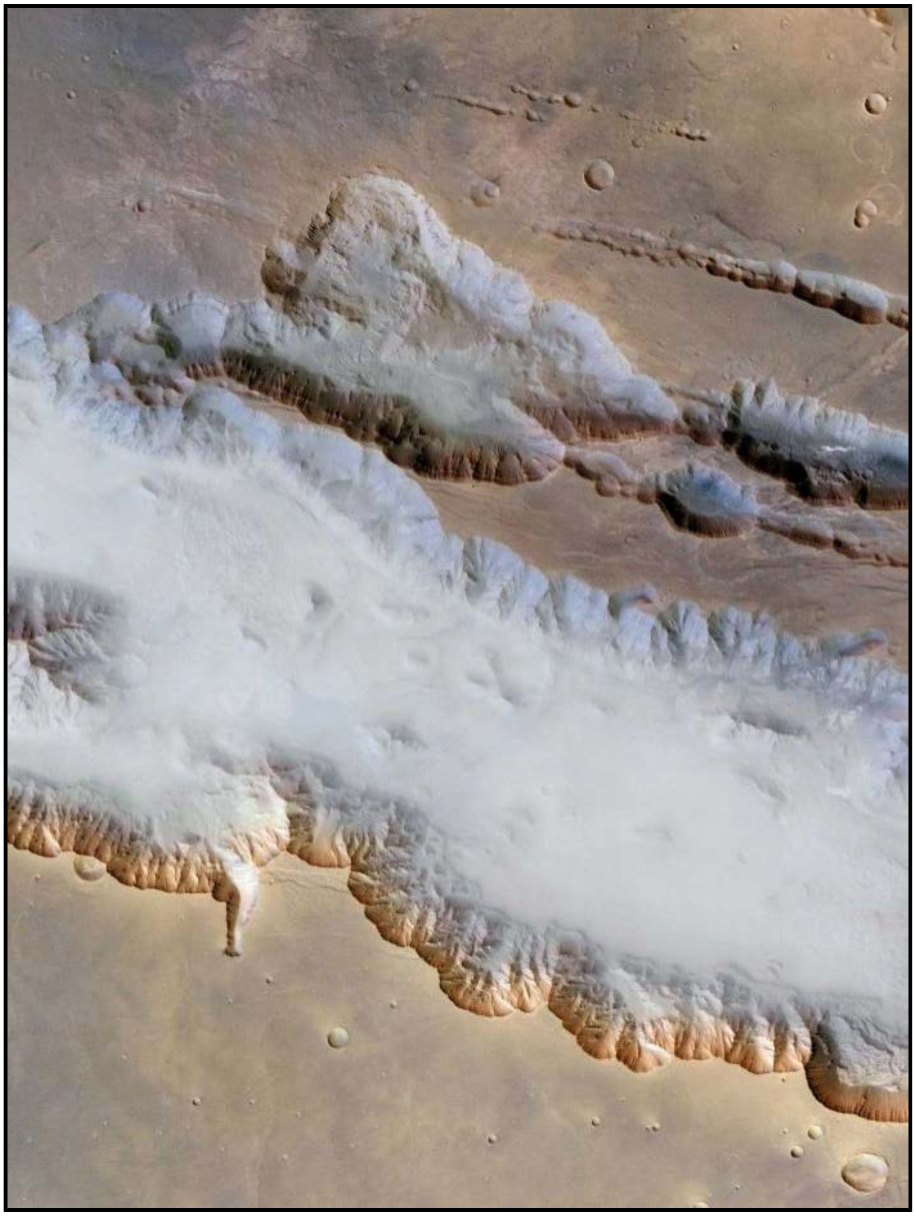
By the way, I have huge respect for Atiyah, whose earlier work revolutionized geometry and physics.

September 29, 2018



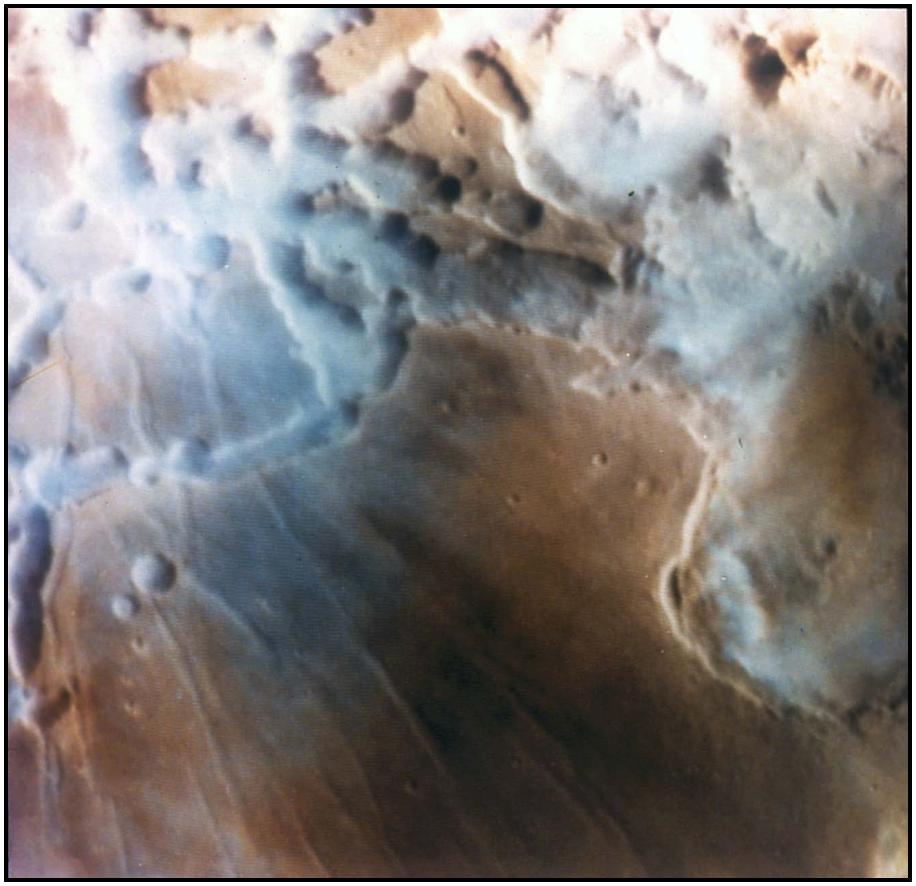
Fog in a Martian canyon on a chilly morning! This is a high-resolution image of Valles Marineris: a huge canyon on Mars, 4000 kilometers long and up to 7 kilometers deep.

Here's another view:



This fog is made of water ice, not liquid water, so it's a bit like a Martian relative of 'pogonip': a dense winter frozen fog in mountain valleys. But it's different: those mountain valleys are *colder* than the uplands, but Valles Marineris is *warmer* than the higher regions nearby! This may indicate that there's source of water in the valley, that makes it more humid. It could be frost submlimated by the morning sun.

And here's a view of fog in Noctis Labyrinthus — a maze-like system of deep, steep-walled valleys between Valles Marineris and the nearby Tharsis upland.



Fog in the Labyrinth of the Night. How poetic! I would like to see this someday.

For my October 2018 diary, go here.

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Diary — October 2018

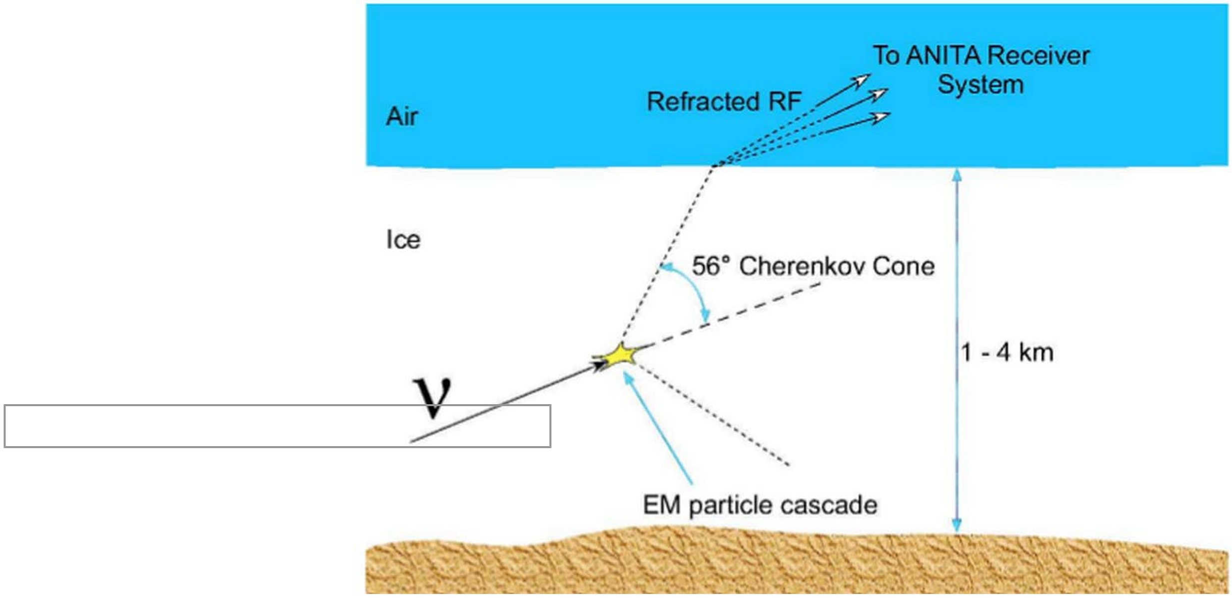
John Baez

October 1, 2018

ANITA, the Antarctic Impulse Transient Antenna, detects ultra-high-energy cosmic neutrinos. Here it is in front of Mount Erebus.

It's seen some amazing things... possibly signs of physics beyond the Standard Model! But first let me say a word about how it works.

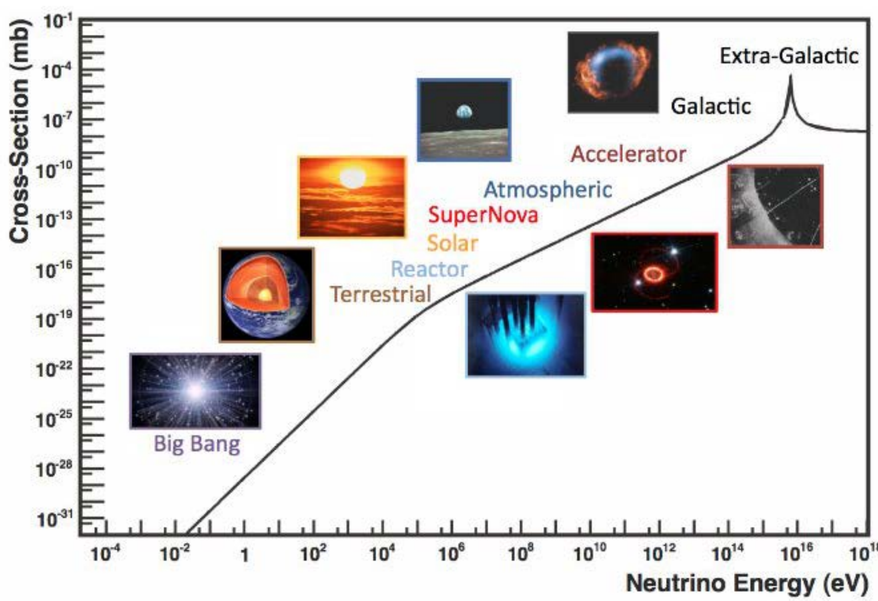
A neutrino shooting through Antarctic ice hits an atom and creates a shower of charged particles moving faster than the speed of light in ice. They create Cherenkov radiation, the electromagnetic analogue of a sonic boom. Some of this radiation is in the form of radio waves. As they leave the ice, these waves refract. It's called the Askaryan effect.



ANITA, hanging from a floating weather balloon, has seen many events like this. But *two* of these pulses came up *through the Earth's crust*, which is not how it's supposed to work!

An ultra-high-energy neutrino shouldn't be able to go through the Earth. Low-energy neutrinos can go through light years of rock, but their rate of absorption goes up with their energy. The pulses seen by ANITA seem to have come from particles with an energy of about 0.6 EeV: that is, 6 10^{17} electron volts. That's an extremely high energy. Such energetic neutrinos should not be able to penetrate lots of rock.

This picture explains it: "[cross section]" is, roughly speaking, proportional to the absorption probability.



ANITA saw one of these events in 2006, and one in 2014. Another Antarctic neutrino detector, [IceCube], has seen similar strange events.

So, something strange is going on. This could be great! But don't believe the theories yet — it's way too early.

For more, read:

- P. W. Gorham *et al*, [Observation of an unusual upward-going cosmic-ray-like event in the third flight of ANITA], March 14, 2018.

- Derek Fox *et al*, The ANITA anomalous events as signatures of a beyond Standard Model particle, and supporting observations from IceCube, September 25, 2018.

The graph of neutrino cross-sections is taken from this paper, which explains why it works this way:

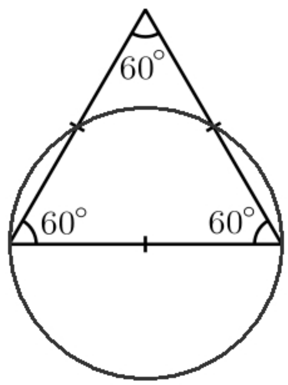
- J. A. Formaggio and G. P. Zeller, From eV to EeV: neutrino cross sections across energy scales, *Rev. Mod. Phys.* **84** (2012), 1307.

Also check out my November 20, 2013 entry on IceCube. It's an amazing experiment.

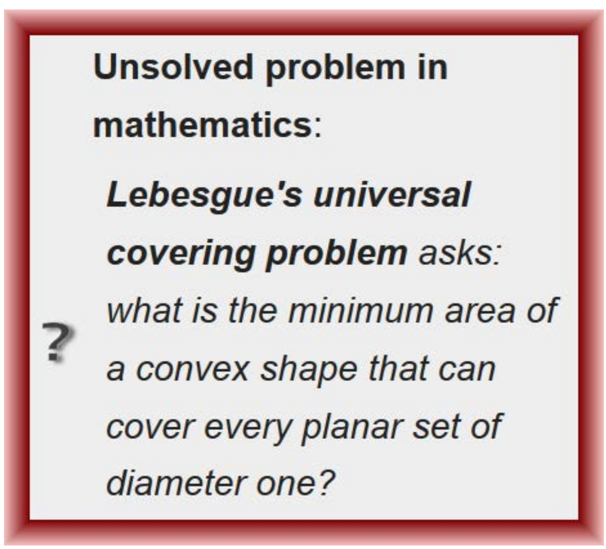
October 8, 2018

More progress on a famous problem!

A subset of the plane has "diameter 1" if the distance between any two points in this set is ≤ 1. For example, an equilateral triangle with edges of length 1 has diameter 1... and it doesn't fit in the circle with diameter 1.



In 1914 the famous mathematician Lebesgue sent a letter to his pal Pál. He challenged Pál to find the convex set with least possible area such that every set of diameter 1 fits inside... at least after you rotate, translate and/or reflect it.



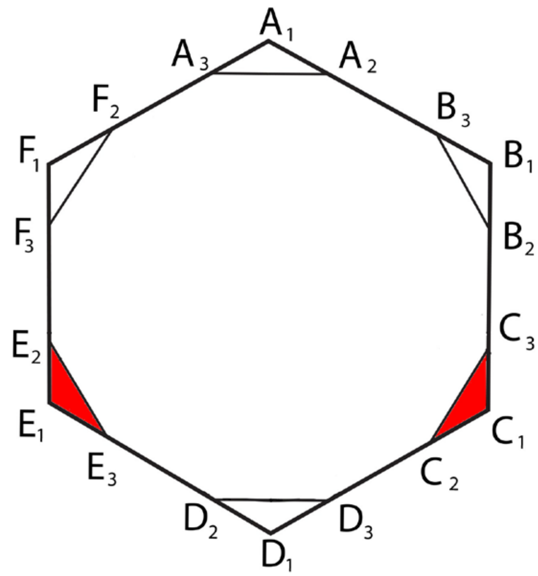
Pál showed the smallest regular hexagon containing a disk of diameter 1 would work. Its area is

0.86602540...

But he also showed you could cut off two corners, and get a solution of area just

0.84529946...

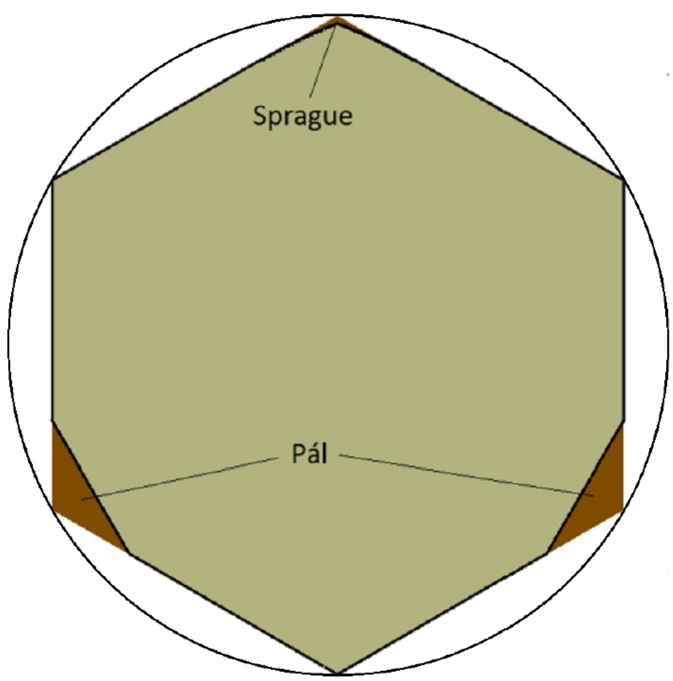
Nice!



In 1936, Sprague showed another piece of the hexagon could be removed and still leave a universal covering.

This piece has area about 0.02. Cutting it out, we get a universal covering of area

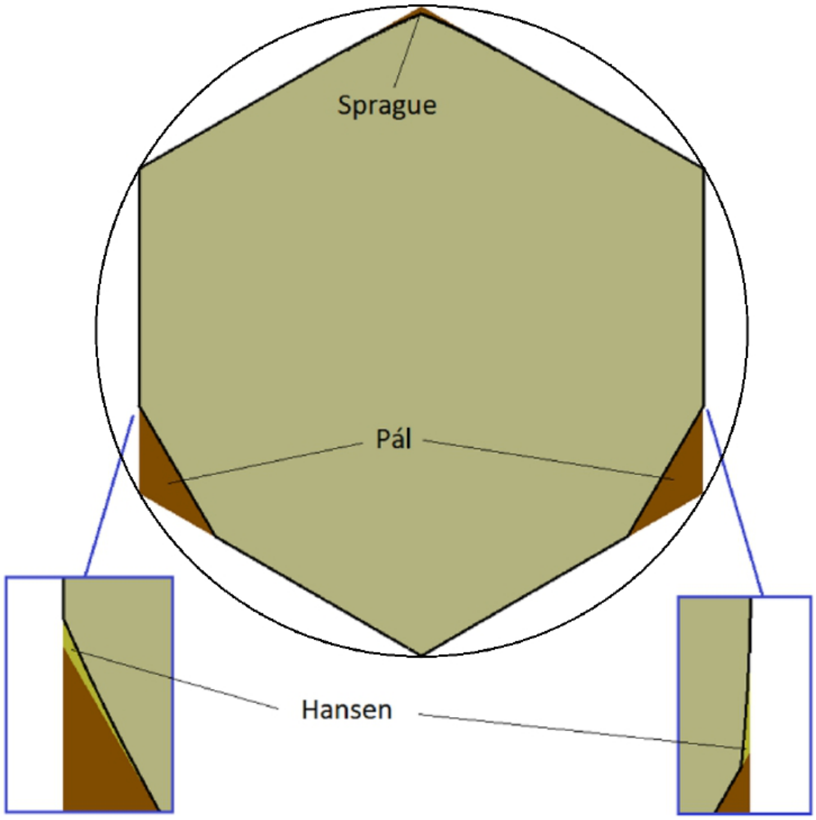
0.84413770843...



In 1992, Hansen removed two microscopic slivers from Sprague's universal covering and got an even better one!

These slivers were absurdly small, with areas of roughly 4 10^{-11} and 8 10^{-21}. He got a universal covering of area

0.844137708398...



At this point two mathematicians joked:

> "...it does seem safe to guess that progress on this problem, which has been painfully slow in the past, may be even more painfully slow in the future."

But that's when Philip Gibbs came in!

He found a way to remove a piece about a million times bigger than Hansen's larger sliver. Its area was a whopping 2 10^{-5}, leaving a universal covering of area

$$0.8441153769...$$

A student and I helped him polish his proof and publish this result:

- John Baez, Karine Bagdasaryan and Philip Gibbs, The Lebesgue universal covering problem, *Journal of Computational Geometry* **6** (2015), 288–299.

I met Philip Gibbs in London last week at a workshop on Noether's theorem. It turned out he's now done even better! He has chopped off another enormous chunk, with area 2 10^{-5}, leaving a universal covering of area just

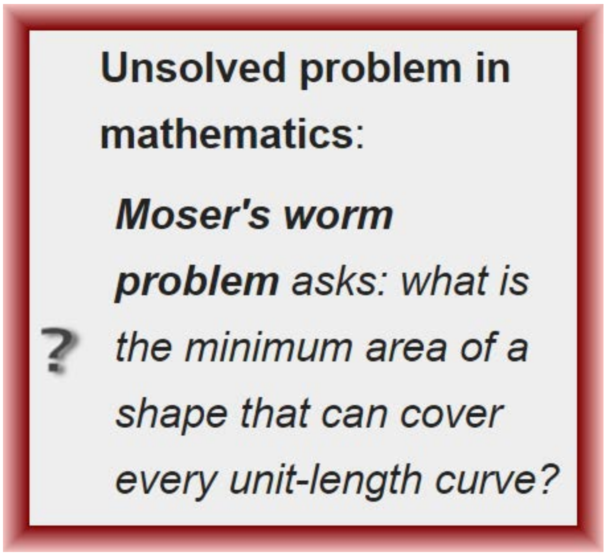
$$0.8440935944...$$

Huge progress! For details, read his paper:

- Philip Gibbs, An upper bound for Lebesgue's universal covering problem, January 22, 2018.

The moral is that even plane geometry holds deep problems connected to optimization.

Lebesgue's universal covering problem, the sofa moving problem, the Moser worm problem, Bellman's lost-in-a-forest problem — we just don't know powerful techniques to tackle these... yet.



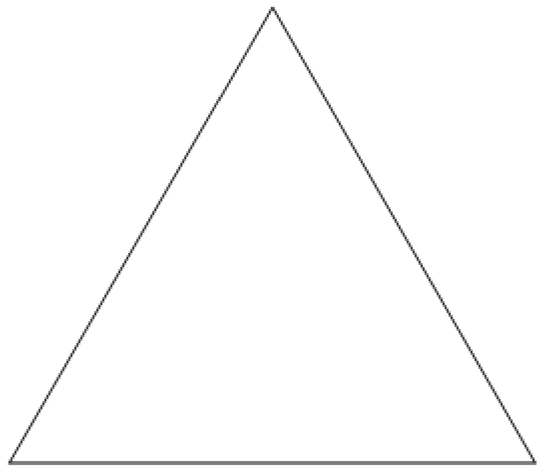
For more on Lebesgue's univeral covering problem, read these:

- John Baez, Lebesgue's univeral covering problem (part 1), *Azimuth*, December 8, 2013.
- John Baez, Lebesgue's univeral covering problem (part 2), *Azimuth*, February 3, 2015.
- John Baez, Lebesgue's univeral covering problem (part 3), *Azimuth*, October 7, 2018.

October 13, 2018

A polygon can be dissected into straight-edged pieces which you can rearrange to get any other polygon of the same area! This is called the Wallace–Bolyai–Gerwien theorem.

In fact, you can always use a "hinged" dissection, as illustrated here by Rodrigo Silveira Camaro:



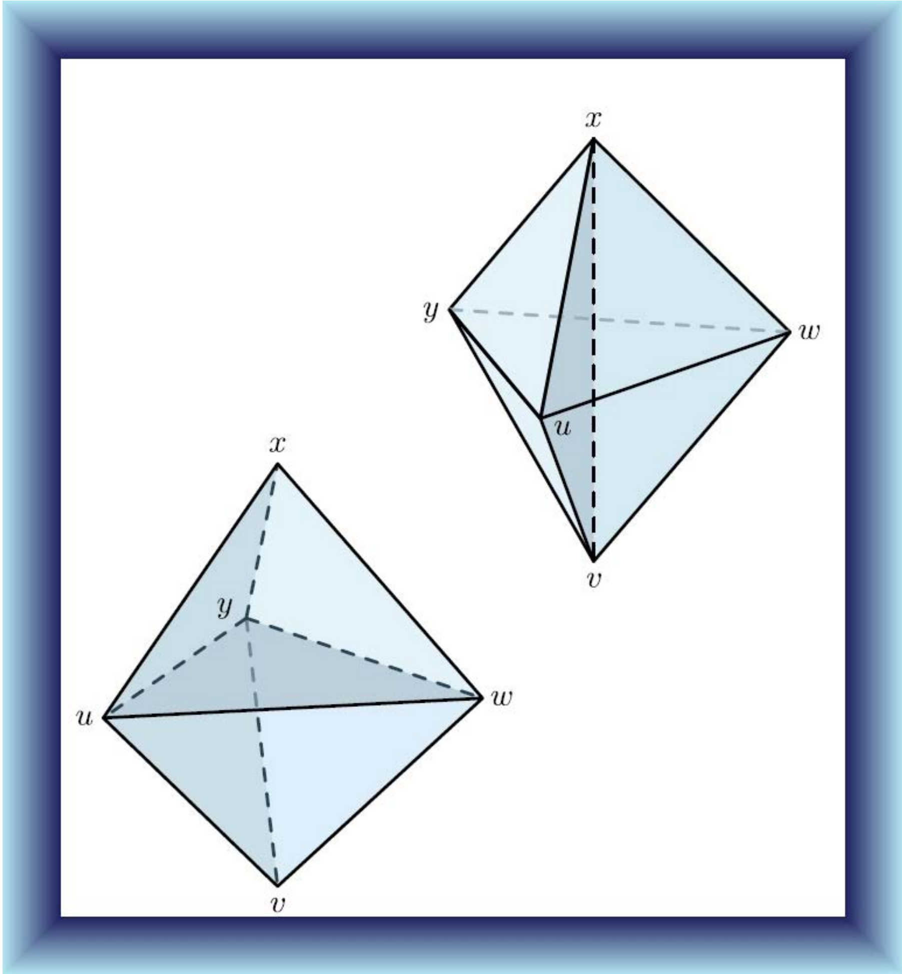
This was proved in 2007:

- Timothy G. Abbott, Zachary Abel, David Charlton, Erik D. Demaine, Martin L. Demaine, and Scott D. Kominers, Hinged dissections exist, *Proceedings of the Twenty-fourth Annual Symposium on Computational Geometry*, ACM, 2008, pp. 110–119.

But what about higher dimensions? Two polyhedra are "scissors-congruent" if you can cut one into finitely many polyhedral pieces, then reassemble them to get the other. In 1900 Hilbert asked: are any two polyhedra with the same area scissors-congruent? His student Max Dehn answered this in less than a year: no!

The cube, and a regular tetahedron with the same volume, are not scissors-congruent. To prove this Dehn found another invariant of scissors-congruence beside the volume!

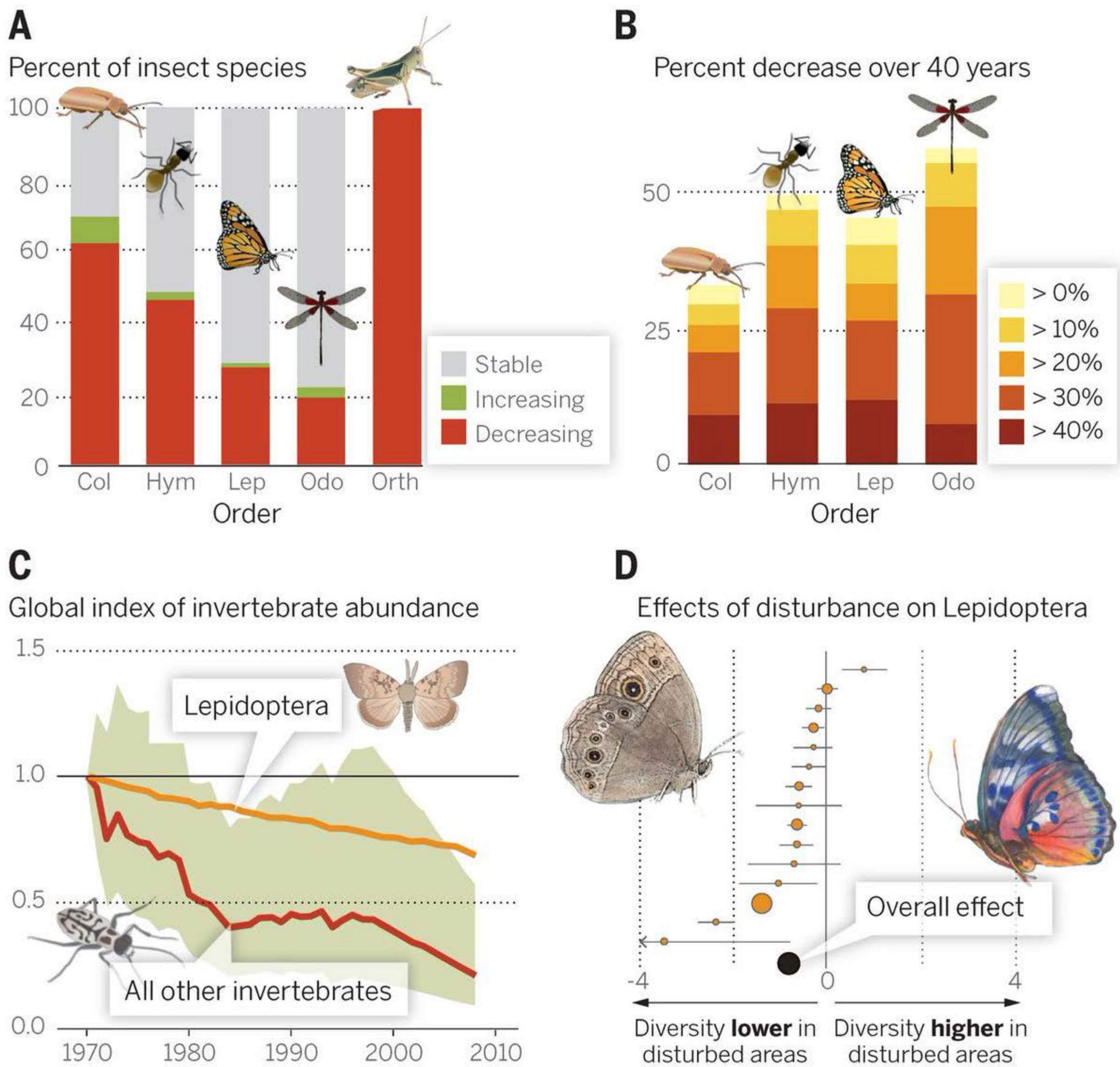
But this is just the beginning of the story, which now involves hyperbolic space, algebraic K-theory, and even ideas connected to quantum gravity, like the famous "pentagon identity" arising from two ways to chop this shape into tetrahedra:



For more:

- Inna Zakaharevich, Perspectives on scissors congruence, *Bulletin of the AMS* **53** (2016), 269–294.

October 15, 2018

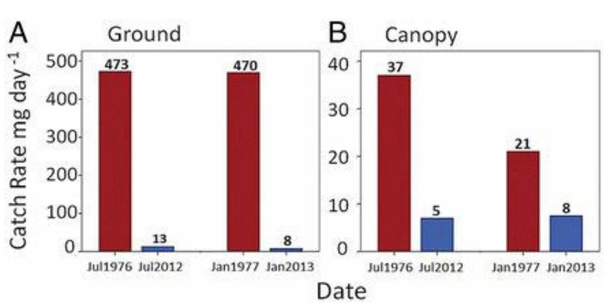


The population of insects has been crashing worldwide. Butterflies and moths are down 35% in the last 40 years. Other invertebrates are down even more:

- Rodolfo Dirzo, Hillary S. Young, Mauro Galetti, Gerardo Ceballos, Nick J. B. Isaac and Ben Collen, [Defaunation in the Anthropocene](), *Science*, **345** (2014), 401–406.

A new study from Puerto Rico is even more disturbing. Between January 1977 and January 2013, the catch rate in sticky ground traps fell 60-fold. Says one expert: "Holy crap".

- Susan Bird, [Insects are disappearing from Puerto Rico's rainforests at an astonishing rate](), *Care2*, October 22, 2018.



Here we see the milligrams of arthropods (like insects) caught in sticky traps on the ground (A) and forest canopy (B) in a rain forest in Puerto Rico. Dramatic collapse! And it hurts the whole ecosystem: lizard populations are halved or worse. For more details, read the actual study:

- Bradford C. Lister and Andres Garcia, Climate-driven declines in arthropod abundance restructure a rainforest food web, *Proceedings of the National Academy of Sciences*, October 30, 2018.

October 18, 2018

JUST WHEN YOU THOUGHT
YOU UNDERSTOOD THE PATTERN

The smallest number with the digits from 1 to 2 arranged in all possible orders has $1! + 2! = 3$ digits: 121.

The smallest number with the digits from 1 to 3 arranged in all possible orders has $1! + 2! + 3! = 9$ digits: 123121321.

The smallest number with the digits from 1 to 4 arranged in all possible orders has $1! + 2! + 3! + 4! = 33$ digits: 123412314231243121342132413214321.

The smallest number with the digits from 1 to 5 arranged in all possible orders has $1! + 2! + 3! + 4! + 5! = 153$ digits.

Does the smallest number with the digits from 1 to 6 arranged in all possible orders have $1! + 2! + 3! + 4! + 5! + 6! = 873$ digits?

No: 872 is enough. Or maybe even fewer — nobody knows yet!

New progress on a hard problem: Greg Egan broke the record for the shortest number with the digits 1 through 7 arranged in all possible orders. He found one with just 5908 digits.

Here is Egan's solution. It's 4 digits shorter than the previous best, and he got it using ideas from Aaron Williams' work on Hamiltonian cycles on Cayley graphs:

```
123456712345617234516723451762345172634517236451723465172345612734516273451267345216734526173452671345267314526734152673451
276345217634527163452761345276314527634152763451273645217364527136452731645273614527364152736451273465217346527134652731465
273416527346152734651273456123745162374512637452163745261374526317452637145263741526374512367452136745231674532167453126745
316274531672453167425316745236174532617453621745361274536172453617425361745236714532671453627145367214536712453671425367145
236741532674153627415367241536742153674125367415236745123764521376452317645321764531276453172645317624531764253176452371645
327164537216453712645371624537164253716452376145327614537261453762145376124537614253761452376415327641537264153762415376421
537641253764152376451237465213746523174653217465312746531724653174265317462531746523714653271465372146537124653714265371462
537146523741653274165372416537421653741265374162537416523746153274615372461537426153746215374612537461523746512374561234751
623475126347521634752613475263147526341752634715263475123647521364752316475321647531264753162475316427531647253164752361475
326147536214753612475361427536147253614752364175326417536241753642175364127536417253641752364715326471536247153642715364721
536471253647152364751234675213467523146753214675312467531426753146275314672531467523416753241675342167543216754231675421367
542163754216735421675341267543126754132675412367541263754126735412675341627543162754136275416327541623754162735416275341672
543167254136725416372541673254167235416725341675234617532461753426175432617542361754263175426137542617354261753462175436217
546321754623175462137546217354621753461275436127546312754613275461237546127354612753461725436172546317254613725461732546172
354617253461752346715324671534267154326715423671542637154267315426713542671534627154362715463271546237154627315462713546271
534672154367215463721546732154672315467213546721534671254367125463712546731254671325467123546712534671523467512347652134765
231476532147653124765314276531472653147625314765234176532417653421765432176542317654213765421736542176354217653412765431276
541327654123765412736541276354127653417265431726541372654173265417236541726354172653417625431762541376254173625417632541762
354176253417652347165324716534271654327165423716542731654271365427163542716534721654372165473216574321657342165732416573214
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657134265713246571326457132654712365741236571423657124365712346571236457123654712635741263571426357124635712643571263457126
354712653471625437162547316257431625734162573146257316425731624573162547136257413625714362571346257136425713624571362547163
257416325714632571643257163425716324571632547162357416235714623571642357162435716234571623547162534716523476153247615342761
543276154237615427361542763154276135427615347261543726154732615743261573426157324615732641573261457326154723615742361572436
157234615723641572361457236154726315742631572463157264315726341572631457263154726135742613572461357264135726143572613457261
354726153476215437621547362157436215734621573642157362415736214573621547632157463215764321567432156473215643721564327156432
175643215763421567342156374215634721563427156342175634215763241567324156372415632741563247156324175632415763214567321456372
145632714563217456321475632145763215476231574623157642315674231564723156427315642371564231756423157624315672431562743156247
315624371562431756243157623415672341562734156237415623471562341756234157623145672314562731456237145623174562314756231457623
154762135746213576421356742135647213564271356421735642137564213576241356724135627413562471356241735624137562413576214356721
435627143562174356214735621437562143576213456721345627134562173456213745621347562134576213547621534761254376125473612574361
257346125736412573614257361245736125476312574631257643125674312564731256437125643172564312756431257634125673412563741256347
125634172563412756341257631425673142563714256317425631472563142756314257631245673124563712456317245631274563124756312457631
254761325746132576413256741325647132564173256413725641327564132576143256714325617432516743251764325174632517436251743265174
325614732516473251467325147632514736251473265147325614372516437251463725143672514376251437265143725614327516432751463275143
627514326751432765143275614325761342567134256173425167342517634251736425173462517342651734256137425163742513674251376425137
462513742651374256134725163472513647251346725134762513472651347256134275163427513642751346275134267513427651342756134257613
245671324561732451673245176324517362451732645173246517324561372451637245136724513762451372645137246513724561327451632745136
274513267451327645132746513274561324751632475136247513264751324675132476513247561324576132547612357461235764123567412356471
235641723564127356412375641235761423567142356174235167423517642351746235174263517423651742356147235164723514672351476235147
263514723651472356142735164273514627351426735142763514273651427356142375164237514623751426375142367514237651423756142357612
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431652743615274365127435612473516247351264735216473526147352641735264713526473152647351246735214673524167352461735246713524
673152467351247635214763524176352471635247613524763152476351247365214736524173652471365247316524736152473651247356124375162
437512643752164375261437526413752643175264371526437512463752146375241637524613752463175246371524637512436752143675241367524
316752436175243671524367512437652143765241376524317652437165243761524376512437561243576123457612354761253476152347651234756
1234
```

But what's the pattern? What's really going on?

Robin Houston noticed that the new winner containing all permutations of $n = 7$ digits has

$$n! + (n-1)! + (n-2)! + (n-3)! + (n-3) = 5908$$

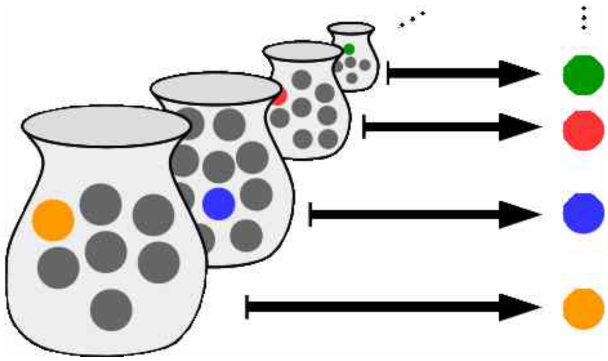
digits. And Egan also found new winners for $n = 8$ and $n = 9$ which also fit this pattern!

However, this formula does *not* correctly predict the length of the winners for $n = 3$ and $n = 4$... so the true pattern is more complicated.

After I wrote the above on Twitter, progress continued, with some surprising twists. I persuaded Erica Klarreich to write an article about the story, so go here for the rest:

- Eric Klarreich, [Mystery math whiz and novelist advance permutation problem](), *Quanta*, November 5, 2018.

October 27, 2018



The axiom of choice says that given any collection of bins, each containing at least one object, it is possible to make a selection of exactly one object from each bin, even if the collection is infinite. It has shocking consequences... but so does its negation!

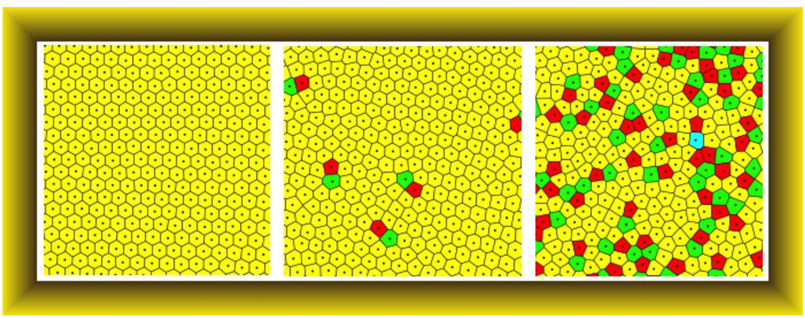
The axiom of choice implies that we can't define lengths, areas, and volumes for all sets in a way that obeys a short list of reasonable-sounding rules. Indeed, we can chop a ball into 5 subsets and rearrange them to get 2 balls: the "Banach-Tarski paradox".

If we assume the *negation* of the axiom of choice, we can make this problem go away. Technically: Solovay found models of ZF¬C (set theory and the negation of axiom of choice) where all subsets of Euclidean space are measurable! But that doesn't mean life is trouble-free.

Indeed, in all known models of ZF¬C, there is a set that can be partitioned into strictly more equivalence classes than the original set has elements! (More precisely, there's no injection from the set of equivalence classes to the original set.)

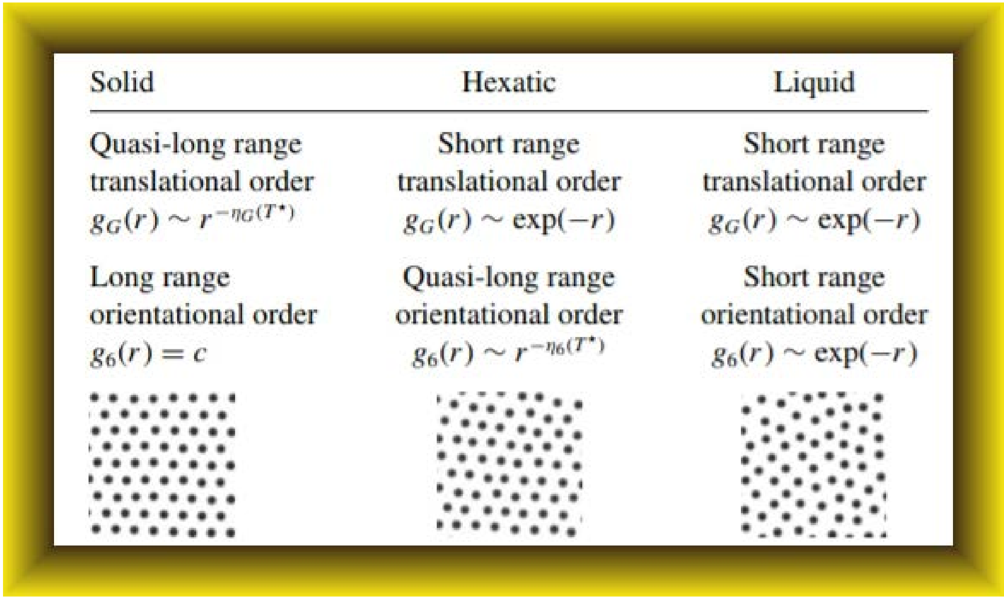
It's hard to find axioms that give just what we want, and nothing freaky.

October 31, 2018



Today I learned: unlike in our universe, in 2d space solids melt in two separate stages! Solid, hexatic, liquid — as shown above.

This has been discussed by theoretical physicists since the 1970s, but in 2013 it was confirmed experimentally using a layer of small magnetized beads. They saw *two* phase transitions as we shake these beads harder and harder, shifting them from a solid phase to a liquid!



Solid	Hexatic	Liquid
Quasi-long range translational order $g_G(r) \sim r^{-\eta_G(T^*)}$	Short range translational order $g_G(r) \sim \exp(-r)$	Short range translational order $g_G(r) \sim \exp(-r)$
Long range orientational order $g_6(r) = c$	Quasi-long range orientational order $g_6(r) \sim r^{-\eta_6(T^*)}$	Short range orientational order $g_6(r) \sim \exp(-r)$

The solid phase has long-range correlations between the *orientation* of the rows of beads: that's what 'long range orientational order' means. The actual *positions* of the beads are also correlated, but these correlations decay according to a power law that's what 'quasi-long range translational order' means.

In the hexatic phase, the orientational correlations decay with distance following a power law!

In the liquid phase, they decay exponentially.

In 3d, crystals can have long-range translational order. In the 1960s, Mermin and Wagner proved this is impossible in 2d if the particle only interact with short-range forces. In the 1970s, Kosterlitz, Thouless, Halperin, Nelson and Young worked out how solids melt in 2d, and discovered that there should be a hexatic phase between solid and liquid.

For details, see:

- Group for Research and Applications in Statistical Physics, Solid-hexatic-liquid transitions in a granular monolayer, June 18, 2013.

For my November 2018 diary, go here.

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home

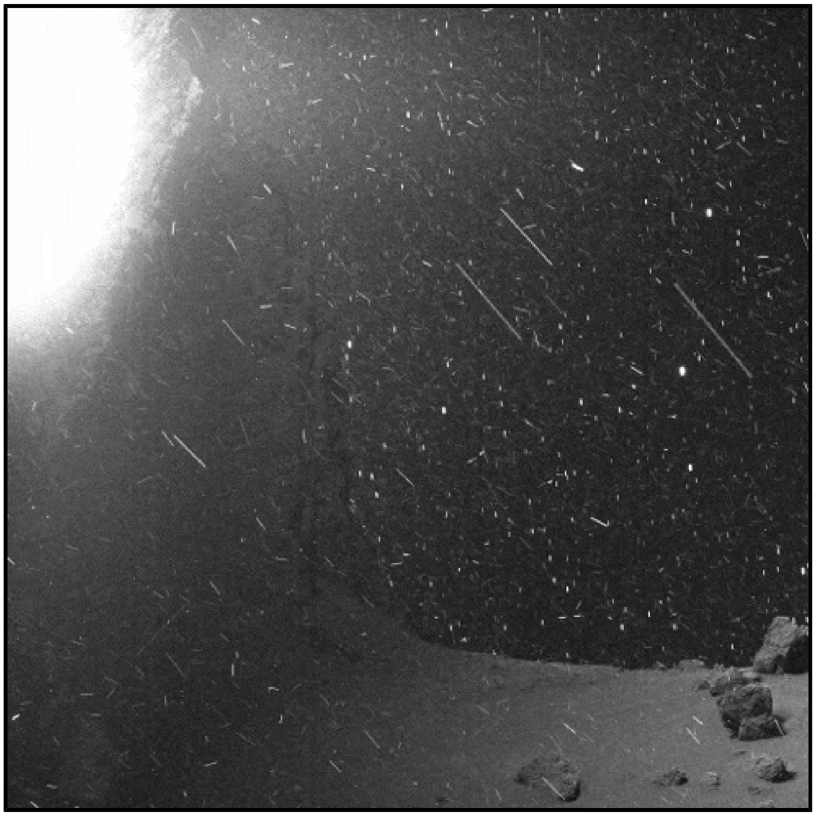
For my October 2018 diary, go here.

Diary — November 2018

John Baez

November 1, 2018

The Cliffs of Hathor



This movie was made from 25 minutes of photos taken by the Rosetta spacecraft when it was several kilometers from comet Churyumov-Gerasimenko. They were taken on June 1st, 2016. On September 30th of that year, Rosetta was deliberately crashed into the comet and the mission ended. These photos were nicely assembled into an animated gif by landru79 on Twitter just recently.

This place is called the Cliffs of Hathor. The "snow" is dust moving slowly; you can also see some stars moving downward in the background, due to the rotation of the comet.

November 2, 2018

Please read this report — it's important:

- World Wildlife Fund, *Living Planet Report 2018*.

Starting on the 47th page of the pdf, you can see information about the crash of mammal, bird, reptile, amphibian and fish populations worldwide:

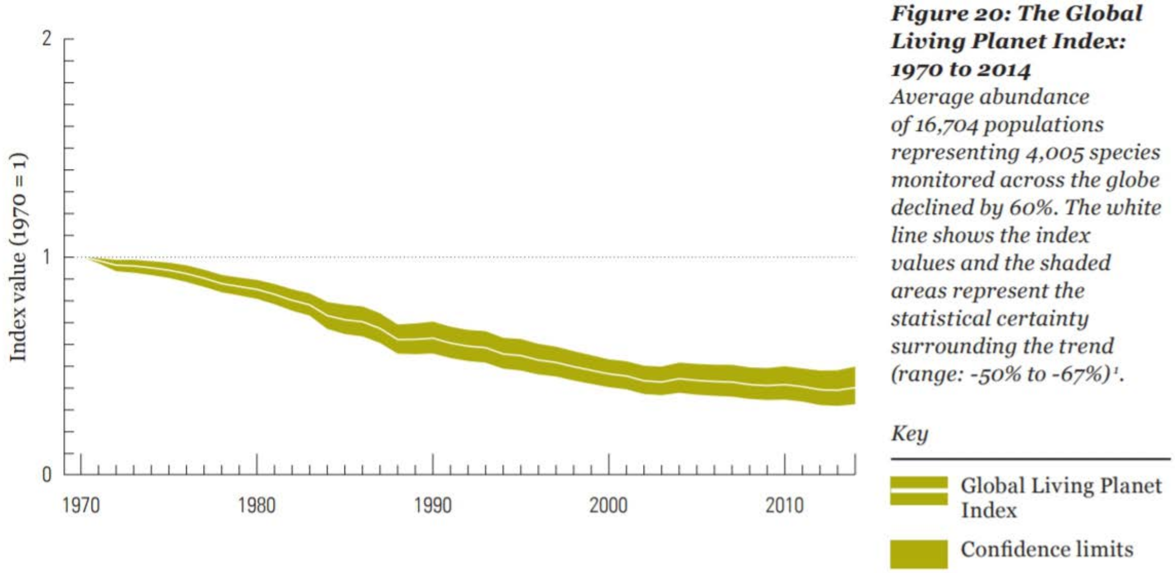


Figure 20: The Global Living Planet Index: 1970 to 2014
Average abundance of 16,704 populations representing 4,005 species monitored across the globe declined by 60%. The white line shows the index values and the shaded areas represent the statistical certainty surrounding the trend (range: -50% to -67%)[1].

The situation is worse for freshwater species: they're down by about 83% since 1970:

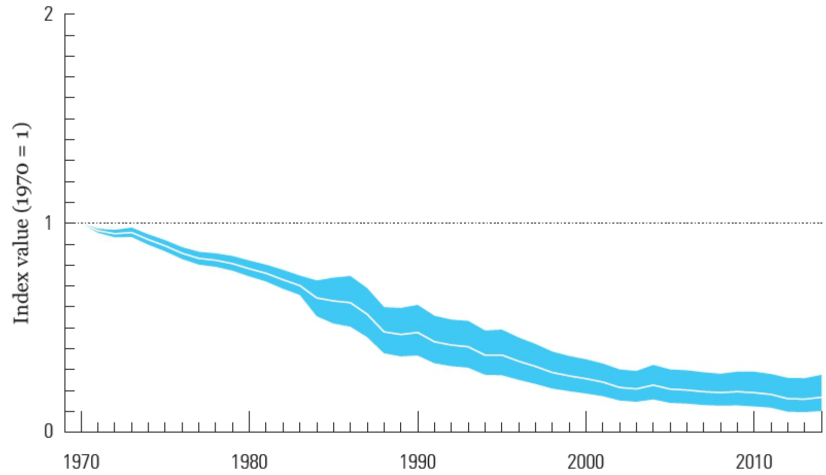
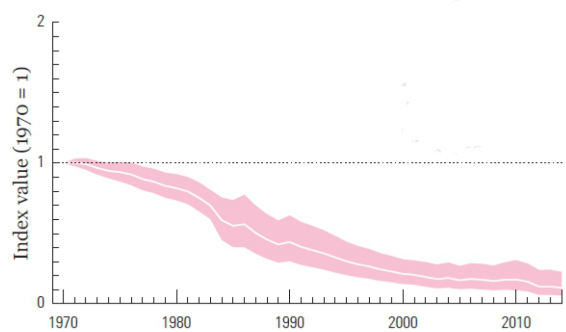


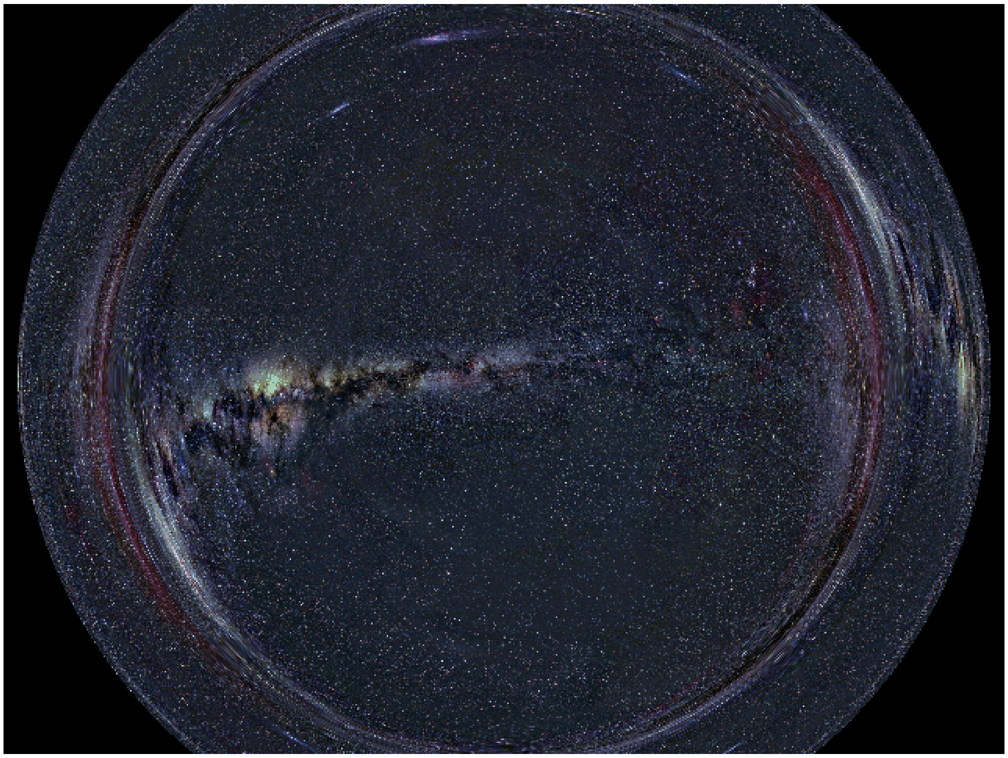
Figure 24: The Freshwater Living Planet Index: 1970 to 2014
The average abundance of 3,358 freshwater populations representing 880 species monitored across the globe declined by 83%. The white line shows the index values and the shaded areas represent the statistical certainty surrounding the trend (range -73% to -90%)[1].

On land, the worst declines are occurring in the "Neotropical" region: Central and South America. Mammals, birds, reptiles and amphibian population have dropped by about 89% since 1970.



The Neotropical Living Planet Index
The average abundance for 1,040 populations (representing 689 species) in the Neotropical biogeographic realm. Vertebrate populations declined by an average of 4.8% annually between 1970 and 2014, translating to an overall decline of 89%. This represents the most dramatic change in any biogeographic realm[1].

November 6, 2018



Slowly lower yourself toward the event horizon of black hole. As you do, look up. Your view of the outside universe will shrink to a point — and become brighter and brighter, tending to infinite brightness!

These effects don't happen if you simply let yourself fall in to the black hole. If you do that, your view of the outside world will not shrink to a point, and the light you see will not be intensified by blueshifting — because you'll be falling along with it!

Andrew Hamilton made this animated gif. See more here:

- Andrew Hamilton, Journey into a Schwarzschild black hole.

November 11, 2018

Good news: starting early next year, Eddie Bernice Johnson will be the first African-American to lead the House Committee on Science, Space, and Technology. She'll also be the first woman to do it. She's the first Democrat to lead this committee since 2011. She's the first who isn't a climate science denier since 2011. And she's the first with some training in *science* since the 1990s: she was chief psychiatric nurse at the Dallas Veteran's Administration Hospital for 16 years.

- Rebecca Leber, Science returns to the House, *Mother Jones*, November 7, 2018.

November 13, 2018

11111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111
11111111111111111111111111111111111111111111111111
11111111111111111 is prime!

This number has 317 digits, all ones. It's prime. 317 is also prime!

That's not a coincidence. A number whose digits are all 1 can only be prime if the number of digits is prime!

This works in any base, not just base ten. Can you see the quick proof?

A prime whose decimal digits are all ones is called a "repunit prime". The largest known repunit prime has 1031 digits.

Decimal repunit primes

R_n is prime for n = 2, 19, 23, 317, 1031, ... (sequence A004023 in OEIS). R_{49081} and R_{86453} are probably prime. On April 3, 2007 Harvey Dubner (who also found R_{49081}) announced that R_{109297} is a probable prime. He later announced there are no others from R_{86453} to R_{200000}. On July 15, 2007 Maksym Voznyy announced R_{270343} to be probably prime, along with his intent to search to 400000. As of November 2012, all further candidates up to $R_{2500000}$ have been tested, but no new probable primes have been found so far.

It has been conjectured that there are infinitely many repunit primes and they seem to occur roughly as often as the prime number theorem would predict: the exponent of the Nth repunit prime is generally around a fixed multiple of the exponent of the $(N-1)$th.

Mathematicians believe there are infinitely many repunit primes, but nobody can prove it yet.

Why does this matter?

The density of primes decreases slowly, like $1/\ln(N)$. So if numbers whose digits have 'no good reason not to be prime', there should be infinitely many of them. This idea gives a probabilistic argument that there should be infinitely many repunit primes.

But what does probability really mean when it comes to prime numbers? God didn't choose them by rolling dice!

This is why silly-sounding puzzles about primes can actually be important: they challenge our understanding of randomness and determinism.

There might be infinitely many true facts about primes that are true just because it's overwhelmingly 'probable' that they're true... but not for any reason we can convert into a proof.

However, even this has not yet been proved.

Clouds of mystery surround us.

November 22, 2018

It's Thanksgiving!

Hamilton's equations

Suppose you have a particle on the line whose position q and momentum p are functions of time, t. If the energy H is a function of position and momentum, **Hamilton's equations** say:

$$\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \\ \frac{dq}{dt} &= \frac{\partial H}{\partial p} \end{aligned}$$

I am thankful for the beauty of mathematics and physics, which always go deeper than I expect.

For example, Hamilton's equations describe the motion of a particle if you know its energy. But they turn out to look a lot like Maxwell's relations in thermodynamics!

Maxwell's relations

There are lots of Maxwell relations, and that's one reason people hate them. But let's just talk about two; most of the others work the same way.

Suppose you have a physical system like a box of gas that has some volume V, pressure P, temperature T and entropy S. Then the first and second **Maxwell relations** say:

$$\begin{aligned} \left.\frac{\partial T}{\partial V}\right|_S &= -\left.\frac{\partial P}{\partial S}\right|_V \\ \left.\frac{\partial S}{\partial V}\right|_T &= \left.\frac{\partial P}{\partial T}\right|_V \end{aligned}$$

Maxwell's relations connect the temperature, pressure, volume and entropy of a box of gas — or indeed, a box of *anything* in equilibrium. Nobody told me they're just Hamilton's equations with different letters and vertical lines thrown in.

So I decided to see what happens if I wrote Hamilton's equations in the same style as the Maxwell relations. It freaked me out at first. What does it mean to take the partial derivative of q in the t direction while holding p constant?

Comparison

Clearly Hamilton's equations resemble the Maxwell relations. Please check for yourself that the patterns of variables are exactly the same: only the names have been changed! So, apart from a key subtlety, Hamilton's equations *become* the first and second Maxwell relations if we make these replacements:

$$\begin{array}{ccccccc} q & \to & S & \quad & p & \to & T \\ t & \to & V & \quad & H & \to & P \end{array}$$

What's the key subtlety? One reason people hate the Maxwell's relations is they have lots of little symbols like $|_V$ saying what to hold constant when we take our partial derivatives. Hamilton's equations don't have those.

So, you probably won't like this, but let's see what we get if we write Hamilton's equations so they *exactly* match the pattern of the Maxwell relations:

$$\left.\frac{\partial p}{\partial t}\right|_q = -\left.\frac{\partial H}{\partial q}\right|_t$$

$$\left.\frac{\partial q}{\partial t}\right|_p = \left.\frac{\partial H}{\partial p}\right|_t$$

But it turns out to be okay. Indeed, this was a useful clue. I thought about it longer and realized what was going on.

You get equations like Hamilton's whenever a system *extremizes something subject to constraints*. A moving particle minimizes action; a box of gas maximizes entropy.

Read the details here:

- John Baez, Classical mechanics versus thermodynamics (part 1), *Azimuth*, January 19, 2012.
- John Baez, Classical mechanics versus thermodynamics (part 2), *Azimuth*, January 23, 2012.

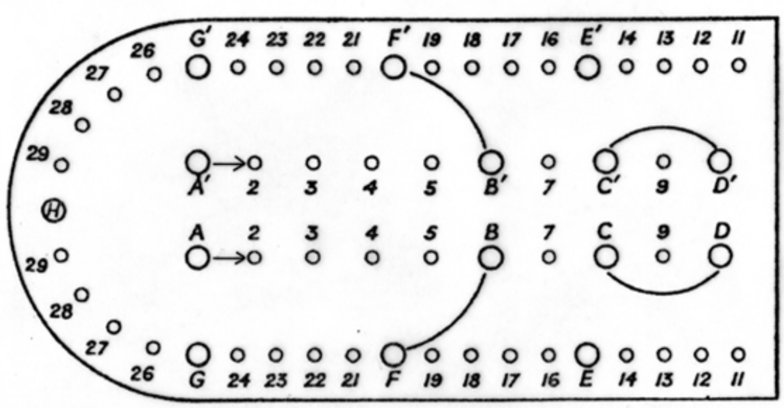
So: whenever you see unexplained patterns in math or physics, write them down in your notebook. Think about them from time to time. Clarify. Simplify.

Soon you'll never be bored. And if you get stuck and frustrated, just ask people. True seekers will be happy to help.

November 24, 2018

The game of "58 holes", or "hounds and jackals", is very ancient. Two players took turns rolling dice to move their pieces forward. This copy comes from Thebes, Egypt. It was made in the reign of Amenemhat IV, during 1814â€“1805 BC, in the [Twelfth Dynasty](#) of the Middle Kingdom. It's now at The Metropolitan Museum of Art.

But why 58 holes? That's a strange number!



The holes come in two groups of 29. Nobody knows the rules for sure! But the Russian game expert Dmitriy Skiryuk argued that the players move their pieces from holes A to 29 and then the large shared hole H, where they exit the board.

If so, each player really has 30 holes! That makes more sense: the number 60 was very important in Egypt and the Middle East. So "58 holes" is a red herring.

You can see Skiryuk's hypothesized rules, the above pictures, and more here:

- Eli Gurevich, [Hounds and jackals](#), *Ancient Games*, October 14, 2017.

The game was really widespread: here's one from a pillaged Iron Age tomb in Necropolis B at [Tepe Sialk](#), Iran. It's now in the Louvre.

But here's something even cooler. The game was just found in Azerbaijan, almost 2000 kilometers from the Middle East — chiseled into a rock by Bronze Age herders!

Read more here:

- Bruce Bower, A Bronze Age game called 58 holes was found chiseled into stone in Azerbaijan, *Science News*, November 16, 2018.

I guess nobody can resist a good game. Someone should popularize this one again.

For my December 2018 diary, go here.

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For my November 2018 diary, go here.

Diary — December 2018

John Baez

December 3, 2018

> **PEIRCE'S LAW:**
>
> $$((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$$
>
> **It's freaky but it follows from the law of excluded middle. The law of excluded middle says either P is true or P is false. If P is true then *anything* implies P, so $(P \Rightarrow Q) \Rightarrow P$ implies P. If P is false, $P \Rightarrow Q$ is true, so $(P \Rightarrow Q) \Rightarrow P$ is just another way of saying P. Either way, Peirce's law holds!**

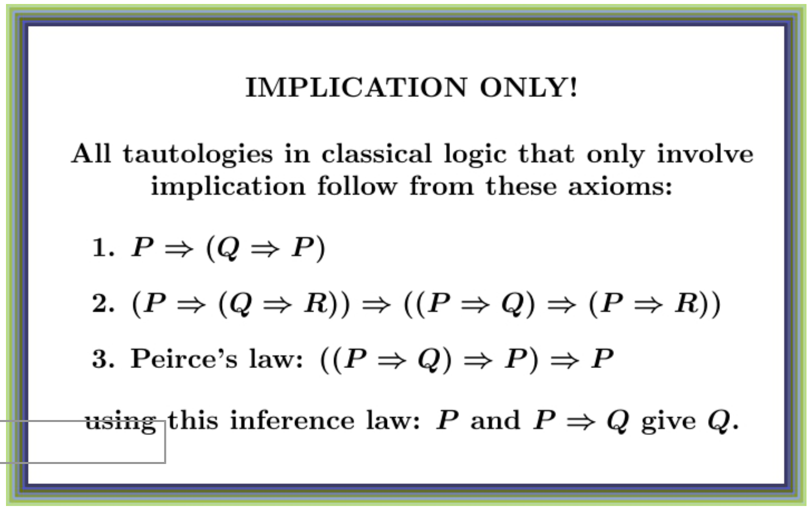
Logic can be counterintuitive. Take Peirce's law: if you can prove that P having *any* consequence would imply P, then you can prove P.

Peirce's law a far-out consequence of the law of excluded middle (also known as *tertium non datur*) and the law saying that a falsehood implies anything (also known as *ex falso quodlibet*). If you don't like Peirce's law, maybe you secretly don't like one of these other laws of classical logic!

Here's an example. Suppose your friend says "If working hard implies I'll get the job done, then I'll be working hard". According to Peirce's laws, this implies he'll be working hard!

To understand this shocking consequence, follow the argument in the box. Either your friend is working hard or not. If he is, then we're done. If not, "working hard implies I'll get the job done" is true, since a falsehood implies anything. So, the statement "If working hard implies I'll get the job done, then I'll be working hard" is equivalent to "I'll be working hard."

December 4, 2018



Logic is all about what implies what... so it's fun to think about 'implicational logic', where the only logical connective we get to use is 'implies'.

Three axioms are all we need! Peirce's law is the spice in the pudding.

Jan Łukasiewicz, the grandfather of 'reverse Polish notation', showed that we only need *one* axiom for implicational logic. But this one axiom is quite weird.

We can define 'or' using 'implies'—do you see how? But we cannot get 'and'. In category theory we can define a closed category to be one with a functor

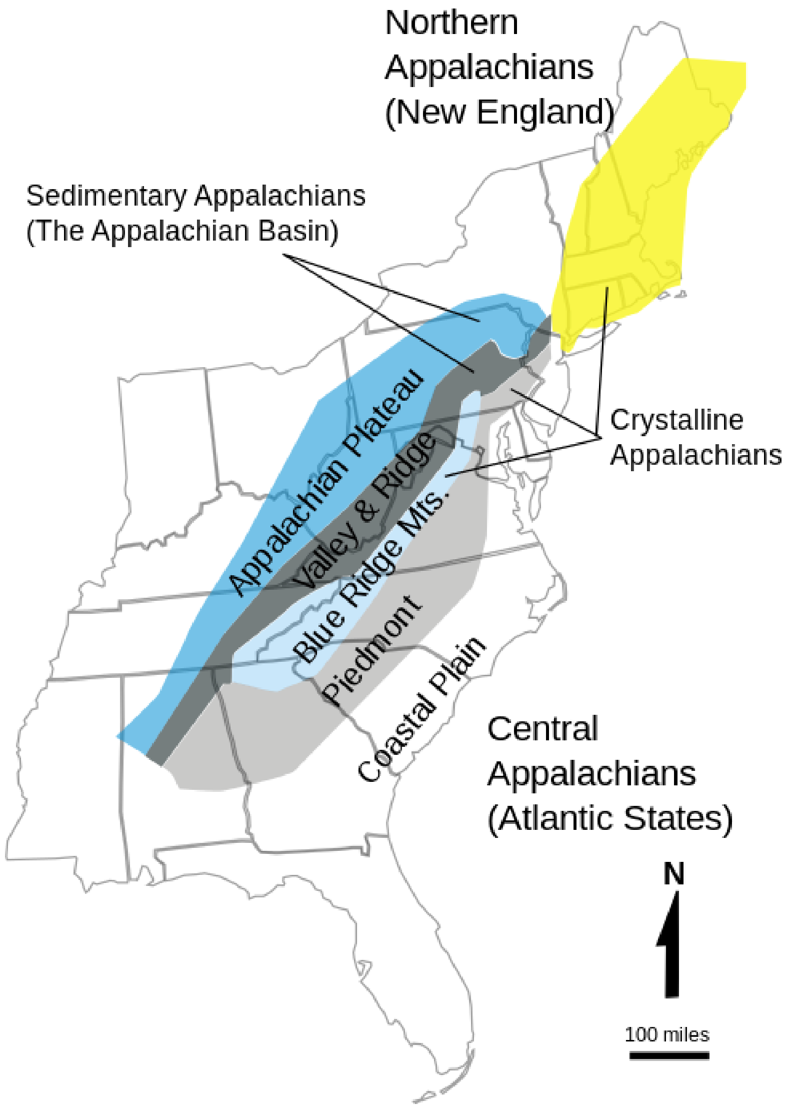
$$: C^{\mathrm{op}} \times C \to C$$

and a morphism from $p \quad q$ to $(r \quad p) \quad (r \quad q)$ for any objects p, q, r, obeying some axioms. This is a categorical version of purely implicational logic.

December 7, 2018

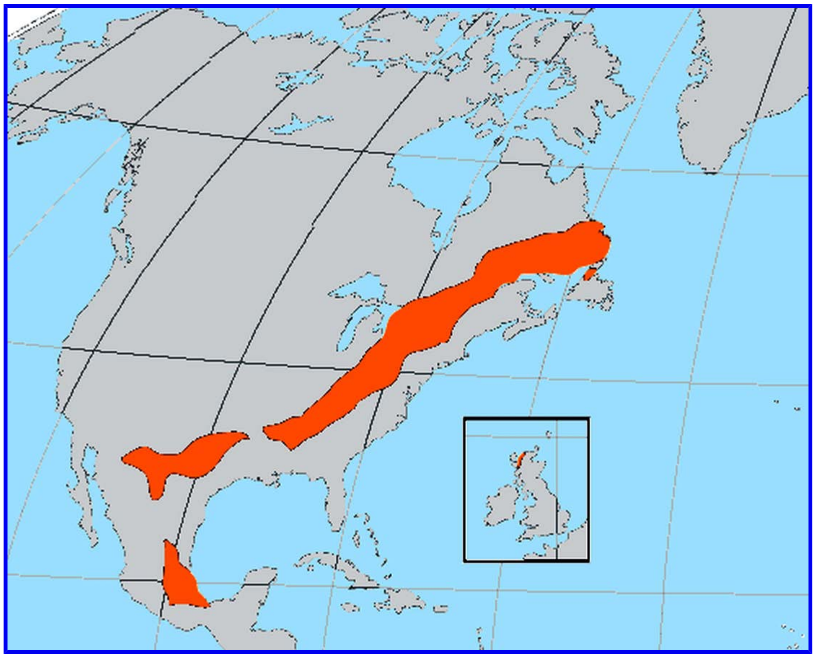
I just listened to some 'Piedmont blues'. Though I grew up nearby, I hadn't known about the American plateau called the Piedmont, named after the Italian one, from the Latin 'foot of the mountains'.

Bounded on the west by the Blue Ridge Mountains, it's very old!



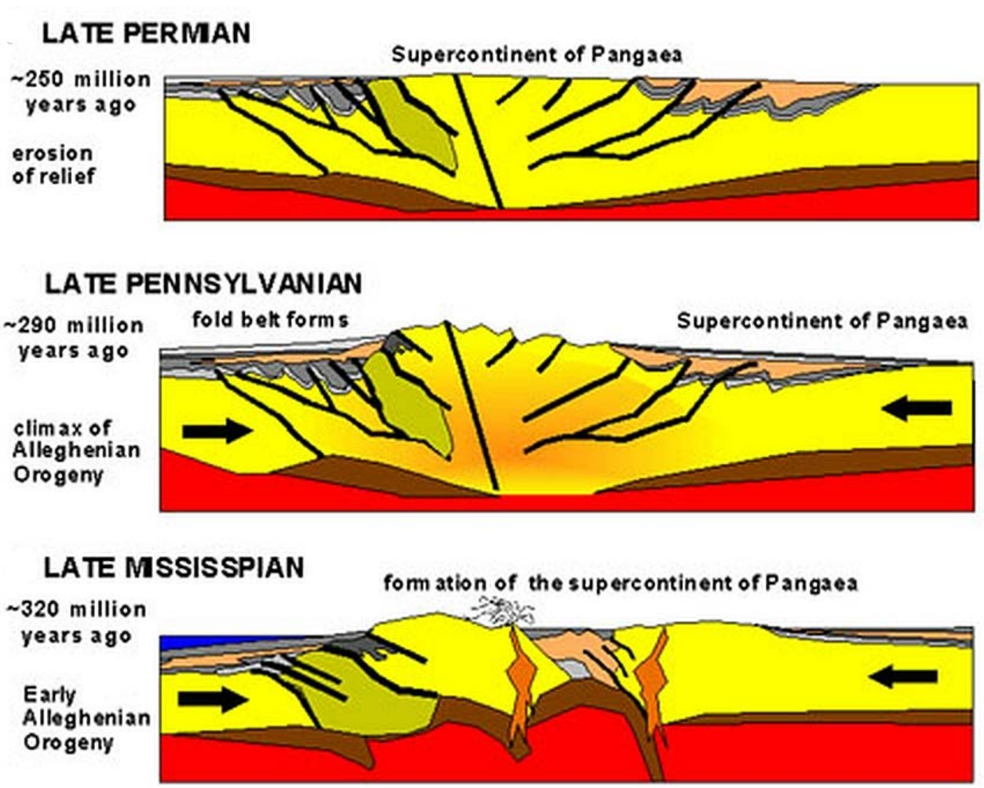
Geologists have identified at least five separate events which have led to sediment deposition in the Piedmont.

One is the 'Grenville orogeny' — the collision of continents that created the supercontinent Rodinia about 1 billion years ago! They're shown here:

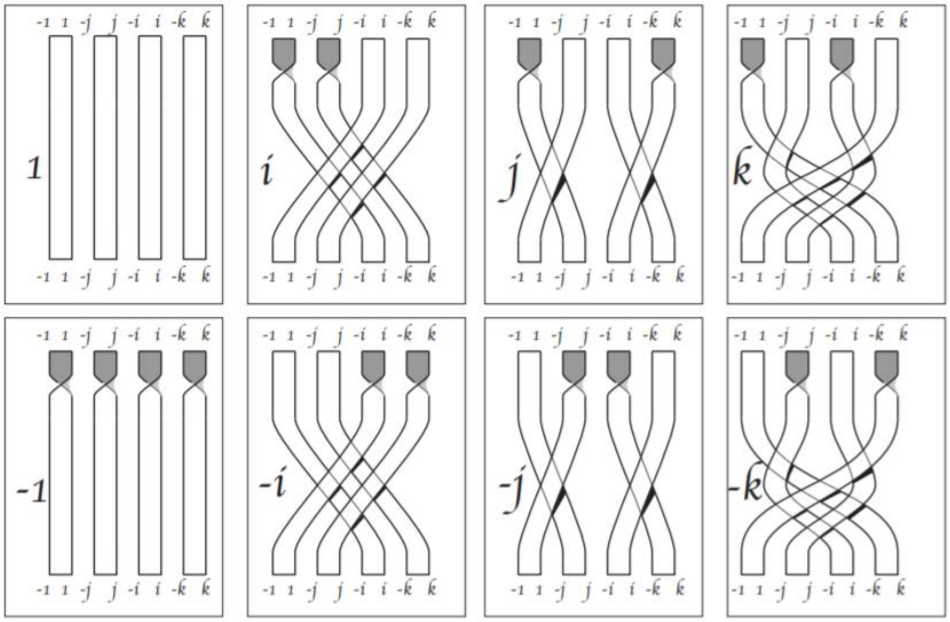


The Piedmont was also formed in the 'Alleghanian oregeny', when the Appalachians rose — as tall as the Rocky Mountains! — around 300 million years ago, when Euramerica hit Gondwana and formed the supercontinent Pangaea.

So the Piedmont blues have deep geological roots!



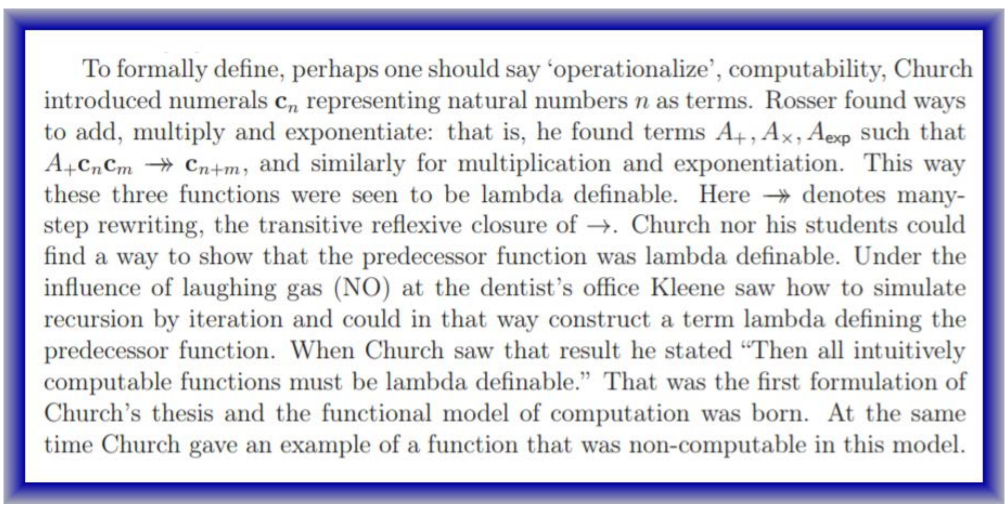
December 21, 2018



You can describe the quaternion group using ribbon braids! Four ribbons that can twist or cross over each other let you describe $\pm 1, \pm i, \pm j, \pm k$. Details here:

- Yongju Bae, J. Scott Carter and Byeorhi Kim, Amusing permutation representations of group extensions.

December 28, 2018



To formally define, perhaps one should say 'operationalize', computability, Church introduced numerals $\mathbf{c}_n$ representing natural numbers n as terms. Rosser found ways to add, multiply and exponentiate: that is, he found terms $A_+, A_\times, A_{\mathsf{exp}}$ such that $A_+\mathbf{c}_n\mathbf{c}_m \twoheadrightarrow \mathbf{c}_{n+m}$, and similarly for multiplication and exponentiation. This way these three functions were seen to be lambda definable. Here $\twoheadrightarrow$ denotes many-step rewriting, the transitive reflexive closure of $\to$. Church nor his students could find a way to show that the predecessor function was lambda definable. Under the influence of laughing gas (NO) at the dentist's office Kleene saw how to simulate recursion by iteration and could in that way construct a term lambda defining the predecessor function. When Church saw that result he stated "Then all intuitively computable functions must be lambda definable." That was the first formulation of Church's thesis and the functional model of computation was born. At the same time Church gave an example of a function that was non-computable in this model.

I'd never heard before that the Church-Turing thesis was born right after Kleene got dosed with laughing gas! This is from here:

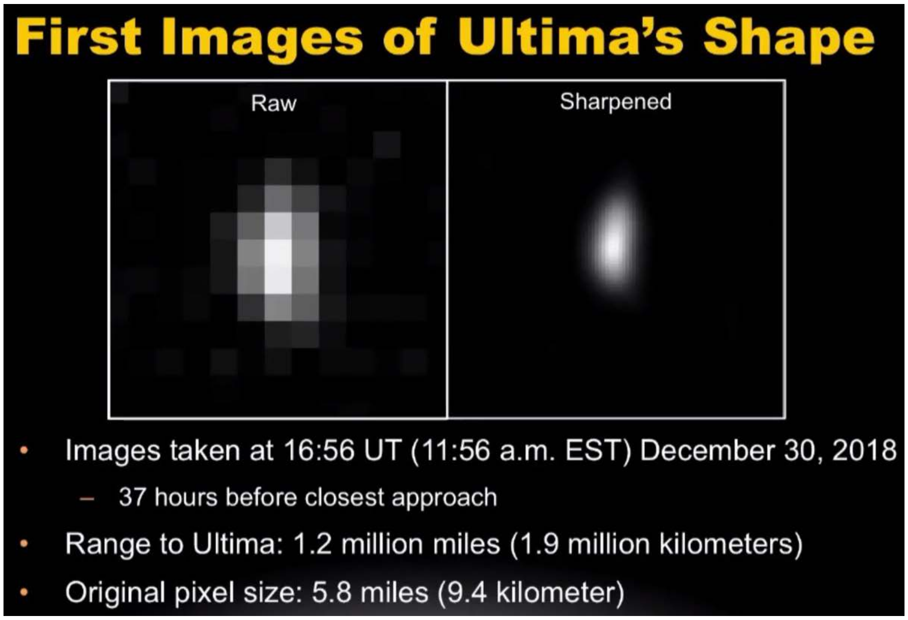
- Henk P. Barendregt, Gems of Corrado Böhm.

The 'predecessor' function subtracts 1 from any natural number except 0, which it leaves alone.

However, there are a few problems with this story. First, laughing gas is nitrous oxide, N_2O, not nitric oxide, NO. More importantly, this story is poised somewhere between fact and legend. For more, including another version of this story as told by Barendregt, go here:

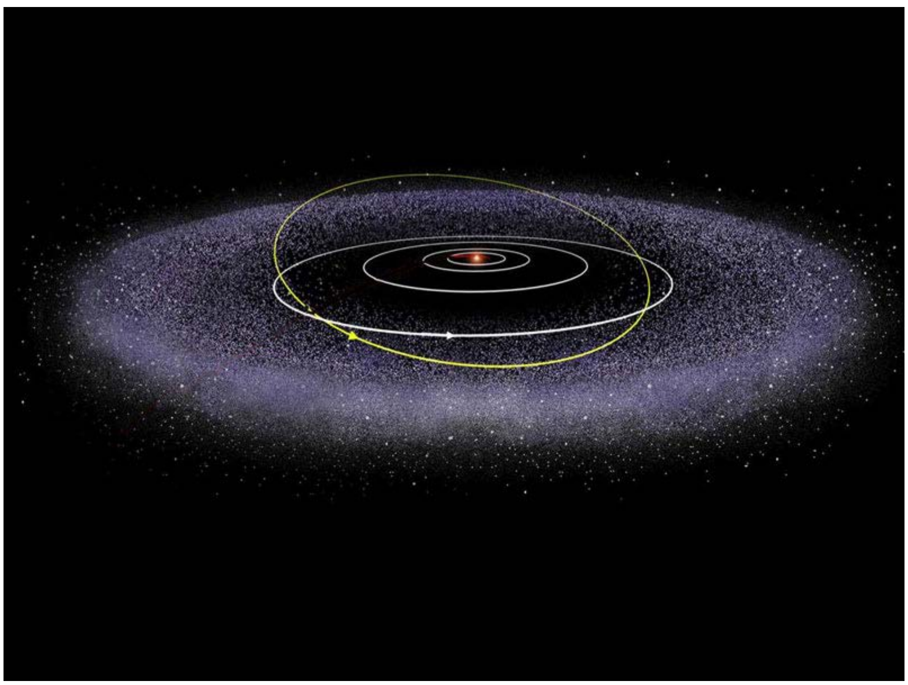
- Youchiza, Did Kleene discover the λ-calculus predecessor function while high on nitrous oxide?

December 31, 2018

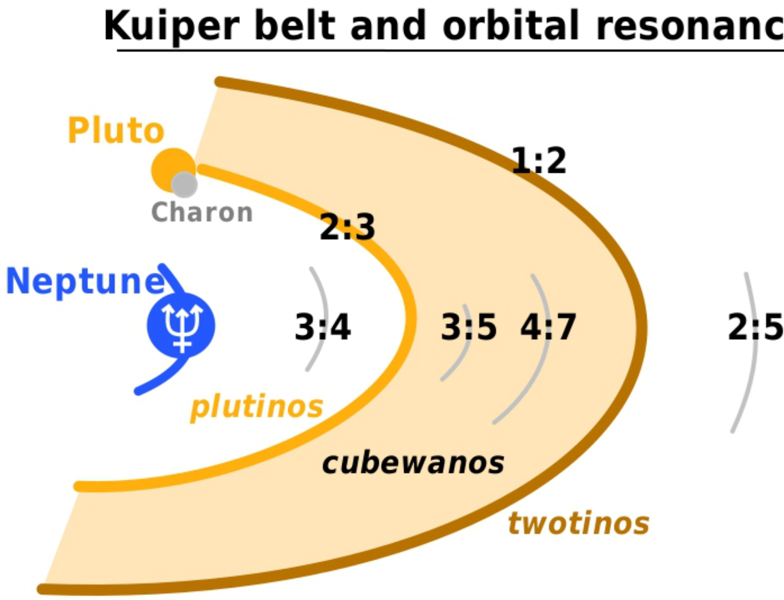


Here's the first real image of Ultima Thule—no longer just a single pixel, it's now 18 pixels!

Soon New Horizons will swoop within 3500 kilometers of this trans-Neptunian object, and get a much better view of its ancient material, perhaps undisturbed for 4 billion years! You see, Ultima Thule lies in the "Kuiper belt", along with many other fragments from the original disk around the Sun that failed to fully coalesce into planets. It's a bit like the asteroid belt — but 20 times bigger across, 200 times more massive, and much older.



Neptune rules over the Kuiper Belt. At the inner edge we find Pluto and other plutinos, which lie in a 2:3 resonance with Neptune, going around twice while Neptune goes around thrice. Over 248 of these are known at present. At the outer edge we have the twotinos, which move in a 1:2 resonance with Neptune. Only 50 of these have been found so far, suggesting that this resonance is less stable than the 2:3 resonance.



Ultima Thule is in between! It's a classical Kuiper belt object, meaning it's not in resonance with Neptune and it doesn't have a highly eccentric orbit like Pluto. Such objects are also called 'cubewanos', since the first to be found was called QB_1.

More specifically, Ultima Thule is a 'cold' Kuiper belt object, meaning it has an inclination of less than 5° and an almost circular orbit. It's believed that these objects were formed very early in the history of the Solar System and have been largely undisturbed ever since.

For my January 2019 diary, go here.

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