For my December 2019 diary, go here.

Diary — January 2020

John Baez

January 1, 2020

A fennec fox, *Vulpes zerda*, in the Sahara. This is the smallest canid in the world.

January 2, 2020

In analysis we generalize sequences to nets in order to handle topological spaces that are too big for sequences to get the job done. A sequence x_i has indices that are natural numbers. In a net, the indices can be in any directed set.

A function between metric spaces is continuous iff it maps every convergent sequence to a convergent sequence. This nice result *fails* for topological spaces. But a function between topological spaces is continuous iff it maps every convergent net to a convergent net!

A metric space is compact iff every sequence has a convergent subsequence. This nice result *fails* for topological spaces. But a topological space is compact iff every net has a convergent subnet!

But you have to be careful: the concept of 'subnet' is subtle. The definition is here:

- Wikipedia, Subnet.

The key nuance is that a net x_i with i ranging over some index set can have a subnet with some other index set... perhaps even a *larger* index set!

This leads to a puzzle that's bothered me for years. There are topological spaces that are compact where not every sequence has a convergent subsequence!

So the puzzle is: find such a space, and a sequence in that space that has no convergent subsequence, but has a convergent subnet!

By general theorems we know such a situation is *possible*, but I wanted to find an example that doesn't use the axiom of choice. The reason is that I wanted to know if there was an example that we can actually *get our hands on*.

I asked this question on Math Stackexchange in last year. After 9 months, Robert Furber gave a promising answer:

- John Baez, Sequence with convergent subnets but no convergent subsequences, *Math StackExchange*, March 22, 2019.

He begins: "Much to my surprise, there is an explicit example, and it comes about at least in part because it seems that the theorem going back and forth between cluster points and convergent subnets does not require the axiom of choice, when done the right way."

He also explains why another style of example *does* require a nonconstructive princple. The natural numbers embeds in its Stone–Čech compactification in such a way that it has no convergent subsequence. The existence of a convergent subnet implies the existence of a nonprincipal ultrafiter! But there are models of Zermelo-Fraenkel set theory where nonprincipal ultrafilters don't exist.

I still don't understand everything Furber wrote, but I'm pleased that finally, after decades of worrying about this problem, I may finally get some satisfaction.

The mill of math grinds slow, but it grinds exceedingly fine.

January 3, 2020

My wife and I just saw a bobcat outside our house!

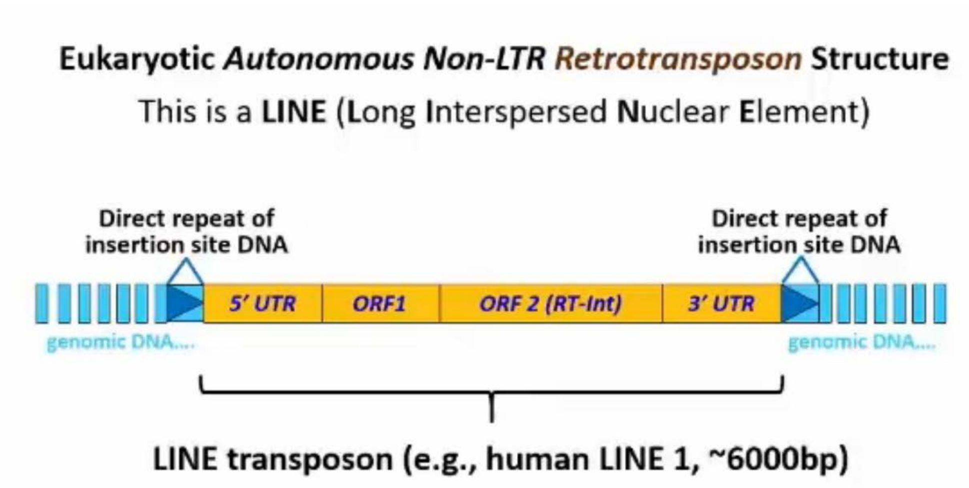
This is the second time she's seen it. It had tufted ears, just as a bobcat should. It stared at her, tail twitching. Then it walked away before we could photograph it.

January 4, 2020

Cute face. Nightmarish claws. It's the world's biggest bat: the Bismarck masked flying fox.

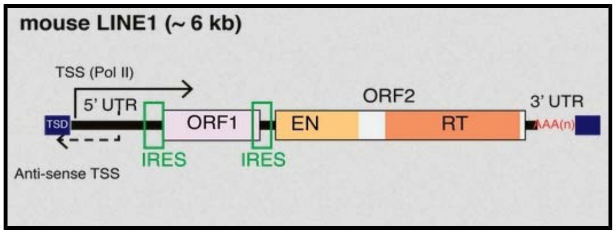
It lives in Papua New Guinea. Not only the females of this species, but also the males can give milk!

And its genome is much more efficient than yours.



About 45% of your DNA consists of transposons: sequences of genes that can copy themselves from one location to another. There are many kinds. Some act as parasites. Some cause diseases. Many become deactivated and lose the ability to hop around.

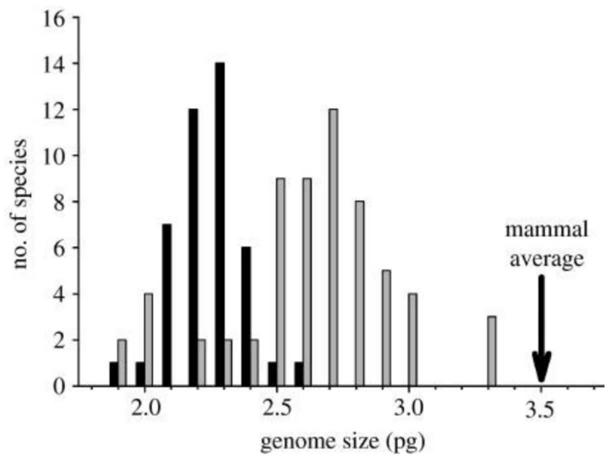
About 17% of your DNA is made of transposons called LINEs. Each one is about 7000 base pairs long. Most have lost the ability to replicate — but about 100 still can. As they cut and paste themselves from here to there, they can disrupt your genes and cause cancer.



There are different lineages of LINEs. In humans, the only kind that can still replicate on its own is called LINE1. The rest became inactive about 200 million years ago.

LINE1 is found in all mammals...

... except Old World bats, called megabats, like the Bismarck masked flying fox!



Here's the genome size of various Old World bats (black) and New World bats (gray) compared to the average for mammals. Bats have smaller genomes... and the Old World bats have completely lost their LINE1 junk DNA.

Why? *Birds* also have smaller genomes, with those that work harder to fly having the smallest. So there's a theory that the need for energy efficiency somehow produces evolutionary pressure to trim down the genome. But nobody really knows.

Here's the paper on the genome sizes of megabats where I got the above figure:

- Jillian D.L. Smith and T. Ryan Gregory, The genome sizes of megabats (Chiroptera: Pteropodidae) are remarkably constrained, *Biology Letters* **5** (2009), 347–351.

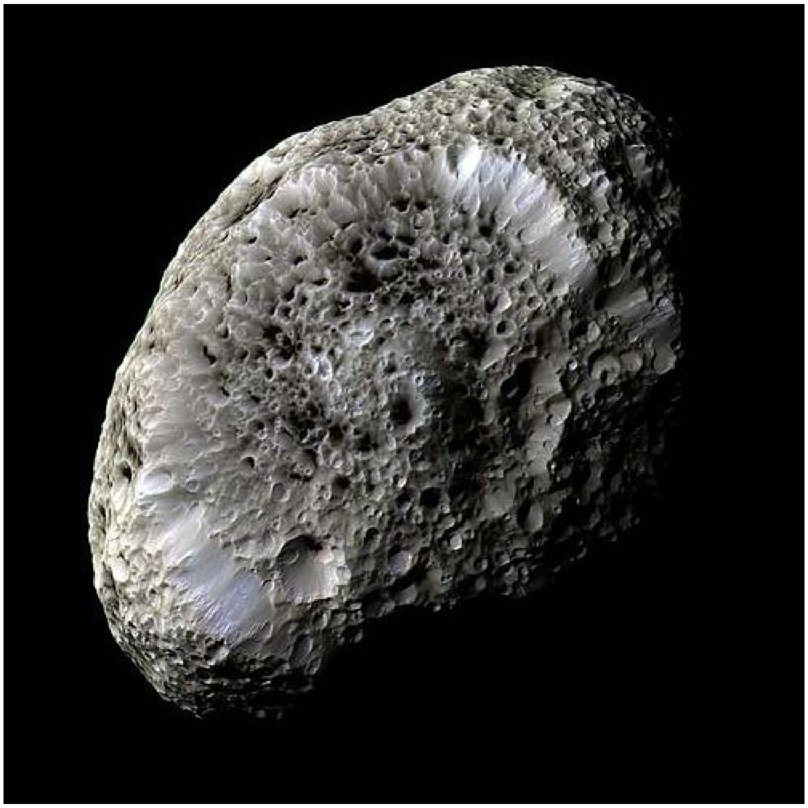
And here's some more information on LINEs:

- Wikipedia, Long interspersed nuclear element.

January 12, 2020

Could the Solar System be unstable? Could a planet eventually be thrown out of the Solar System?

People have done a lot of work on this problem. It's hard. The Solar System is chaotic in a number of ways.



Saturn's moon Hyperion wobbles chaotically thanks to interactions with Titan. It has a Lyapunov time of 30 days. That is, a slight change in its rotation axis gets magnified by a factor of *e* after 30 days. You can't really predict what it will do a year from now.

Pluto's moon Nix also rotates chaotically, thanks to its elongated shape and its interactions with Pluto's larger moon Charon. You could spend a day on Nix where the sun rises in the east and sets in the north! Watch the video to see how weird it is.

But what about planets?

Pluto is locked in a 2:3 resonance with Neptune. Apparently this creates chaos: uncertainties in Pluto's position in its orbit grow by a factor of e every 10â€“20 million years. This makes long-term simulations of the Solar System harder.

In 1989, Jacques Laskar showed that the Earth's orbit is chaotic. An error as small as 15 meters in measuring the position of the Earth today would make it completely impossible to predict where the Earth would be in its orbit 100 million years from now!

The planet Mercury is especially susceptible to Jupiter's influence. Why? Mercury's perihelion, the point where it gets closest to the Sun, precesses at a rate of about 1.5 degrees every 1000 years. Jupiter's perihelion precesses just a little slower.

In some simulations, Jupiter's gravitational tugs accumulate and pull Mercury off course 3-4 billion years from now. Astronomers estimate there's about a 1% probability that it could collide with Venus, the Sun, or Earth — or even be ejected from the Solar System!

- Jacques Laskar and Mickaël Gastineau, Existence of collisional trajectories of Mercury, Mars and Venus with the Earth, *Nature* **459** (2009), 817–819.

In this paper, Laskar and Gastineau simulated 2500 futures for the Solar System, changing the initial position of Mercury by about 1 meter. In 20 cases, Mercury went into a dangerous orbit! In one it passed within 6500 kilometers of Venus. In one it collided with Venus, and in three it collided with the Sun. In one it led to Mars coming within 800 kilometers of Earth 3.3443 billion years from now. This would probably disrupt of Mars, with large fragments hitting Earth.

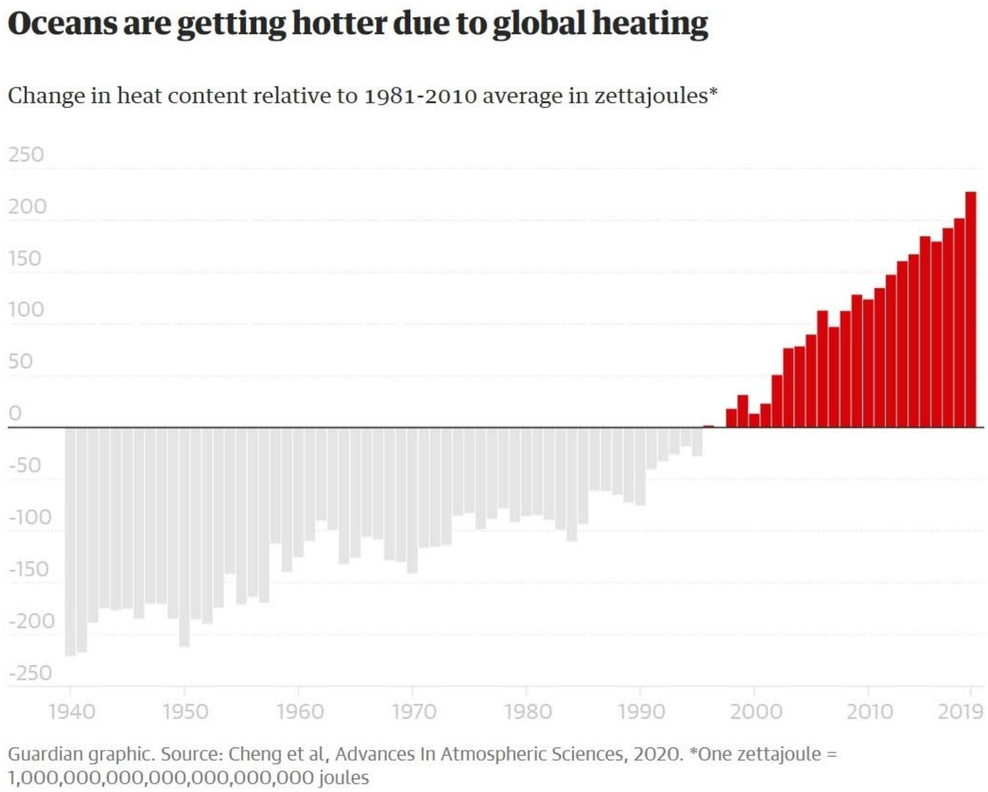
They were very interested in this last scenario, so they studied 201 slight variants starting at 3.344299 billion years from now, each with slightly different positions of Mars. In five of these Mars was ejected from the Solar System, while the remaining 196 led to collisions: 33 between Mercury and the Sun, 48 between Mars and the Sun, 43 between Mercury and Venus, 1 between Mercury and the Earth, 18 between Venus and the Earth, 23 between Venus and Mars, and 29 between the Earth and Mars.

The most surprising one was a collision between Venus and the Earth that took place in a 5-stage process. First Mercury's orbit became more eccentric through its perihelion resonance with Jupiter, as described. This then increased the eccentricity of the orbits of Venus, the Earth and Mars. When the orbits of the Earth and Mars became very eccentric, it became possible for them to collide. But in the case at hand, Mars didn't hit the Earth. Instead, Mars made the orbit of Venus more eccentric until *Venus* hit the Earth!

But here's the good news: the work of Laskar and Gastineau — and also another team — shows that nothing dramatic should happen to the planets' orbits for the next billion years. So we can worry about other things.

In the *really* long term, most of the stars in the Milky Way will be ejected. Through random encounters, individual stars will pick up enough speed to reach escape velocity. The whole Galaxy will slowly 'boil away'. It will dissipate in about 10^{19} years.

January 13, 2020



This chart shows the change in the oceans' heat energy measured in zettajoules. A zettajoule is 10^{21} joules.

It's incredible how much better fossil fuels are at making the Earth retain solar energy than they are at producing *useful* energy. Let's think about what a zettajoule means!

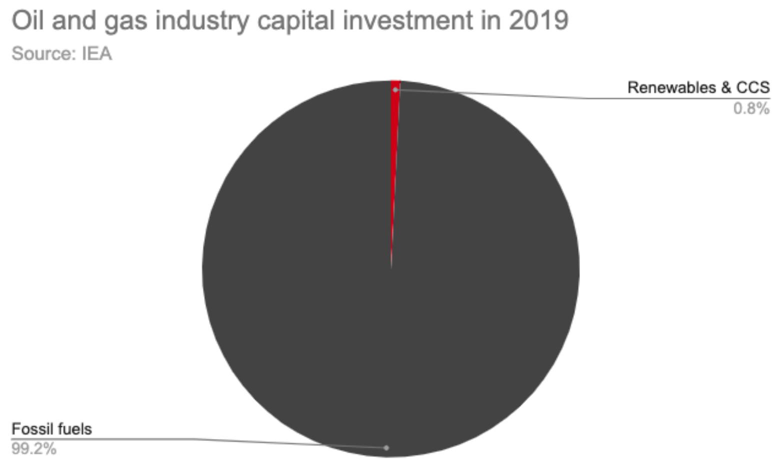
- If you convert one kilogram of mass into energy, you create 0.00009 zettajoules of energy.
- The world's largest nuclear bomb ever tested, the Tsar Bomba, released 0.00021 zettajoules of energy.
- The energy released by the explosion of Krakatoa was about 0.0008 zettajoules.
- The total amount of electrical energy used by humans in 2008 was 0.064 zettajoules.
- The total of *all* kinds of energy used by humans in 2010 was 0.5 zettajoules.
- The estimated energy in the world's oil reserves is about 8 zettajoules.
- The amount of energy in sunlight hitting the Earth each day is 15 zettajoules.
- The estimated energy in the world's coal reserves is about 24 zettajoules.
- The heat energy that's gone into the oceans from global warming since 2000 is about 200 zettajoules.

It's interesting how global warming is the evil twin of solar energy: both rely on the immense power of the Sun.

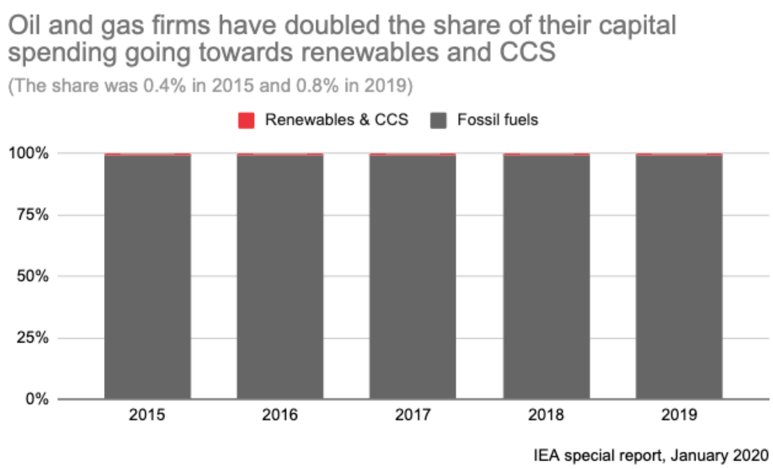
January 20, 2020

I see a lot of TV ads from oil and gas companies about how they're working on renewable energy. They make me feel so good. Women and men working together for a better future!

Uh-oh.



Since 2015 these companies have *doubled* their spending on renewable energy and carbon capture and storage! That makes me feel so good!



Uh-oh.

Let's face it: oil and gas firms are in business to sell us oil and gas. This might change someday... around when it stops being profitable to them.

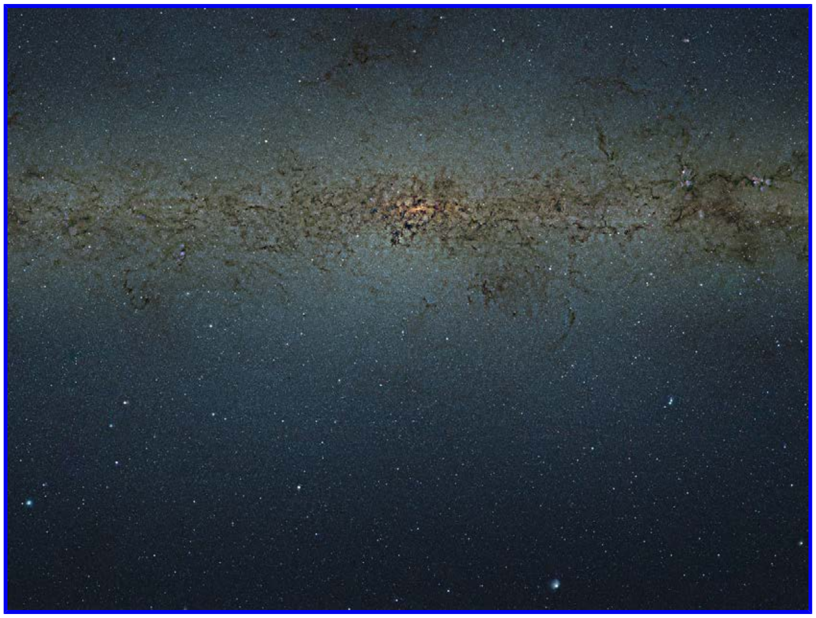
We should make that day come soon.

For the full report from the International Energy Agency, go here:

- IEA, The oil and gas industry in energy transitions.

January 24, 2020

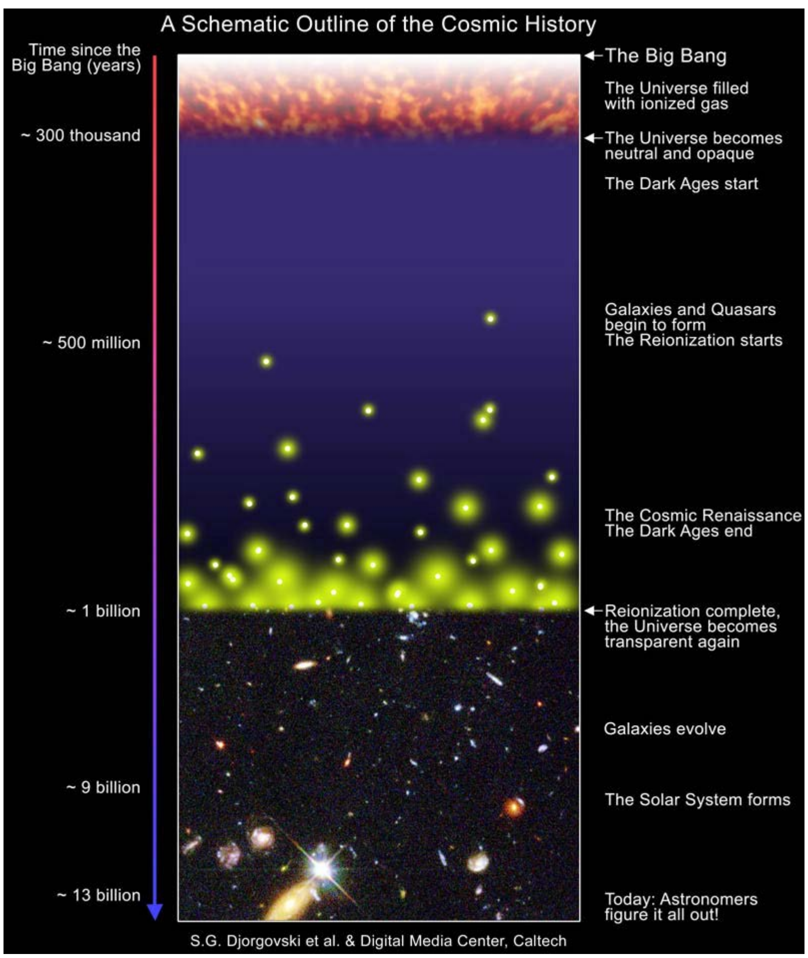
Click to see a zoomable image of the Milky Way with 84 million stars:



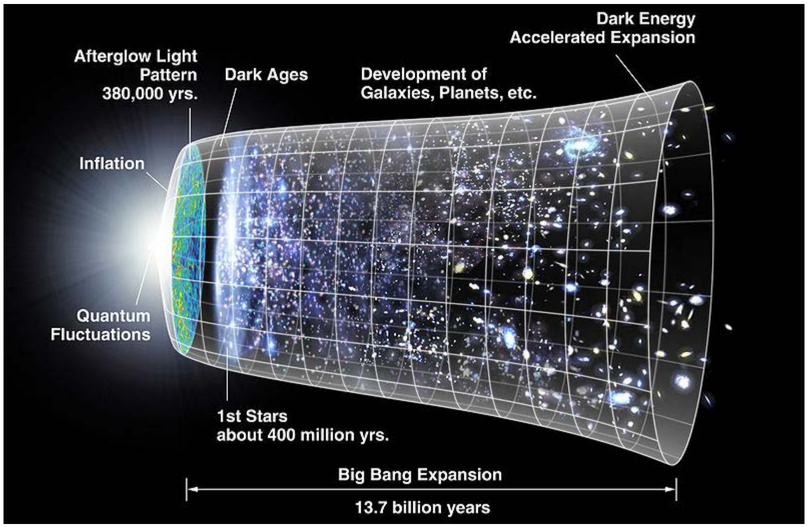
But stars contribute only a tiny fraction of the total entropy in the observable Universe. If it's random information you want, look elsewhere!

First: what's the 'observable Universe', exactly?

The further you look out into the Universe, the further you look back in time. You can't see through the hot gas from 380,000 years after the Big Bang. That 'wall of fire' marks the limits of the observable Universe.



But as the Universe expands, the distant ancient stars and gas we see have moved even farther away, so they're no longer observable. Thus, the so-called 'observable Universe' is really the 'formerly observable Universe'. Its edge is 46.5 billion light years away now! This is true even though the Universe is only 13.8 billion years old. (A standard challenge in understanding general relativity is to figure out how this is possible, given that nothing can move faster than light.)



What's the total number of stars in the observable Universe? Estimates go up as telescopes improve. Right now people think there are between 100 and 400 billion stars in the Milky Way. They think there are between 170 billion and 2 trillion galaxies in the Universe.

In 2009, Chas Egan and Charles Lineweaver estimated the total entropy of all the stars in the observable Universe at 10^{81} bits. You should think of these as qubits: it's the amount of information to describe the quantum state of *everything* in all these stars.

But the entropy of interstellar and intergalactic gas and dust is about 10 times more the entropy of stars! It's about 10^{82} bits.

The entropy in all the photons in the Universe is even more! The Universe is full of radiation left over from the Big Bang. The photons in the observable Universe left over from the Big Bang have a total entropy of about 10^{90} bits. It's called the 'cosmic microwave background radiation'.

The neutrinos from the Big Bang also carry about 10^{90} bits — a bit less than the photons. The gravitons carry much less, about 10^{88} bits. That's because they decoupled from other matter and radiation very early, and have been cooling ever since. On the other hand, photons in the cosmic microwave background radiation were formed by annihilating electron-positron pairs until about 10 seconds after the Big Bang. Thus the graviton radiation is expected to be cooler than the microwave background radiation: about 0.6 kelvin as compared to 2.7 kelvin.

Black holes have *immensely* more entropy than anything listed so far. Egan and Lineweaver estimate the entropy of stellar-mass black holes in the observable Universe at 10^{98} bits. This is connected to why black holes are so stable: the Second Law says entropy likes to increase.

But the entropy of black holes grows *quadratically* with mass! So black holes tend to merge and form bigger black holes — ultimately forming the 'supermassive' black holes at the centers of most galaxies. These dominate the entropy of the observable Universe: about 10^{104} bits.

Hawking predicted that black holes slowly radiate away their mass when they're in a cold enough environment. But the Universe is much too hot for supermassive black holes to be losing mass now. Instead, they very slowly *grow* by eating the cosmic microwave background, even when they're not eating stars, gas and dust.

So, only in the far future will the Universe cool down enough for large black holes to start slowly decaying via Hawking radiation. Entropy will continue to increase... going mainly into photons and gravitons! This process will take a very long time. Assuming nothing is falling into it and no unknown effects intervene, a solar-mass black hole takes about 10^{67} years to evaporate due to Hawking radiation — while a really big one, comparable to the mass of a galaxy, should take about 10^{99} years.

For more details, go here:

- Chas A. Egan and Charles H. Lineweaver, A larger estimate of the entropy of the universe, *The Astrophysical Journal* **710** (2010), 1825.

Also read my page on information.

For my February 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

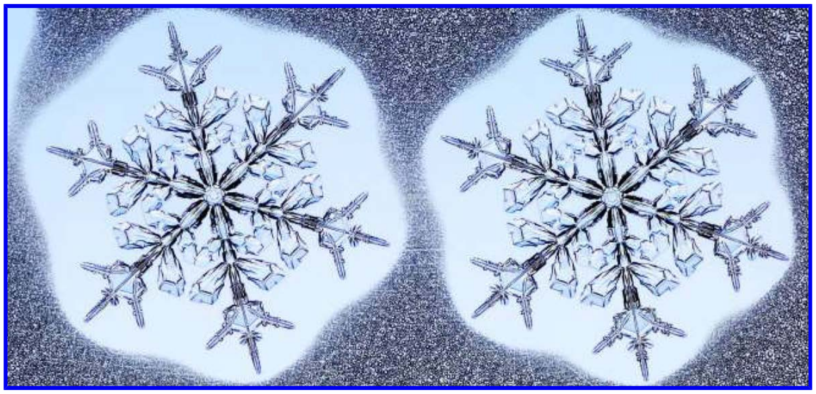
home

[For my January 2020 diary, go here.]

Diary — February 2020

John Baez

February 1, 2020

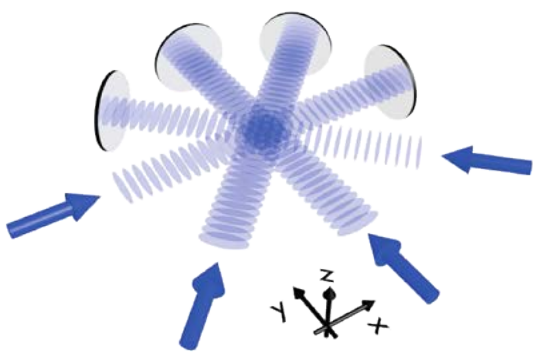


The world's expert on snowflakes has written a 540-page book on them. Now he's giving it away for free here:

- Kenneth Libbrecht, *Snow Crystals*.

He has figured out how to grow identical twin snowflakes, like those shown here.

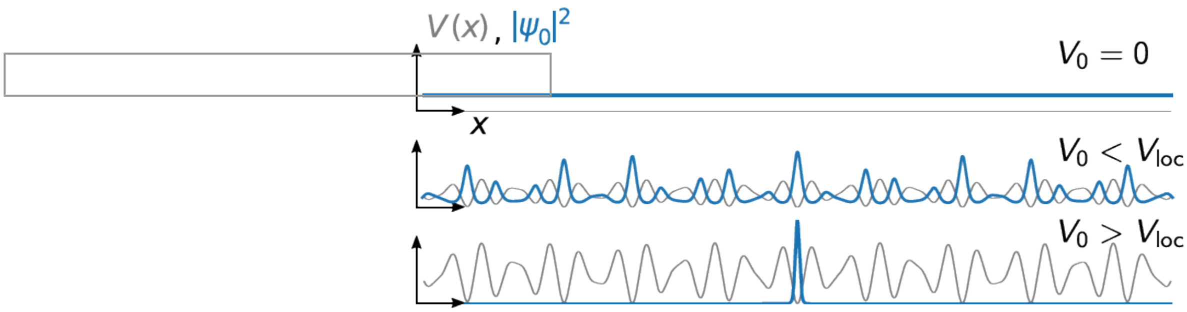
February 2, 2020



Condensed matter physics is so cool! Bounce 4 laser beams off mirrors to make an interference pattern with 8-fold symmetry. Put a Bose–Einstein condensate of potassium atoms into this 'optical lattice' and you get a *superfluid quasicrystal!* But that's not all....

As you increase the intensity of the lasers, the Bose-Einstein condensate (in blue) suddenly collapses from a quasicrystal to a 'localized' state where all the atoms sit in the same place!

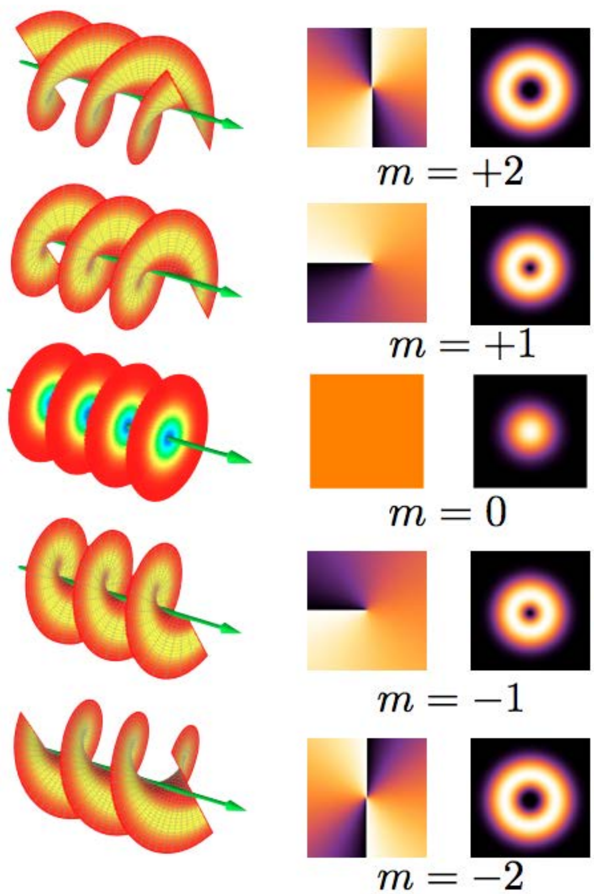
Here the gray curve V_0 is the potential formed by the lasers, while the blue curve ψ_0 is the wavefunction of the Bose–Einstein condensate:



When V_0 exceeds a certain value, localization occurs. For more, see:

- John Baez, Superfluid quasicrystals, *Azimuth*, January 31, 2020.

February 5, 2020



An 'optical vortex' is a beam of light that turns like a corkscrew as it moves. It's dark at the center.

You can use an optical vortex to trap atoms! They move along the dark tube at the center of the vortex.

Photons have spin angular momentum, and in circularly polarized light this equals +1 or -1. An optical vortex is different: it exploits the fact that photons can also have orbital angular momentum! This can be any integer *m*, as shown above.

So, some hotheads call an optical vortex a 'photonic quantum vortex'. But you can study optical vortices without quantum mechanics, using the classical Maxwell equations! The electromagnetic field is described using a complex function that in cylindrical coordinates is $\exp(im\theta)$ times some function that vanishes at $r = 0$: the dark center. One class of functions like this are the 'hypergeometric-Gaussian modes':

$$u_{pm}(\rho,\theta,Z)=\sqrt{\frac{2^{p+|m|+1}}{\pi\Gamma(p+|m|+1)}}\frac{\Gamma(1+|m|+\frac{p}{2})}{\Gamma(|m|+1)}\,i^{|m|+1}\;\cdot$$
$$Z^{\frac{p}{2}}\,(Z+i)^{-(1+|m|+\frac{p}{2})}\,\rho^{|m|}\;\cdot$$
$$\exp\left(-\frac{i\rho^2}{(Z+i)}\right)e^{im\phi}\,{}_1F_1\left(-\frac{p}{2},|m|+1;\frac{\rho^2}{Z(Z+i)}\right)$$

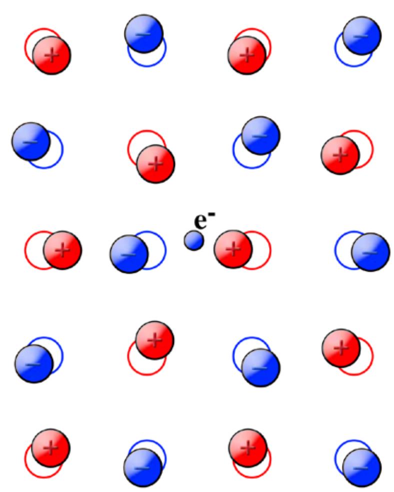
Rather complicated! More importantly, the phase of the electromagnetic field, $\exp(im\theta)$, is undefined at the center of the optical vortex. It turns around m times as you go around the vortex. So this number m has to be an integer. It's a simple example of a 'topological charge'.

People make optical vortices using many different technologies, including spiral-shaped pieces of plastic, 'computer-generated holograms', and computer-controlled liquid crystal gadgets called ''.

February 6, 2020

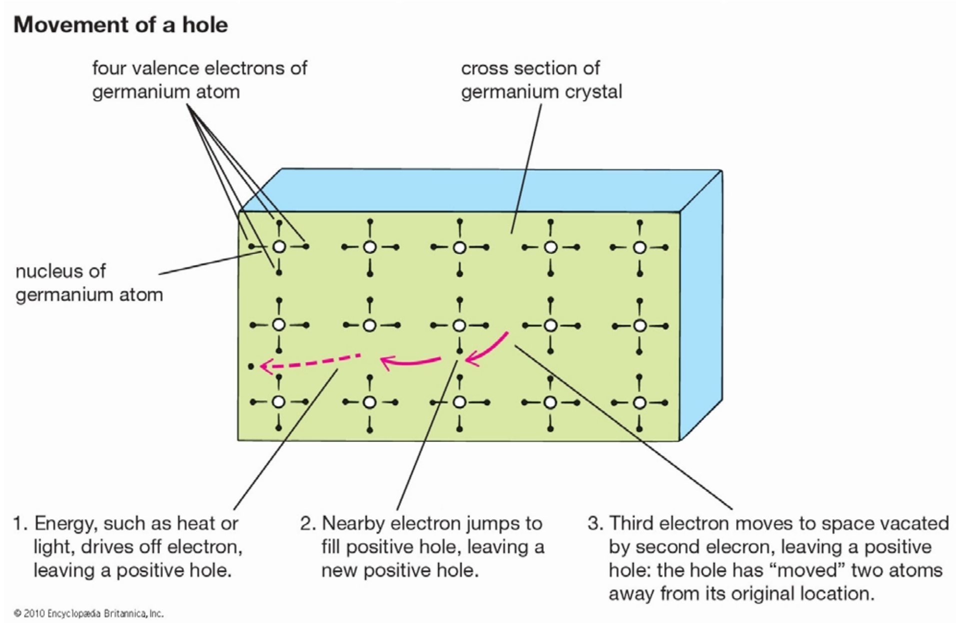
What's the difference between a polaron and a polariton?

When an electron moves through a crystal, it repels other electrons and attracts the protons. The electron together with this cloud of distortion acts like a particle in its own right: a 'polaron'.

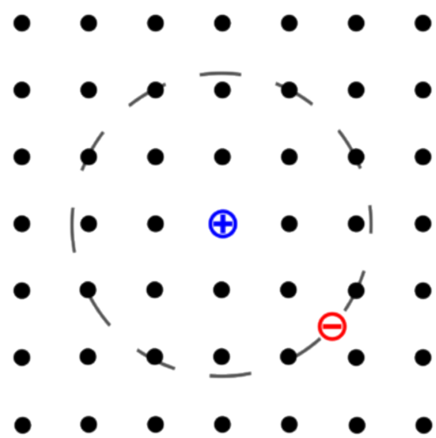


A polariton is more complicated.

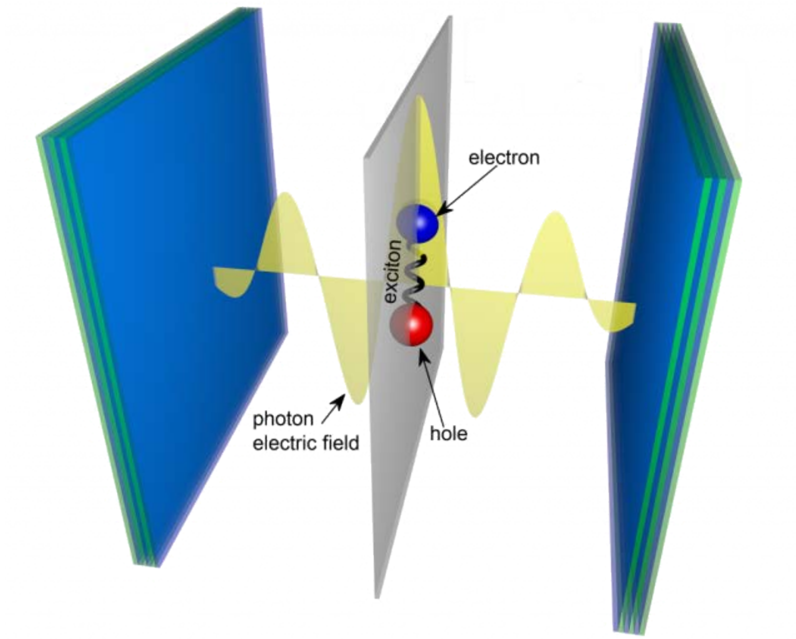
First, when an electron in a crystal is knocked out of place, it leaves a 'hole'. This hole can move around — and it acts like a positively charged particle!



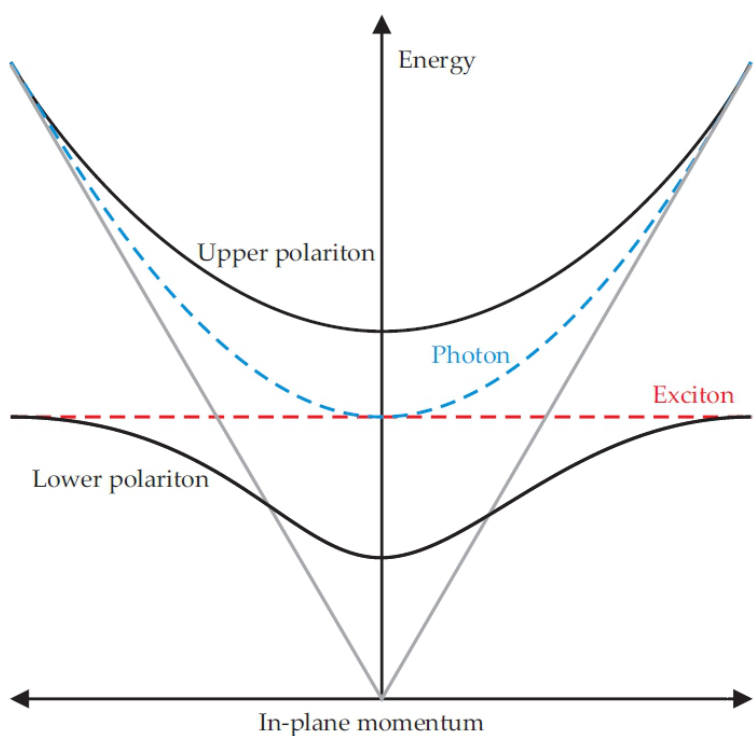
Since electrons are negative and holes are positive, they attract each other! An electron orbiting a hole acts like a hydrogen atom. It's called an 'exciton'. It can move around! But after a while, the electron falls into the hole.



Finally, an exciton can attract a photon! They can stick to each other a form a new particle called a 'polariton'!

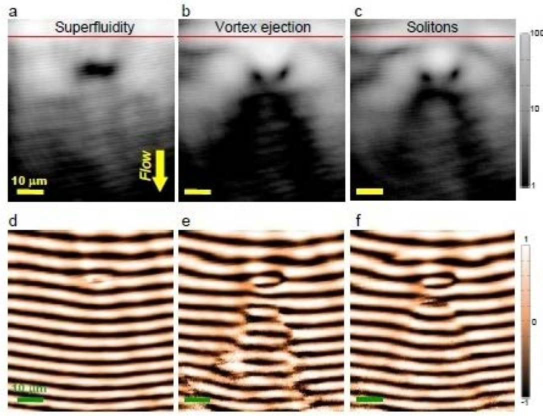


Polaritons are exciting to me because they're made of an electron, an *absence* of an electron, and light. Here's the dispersion relation (the relation between energy and moment) for polaritons as compared with that for photons and excitons.



You'll notice there are two kinds of polaritons. These form another basis of the space of quantum states spanned by the photon and exciton.

February 7, 2020



Scientists have made 'liquid light' by blending light and matter. It can be superfluid and flow smoothly past an obstacle (left), or an ordinary fluid that forms eddies as it flows past (middle), or it can form a sonic boom (right).

That's right: *a sonic boom in liquid light!*

As I explained on [February 6](), a [polariton]() is a particle that's a blend of light and matter. More precisely, it's a quantum superposition of a photon and an [exciton](), which is an electron-hole pair.

Scientists made a fluid of polaritons! Then they made it flow. The polaritons only last for 4-10 picoseconds (trillionths of a second). But that was long enough to watch the fluid do all the usual things fluids do: turbulence, sonic booms, etc.

Try my blog for more:

- John Baez, [Liquid light](), *Azimuth*, November 28, 2011.

February 8, 2020

The magic of condensed matter physics: by carefully crafting materials, you can make familiar particles behave in strange new ways.

> Imagine, if you will, a collection of many photons. Now imagine that they have mass, repulsive interactions, and number conservation. The photons will act like a gas of interacting bosonic atoms, and if cooled below a critical temperature, they will undergo a well-known phase transition: Bose–Einstein condensation. You will have a "superfluid of light."

You can effectively manage to adjust the mass of photons by trapping it between two parallel mirrors. Its frequency in the transverse direction affects its energy as if it had a mass. Now you have a massive photon in 2 dimensions!

This lets you do some interesting things.

> Now imagine that you can choose the photons' mass. Then, because the critical temperature for Bose–Einstein condensation depends on particle mass and density, you can create the condensed state even at room temperature.
>
> That is not an idle dream.

To get your massive photons to interact, you should get them to interact strongly with the material between your parallel mirrors.

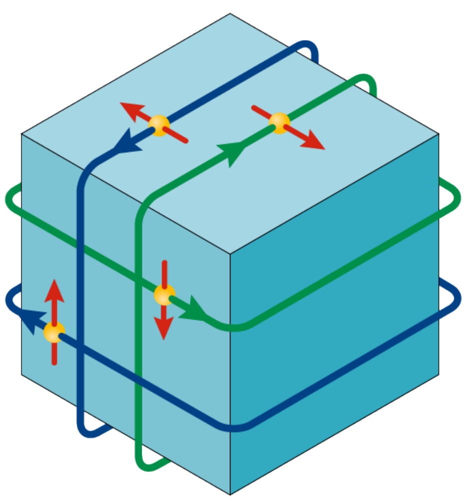
> That is not an idle dream. Condensed-matter physicists have a long history of inventing novel quasiparticles, such as massless electrons, particles with fractional charge, and particles with spins detached from their charges. Two decades ago researchers began to engineer hybrid particles of light and matter, called polaritons, that could be used to realize the Bose–Einstein condensates of light described above. Today, the study of polariton condensates has come of age: Researchers have advanced beyond merely demonstrating that they exist to demonstrating ways to harness them in optical devices.

The quotes are from this absolutely delightful article:

- David W. Snoke and Jonathan Keeling, The new era of polariton condensates, *Physics Today* **70** (2017), 54–60.

They describe, step by step, how to make a polariton condensate and why it works.

February 11, 2020



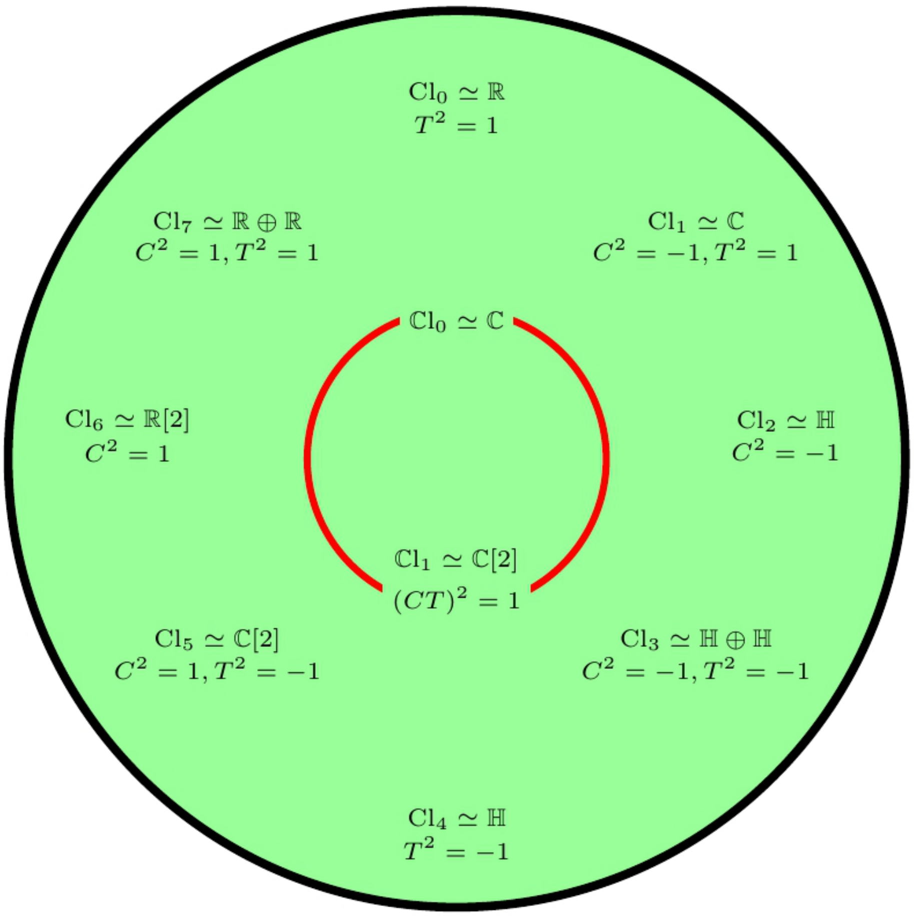
A 'topological insulator' is insulating on the inside, but its surface conducts electricity. More importantly, electrons on the surface have their spin locked at right angles to their momentum, so they come in two kinds.

(This is the simplest type of topological insulator — there are others.)

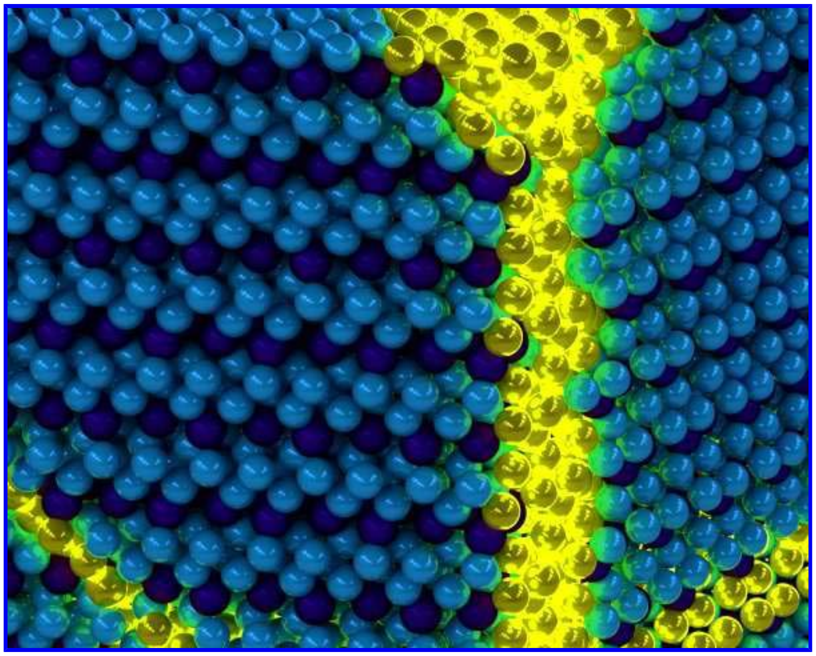
So far, most topological insulators have been made with bismuth compounds like Bi_2Se_3 and Bi_2Te_3. Someday they may have applications in 'spintronics': a version of electronics where information is encoded in electron spins. A spin, after all, is nature's own qubit!

But right now, a lot of interest in topological insulators comes from the math. Their classification, the 'tenfold way', unifies the 8 types of real and 2 types of complex Clifford algebras! For more about it, read this:

- John Baez, The tenfold way.



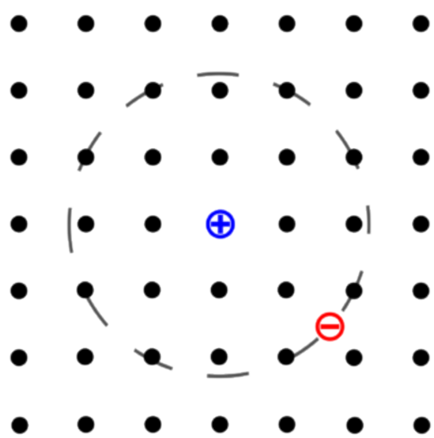
February 12, 2020



Condensed matter physicists are exciting. They're exciting electrons, knocking them out of their usual places in crystals, leaving holes. The electrons and holes orbit each other, forming 'excitons'.

They've been trying to make a metal out of excitons: 'excitonium'!

When a hole is much heavier than an electron, it stands almost still when an electron orbits it. So, they form an exciton thatâ€™s very similar to a hydrogen atom!



Hydrogen comes in many forms. At high densities, like the core of Jupiter, it becomes a *metal*.

Any ideas?

In 1978 the Russian physicist Abrikosov wrote a short and very creative paper in which he raised the possibility that excitons could form a crystal similar to metallic hydrogen! He called this new state of matter 'metallic excitonium'.

Can we actually make it? I don't think anyone has made metallic excitonium yet — correct me if I'm wrong. But in 2016, researchers made something equally exciting! An electron is a fermion and so, therefore, is a hole. Two fermions make a boson — so an exciton is a boson.

Any ideas?

At low temperatures, identical bosons like to be in the *exact same state*. This is called a 'Bose-Einstein condensate'. And in 2016, researchers made a Bose-Einstein condensate of excitons! At shockingly high temperatures, too.

Here's the paper:

- Anshul Kogar, Melinda S. Rak, Sean Vig, Ali A. Husain, Felix Flicker, Young Il Joe, Luc Venema, Greg J. MacDougall, Tai C. Chiang, Eduardo Fradkin, Jasper van Wezel and Peter Abbamonte, Signatures of exciton condensation in a transition metal dichalcogenide, *Science* **358** (2017), 1314–1317.

And here's something I wrote:

- John Baez, Metallic excitonium, *Azimuth*, December 10, 2017.

February 20, 2020

MOMENTUM IN SPECIAL RELATIVITY

If an object has mass m and velocity $\vec{v}$, its momentum is

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}}$$

where c is the speed of light.

This appeared in a paper published by Max Planck in 1906 —
the first paper on special relativity by someone besides Einstein!

Max Planck was the first established physicist to embrace Einstein's work on special relativity. He worked out some important consequences.

His formula for momentum almost matches Newton's for speeds much slower than light. But it gives dramatically different answers at high speeds.

Planck published the paper containing this formula shortly after a physicist named Walter Kaufmann had done experiments that seemed to confirm a *different* formula for momentum, due to Max Abraham! But Planck wrote:

> However, in view of the complicated theory of these experiments I would not completely exclude the possibility, that the principle of relativity on closer elaboration might just prove compatible with the observations.

You can read his paper, translated into English, here:

- Max Planck, The principle of relativity and the fundamental equations of mechanics, *Verhandlungen Deutsche Physikalische Gesellschaft* **8** (1906), 136–141.

It's short and sweet. Equation 6) contains the new formula for momentum, built into the new relativistic version of Newton's $F = dp/dt$.

Later, in 1914, Planck helped Einstein get a research position in Berlin.

February 21, 2020

> **I was sitting in a chair in the patent office at Bern when all of sudden a thought occurred to me: If a person falls freely he will not feel his own weight. I was startled. This simple thought made a deep impression on me. It impelled me toward a theory of gravitation.**
>
> ***Albert Einstein recalling his thought in November 1907***

In 1907 Einstein tried to combine special relativity with gravity — and very soon he realized gravity would make clocks tick slower, and would bend light.

It took him until 1915 to find the equations of general relativity. He needed the right kind of math.

In 1912 an old college friend, a mathematician named Marcel Grossman, helped Einstein get a job in Zurich. That year, Grossman told Einstein that the math he needed for describing gravity had been invented by Riemann. He warned Einstein that it was a "terrible mess".

Einstein and Marcel Grossman

> **I am now working exclusively on the gravitation problem and believe that I can overcome all difficulties with the help of a mathematician friend of mine here. But one thing is certain: never before in my life have I toiled anywhere near as much, and I have gained enormous respect for mathematics, whose more subtle parts I considered until now, in my ignorance, as pure luxury. Compared with this problem, the original theory of relativity is child's play.**
>
> ***Albert Einstein writing about Marcel Grossman in a letter to Arnold Sommerfeld, October 29, 1912***

Grossman was not an expert on Riemannian geometry, but he and Einstein quickly learned the subject together. They came out with a paper applying it to gravity in 1913.

They ran into a big problem, though, which Einstein only surmounted later. They realized: if the equations of gravity are 'generally covariant' — preserved by all coordinate transformations — you cannot use complete knowledge of what's happening at $t = 0$ to predict what will happen at a point in the future with specific coordinates (t, x, y, z). After all, a coordinate transformation could change the coordinates of that point to some other coordinates (t', x', y', z'). Nature can't guess what coordinates you are using!

Einstein and Grossman erroneously concluded that the equations of gravity should not be generally covariant. Only later did Einstein realize that they *should* be!

It turns out to be *okay* that we can't predict what will happen at a point with coordinates (t, x, y, z). All a theory needs to predict is what we can observe. Coordinates are not something we observe.

This realization freed Einstein, and he found the correct equations of general relativity on November 25, 1915. It was still a long road to our current understanding of black holes, the Big Bang and gravitational waves. But that's another story.

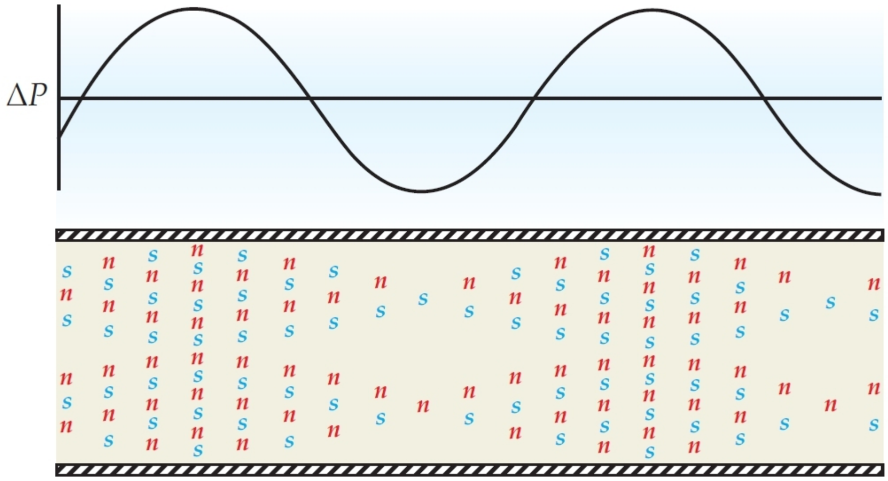
February 26, 2020

What's even cooler than sound? Second sound!

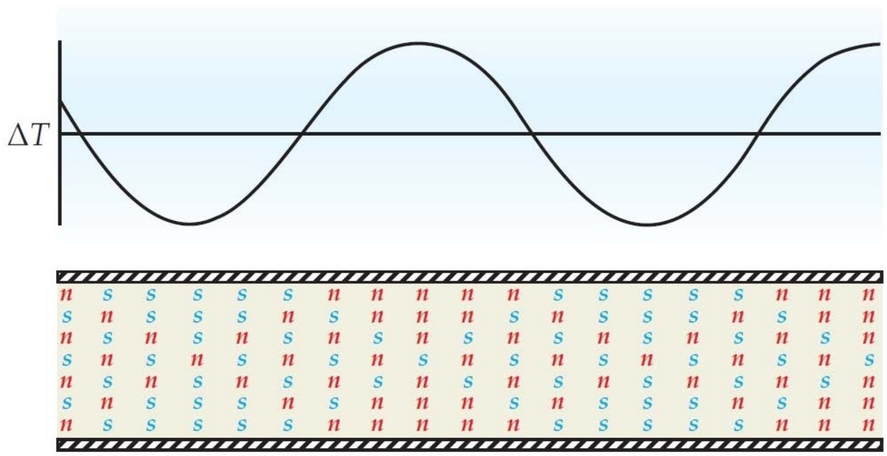
Usually heat spreads out like the picture on top. Second sound when heat moves in *waves*.

You see, ordinary sound — 'first sound' — is waves of *pressure*. 'Second sound' is waves of *temperature*.

Liquid helium is a mix of normal liquid and superfluid. In ordinary sound in liquid helium, a wave of high pressure has more of both normal liquid (n) and superfluid (s):



But in second sound, a wave of high temperature has more normal fluid and less superfluid. The total pressure is constant:



Here's a great introduction to second sound in liquid helium:

- Russell J. Donnelly, The two-fluid theory and second sound in helium, *Physics Today* (October 2009), 34–39.

Second sound in liquid helium moves much slower than ordinary first sound: it moves faster at lower temperatures, with a top speed of about 20 meters per second. Second sound has also been seen in graphite at much higher temperatures: 120 kelvin! But it dies out after traveling just a few microns.

February 27, 2020

Luitzen Egbertus Jan Brouwer was born 139 years ago today, in 1881. He invented 'intuitionism', an approach to mathematics where the law of excluded middle ("p or not p") doesn't hold. So today I wrote about intuitionism and topos theory:

- John Baez, Topos theory (part 8), *Azimuth*, February 27, 2020.

Define a "time-dependent set" to be a set $X(n)$ for each natural number n, together with functions $X(n) \to X(n+1)$.

For example, $X(n)$ could be the set of solutions to some equation that you know on the nth day of your research.

As days pass you can find new solutions, and also prove that two solutions you knew are actually equal. You can never *lose* solutions, or discover that two solutions you thought were equal are *different*. So this is a simple model of an infallible but not omniscient mathematician.

There is a category of time-dependent sets, and it's a topos. This means you can do all of mathematics and logic in this category — like you can with sets. But logic works differently in the topos of time-dependent sets, because you learn new truths as time goes on!

Instead of a mere set of truth values,

$$\{\text{true}, \text{false}\}$$

there's a time-dependent set of truth values, called Ω. For each time n, $\Omega(n)$ has infinitely many elements: known today, known tomorrow, known the next day, etc.,... and never known.

In my article, I don't yet say much about how logic works with time-dependent truth values — like why the law of excluded middle fails. Instead, I show how to *derive* time-dependent truth values from the category of time-dependent sets.

In previous articles I explained why any presheaf category is a topos and how to figure out the truth values in such a category. Here I'm illustrating how that works for time-dependent sets.

Later I'll get into Heyting algebras and the law of excluded middle!

February 29, 2020

Condensed matter physics is so full of surprises. I am constantly awed by it. I just learned:

You can make particles that act like massless particles moving at the speed of light when they move in one direction — but like *massive* particles that move *slower* than light when they move in the orthogonal direction!

In classical mechanics, at low momenta, ignoring relativistic effects, the energy of a massive particle grows *quadratically* with the magnitude of its momentum:

$$E = \frac{p^2}{2m}$$

where m is the particle's mass.

This doesn't make sense for a particle of mass zero. That's because a massless particle moves at the speed of light and we need special relativity to understand it. In special relativity, the energy of a massless particle depends in a very different way on its momentum: it grows *linearly* with the magnitude of the momentum! More precisely,

$$|E| = c|\vec{p}|$$

where c is the speed of light.

All these formulas are still true in quantum mechanics, but we have more.

In quantum mechanics, energy is proportional to the rate at which the wavefunction wiggles in time: that is, its frequency ω. Similarly, momentum is proportional to the rates at which the wavefunction wiggles in the 3 spatial directions: that is, its wave vector $\vec{k} = (k_x, k_y, k_z)$. In both cases the constant of proportionality is Planck's constant:

$$E = \hbar\omega, \quad \vec{p} = \hbar\vec{k}$$

We can combine these our earlier formulas. The result is that at low momenta, ignoring relativistic effects, the frequency of the wavefunction of a massive particle is a *quadratic* function of the magnitude of the wave vector:

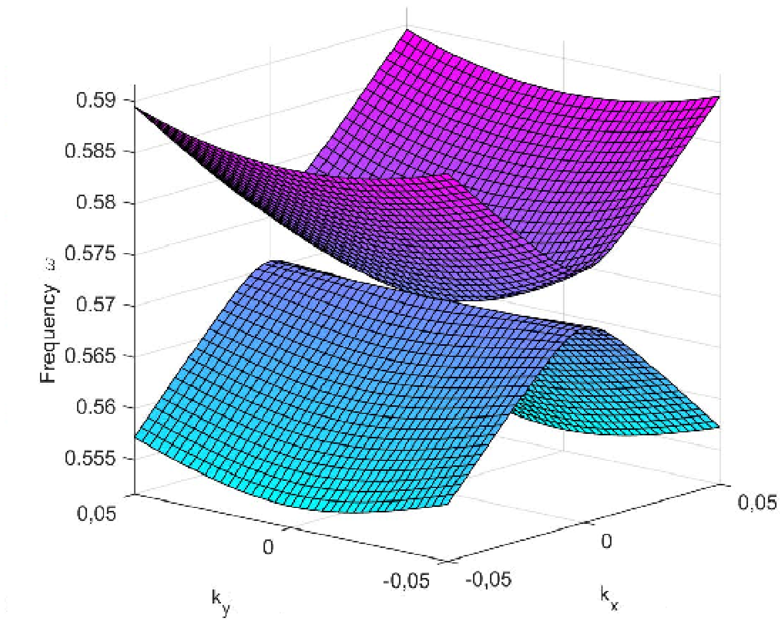
$$\omega = \hbar \frac{k^2}{2m}$$

but for a massless particle, the frequency depends *linearly* on the magnitude of the wave vector, or more precisely

$$|\omega| = c|\vec{k}|$$

So that's how massive and massless particles are different. But you can make the energy of a photon, or electron, or a quasiparticle of your choice, be almost *any* function of its momentum if you're good enough at making strange materials. The reason is that its *effective* energy and momentum depend on its interactions with the material.

So, if you're clever enough, you can make the relation between energy and momentum look like this:



Note that if you set k_y to zero you get

$$|\omega| \quad k_x^2$$

like a massive particle at low momentum, except that ω takes both signs. But if you set k_x to zero you get

$$|\omega| \quad |k_y|$$

like a massless relativistic particle.

The relation between a particle's frequency and its wave vector is called its dispersion relation. The unusual dispersion relation in the picture above is called a 'semi-Dirac cone'.

One way to create a semi-Dirac cone is described here:

arXiv.org > physics > arXiv:1312.0201

Physics > Optics

Semi-Dirac dispersion relation in photonic crystals

Ying Wu

(Submitted on 1 Dec 2013)

> A semi-Dirac cone refers to a peculiar type of dispersion relation that is linear along the symmetry line but quadratic in the perpendicular direction. Here, I demonstrate that a photonic crystal consisting of a square array of elliptical dielectric cylinders is able to produce this particular dispersion relation in the Brillouin zone center. A perturbation method is used to evaluate the linear slope and to affirm that the dispersion relation is a semi-Dirac type. Effective medium parameters calculated from a boundary effective medium theory not only explain the unexpected topological transition in the iso-frequency surfaces occurring at the semi-Dirac point, they also offer a perspective on the property at that point, where the photonic crystal behaves as a zero-refractive-index material along the symmetry axis but functions like at a photonic band edge in the perpendicular direction.

Wu created a photonic crystal consisting of square array of elliptical cylinders of plastic with a high dielectric constant in air. But in condensed matter physics there are other very different ways to make dispersion relations with semi-Dirac cones.

Wu's paper is here:

- Ying Wu, Semi-Dirac dispersion relation in photonic crystals, *Optics Express* **22** (2014), 1906–1917.

For my March 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

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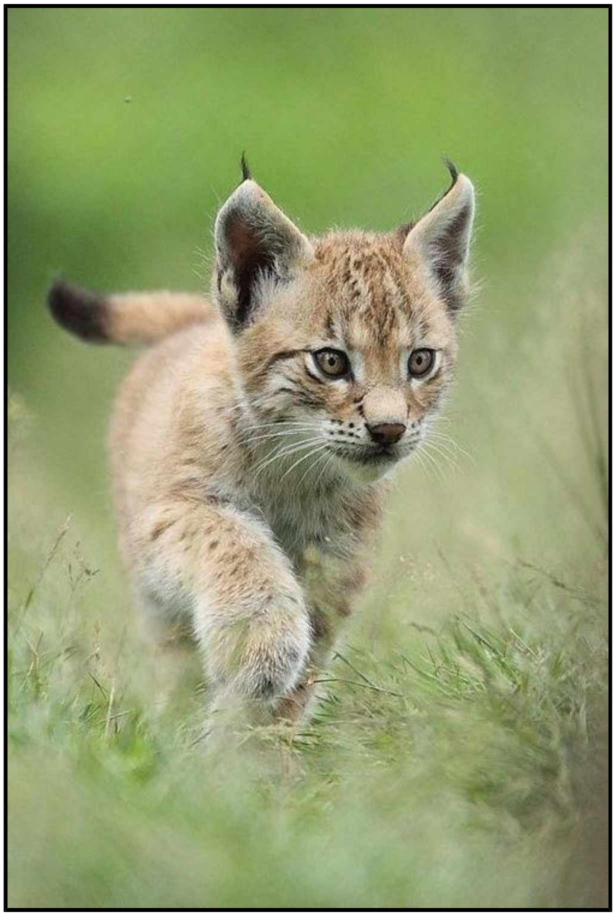
[For my February 2020 diary, go here.](#)

Diary — March 2020

John Baez

March 1, 2020

Lynxes, from HourlyLynxes:



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March 2, 2020

Why is there just 1 thing that acts like 1?

2 nice things about 1:

$$1x = x \text{ (it's a 'left unit')}$$

and

$$x1 = x \text{ (it's a 'right unit')}$$

There are number systems with many left units, or many right units. But if there's a left unit *and* a right unit, there's just one.

For any set you can define a funny multiplication like this:

$$ab = b$$

Then *every* element is a left unit! This multiplication is even associative.

Similarly, if you define

$$ab = a$$

then every element is a right unit.

But suppose you have a multiplication with a left unit, say L. Then there can be at most one right unit! For suppose you have two right units, say R and R'. Then

$$R = LR = L = LR' = R'$$

Similarly, if you have a multiplication with a right unit R, there can be at most one left unit. For suppose you have two left units, say L and L'. Then

$$L = LR = R = L'R = L'$$

So if your multiplication has both a left unit L and a right unit R, they are both unique.

Furthermore, they are equal! Why? We've already seen why:

$$L = LR = R$$

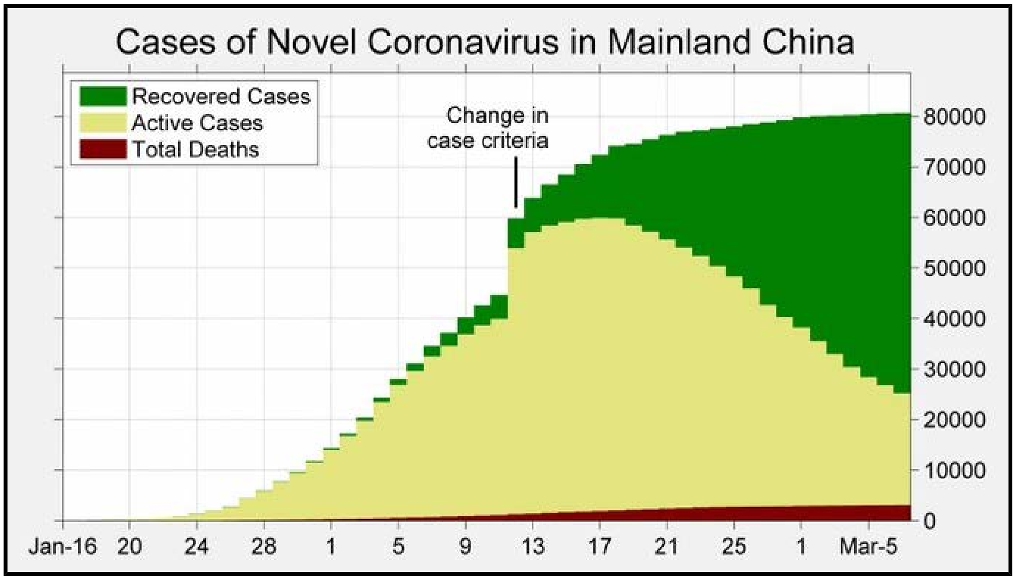
In the battle of left and right units, both win... so they must be equal!

So: if you have a binary operation with a left unit and a right unit, they are both unique — and they're equal.

If we call the binary operation 'multiplication', then it makes sense to call this unique left and right unit '1'.

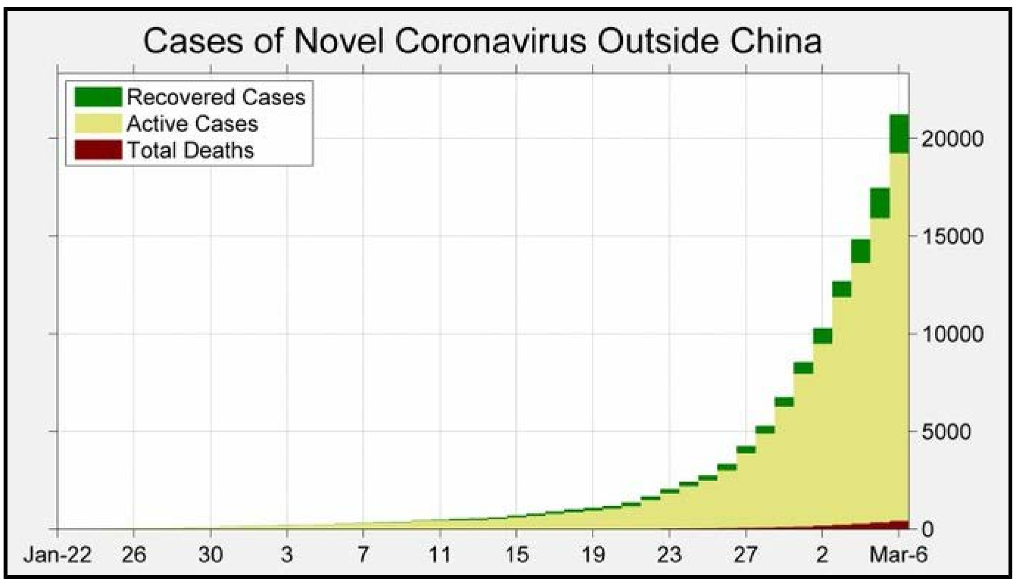
After all, there's just one.

March 7, 2020

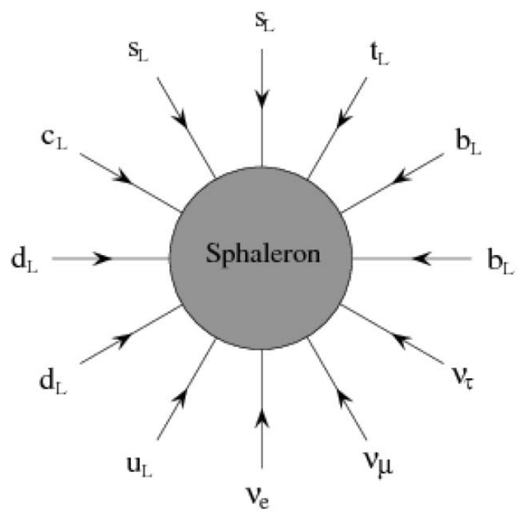


The drastic measures taken by China to contain coronavirus seem to be working there, if their data can be trusted.

However, *outside* China the disease is spreading rapidly.



March 8, 2020



Why is there more matter than antimatter? Nobody knows, but the Standard Model does allow a process where 9 quarks and 3 leptons all annihilate each other! This 'sphaleron' process can also turn 9 antiquarks into 3 leptons. Is that where the antimatter went?

The sphaleron process is nonperturbative, so you can't understand it using the usual Feynman diagrams in the Standard Model. The minimum energy required to trigger it is about 9 TeV, but it's hard to get enough particles to collide to make it happen!

The early universe was very hot. As it cooled there was probably an electroweak phase transition (EWPT): bubbles formed, in which the electromagnetic and weak forces became different. Sphalerons *might* form at the bubble walls, preferentially destroying antimatter.

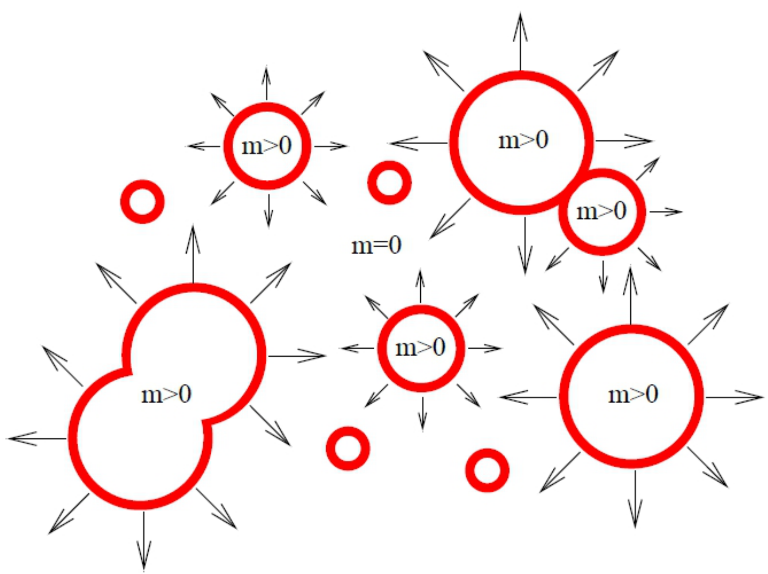


Fig. 11. Bubble nucleation during a first-order EWPT.

Alas, the best calculations people have done so far do not predict a 'first-order' electroweak phase transition &mdash a phase transition involving latent heat, like freezing water. So, the bubble walls would not be discontinuous enough to create lots of sphalerons.

So, physicists are looking for earlier phase transitions to explain 'baryogenesis': the creation of more baryons (protons, neutrons, etc.) than antibaryons.

Here's an excellent introduction:

- James M. Cline, Baryogenesis.

If reading a paper on baryogenesis sounds too ambitious, maybe some Wikipedia articles will be enough to satisfy your thirst for knowledge:

- Wikipedia, Baryogenesis.
- Wikipedia, Sphaleron.

If you know some differential geometry, you may enjoy reading how sphalerons arise from Morse theory — they are really saddle points:

- N. S. Manton, The inevitability of sphalerons in field theory, *Philosophical Transactions of the Royal Society A* **377** (2019): 20180327.

March 11, 2020

Today, Wednesday, Dean Ulrich of the College of Natural and Agricultural Sciences here at U. C. Riverside announced some coronavirus measures:

- Winter quarter final exams will not be held in person.
- In-person instruction for Spring quarter will be suspended through April 3, 2020.
- As of today, campus is not closed.

Since final exams start on Monday next week, we're all having to rush to figure out how to give final exams online!

March 15, 2020

I think I've figured out how to create a multiple choice final that my undergraduate calculus students can take on iLearn, U. C. Riverside's electronic system for delivering homework, etc. The difficulty is that it doesn't do LaTeX, so it's hard to present questions and answers involving equations. I realized I can upload images that I create using LaTeX to give questions involving equations. For some idiotic reason it's impossible to upload images for answers, but I can link to images stored elsewhere. It's a lot of work.

Meanwhile, a dramatic announcement from U. C. Riverside! Three days ago classes were going to be done electronically for the first week of the next quarter. Now that's changed:

- All instruction (including but not limited to labs, studios, and directed studies) will be delivered remotely for the entirety of the spring quarter. This is a change from the April 3 end date for remote instruction.
- On campus housing and residential dining services remain open and residents can elect to remain on campus if needed.

March 21, 2020

Having been home all week, Lisa and decided to get dinner at a local ribs joint, Smokey Canyon. Takeout, of course, since restaurants in California are all closed except for takeout orders. We decided to combine this with a trip to the supermarket, Ralphs in the same mall. For the first time I tried wearing a mask and gloves, as my nurse had recommended. So, I donned these before entering Ralphs.

Since we got there after 7:30 on a Saturday night it's perhaps not surprising that the hoarders had taken all the paper towels and bleach and chicken. We managed to get hand lotion and peanut butter and something else. I forget to look for vinegar. The clerk said that they have a special 'seniors hour' from 6:00 to 7:00 to let people over 65 get the first crack at the goods. So, Lisa may try that.

At the ribs joint there was one woman eating dinner at the bar — maybe a friend of the management? The waitress took a little while to bring out our order. I took off my mask because it made me uncomfortable and there were just this one woman and two employees there.

At home I tried to take the ribs and onion rings and sweet potato fries out of the packaging without letting the packaging contaminate our house too much, and I wondered how much this was even possible. Of course I think there's a low probability that the packaging had COVID-19 viruses on it, since only a small number of cases have been detected in Riverside so far, but I figure I should get in the habit of being very careful, since things will only get worse. The whole idea of eating food that someone else has prepared becomes a lot less attractive to me under these cicumstances.

For my April 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

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For my March 2020 diary, go here.

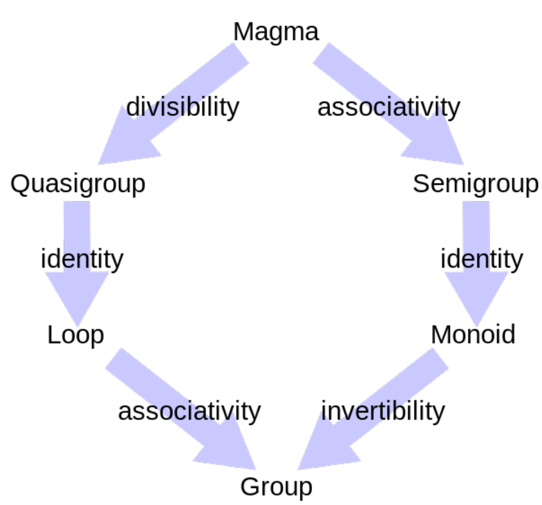
Diary — April 2020

John Baez

April 2, 2020

This is a 'cross fox': a red fox with some melanistic traits.

April 4, 2020



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I stand up for all downtrodden, oppressed mathematical objects. Consider the humble **commutative semigroup**. This is a set with a binary operation $+$ obeying

$$x + y = y + x$$

and

$$(x + y) + z = x + (y + z)$$

for all elements x, y, z. Too simple for any interesting theorems? No!

We can describe commutative semigroups using generators and relations. I'm especially interested in the finitely presented ones.

We can't hope to classify all commutative semigroups, not even the finitely presented ones. But there's plenty to say about them. n

For starters, some examples. Take any set of natural numbers. If you take all finite sums of these you get a commutative semigroup. So here's a finitely generated one:

$$\{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, \ldots\}$$

See the generators? This kind of example is called a numerical semigroup, although by convention people decree that 0 must be an element of any numerical semigroup, since if it's not you can always put it in without changing anything else.

We can also put in extra relations. So there's a commutative semigroup like this:

$$\{5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22\}$$

with addition as usual except if you'd overshoot 22 you decree the sum to be 22, e.g. $21 + 5 = 22$.

Now it's time to bring a bit of order to this wilderness! Given any commutative semigroup C we can impose the relations $a + a = a$ for all a. The result is called a **semilattice**. Let's call it C'. There's a homomorphism

$$p: C \to C'$$

Note that if $p(x) = a$ and $p(y) = a$ then

$$p(x + y) = a + a = a$$

So the set of x in C that map to a given element a in the semilattice C' is closed under addition! It's a sub-semigroup of C.

In short, given a commutative semigroup C it maps onto a semilattice

$$p: C \to C'$$

and each 'fiber'

$$\{x : p(x) = a\}$$

is a commutative semigroup in its own right.

And these fibers are especially nice: they're 'archimedean semigroups'. A semigroup is **archimedean** if it's commutative and for any x, y we have

$$x + \quad + x = y + z$$

for some z and some number of times of adding x. Can you guess why this property is called 'archimedean'? Hint: it's true for the positive real numbers!

I'll let you check that the fibers of the map from a commutative semigroup C to its semilattice C' are archimedean. It's a fun way to pass the time when you're locked down trying to avoid coronavirus. So, people say "any commutative semigroup is a semilattice of archimedean semigroups".

So, to a large extent we've reduced the classification of commutative semigroups to two cases:

1. semilattices
2. archimedean semigroups

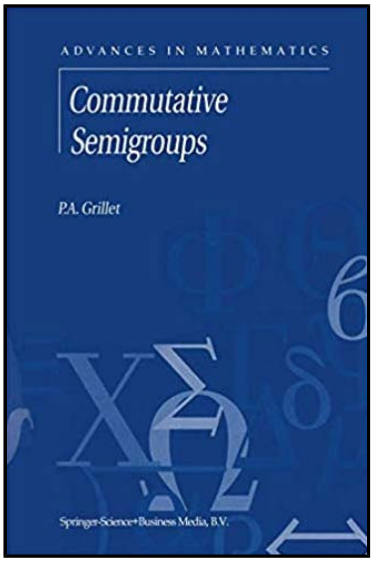
Semilattices are nice because they always have a partial order $\leq$ where $a + b$ is the least upper bound of a and b. Archimedean semigroups are a different story. For example, every abelian group is archimedean.

To see how the story continues, go to this great post:

- Tim Campion, What are the main structure theorems on finitely generated commutative monoids?, *MathOverflow*, April 4, 2020.

Or if you get serious, this book:

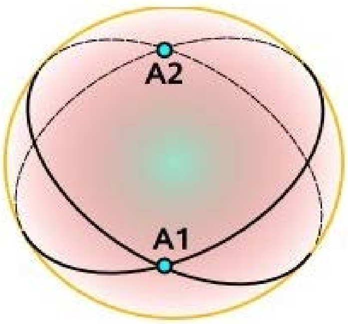
- Pierre A. Grillet, *Commutative Semigroups*, Springer, Berlin, 2013.



April 5, 2020

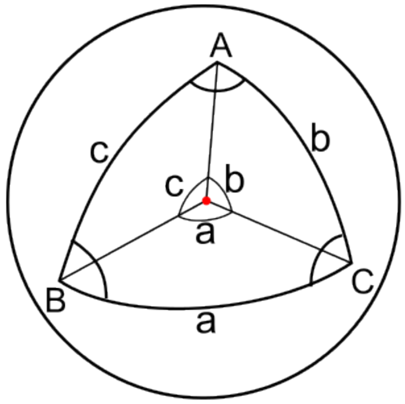
People often say non-Euclidean geometry was discovered in the 1800s, but spherical geometry goes back to the ancient Greeks. It's important in astronomy, because the sky is a sphere!

In spherical geometry, the parallel postulate breaks down.



Spherical trigonometry is more beautiful than plane trigonometry because the sides of a triangle are also described by angles!

This spherical triangle has 3 angles A, B, C and 3 sides, whose lengths are conveniently described using the angles a, b, c:



You can see this beautiful symmetry in the 'law of sines':

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Here A, B, C are the angles of a spherical triangle and a, b, c are the sides, measured as angles. Or the other way around: it's still true if we switch A, B, C and a, b, c!

Spherical geometry is also beautiful because it *contains* Euclidean geometry. Just take the limit where your shape gets very small compared to the sphere!

For example, if the sides a, b, c of a spherical triangle become smaller and smaller,

$$\frac{\sin a}{a}, \frac{\sin a}{b}, \frac{\sin c}{c} \to 1$$

so we get the familiar law of sines in Eucliean geometry:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The law of cosines in spherical geometry is more complicated:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

But you can use it to prove the law of sines. And you just need to remember one of these 3 equations.

Puzzle. In the limit where $a, b, c \to 0$ show the spherical laws of cosines gives the usual Euclidean rule of cosines.

In 100 AD the Greek mathematician Menelaus of Alexandria wrote a 3-volume book *Sphaerica* that laid down the foundations of spherical geometry. He proved a theorem with no planar analogue: two spherical triangles with the same angles are congruent! And much more.

Menelaus' book was later translated into Arabic. In the Middle Ages, astronomers used his results to determine holy days on the Islamic calendar. In the 13th century, Nasir al-Din al-Tusi discovered the law of sines in spherical trigonometry!

Later mathematicians discovered many other rules in spherical trigonometry. For example, these additional laws:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

So in what sense did people only 'invent non-Euclidean geometry' in the 1800s?

Maybe this: to get the axioms of Euclidean geometry except for the parallel postulate to apply to spherical geometry, we need to decree that opposite points on the sphere count as the same. Then distinct lines intersect in at most one point!

Or maybe just this: people were so convinced that the axioms of Eucidean geometry described the geometry of the *plane* that they wouldn't look to the sky for a nonstandard model of these axioms.

Geometry where we identify opposite points on the sphere is called 'elliptic geometry':

- Wikipedia, Elliptic geometry.

Spherical trigonometry is full of fun stuff, and you can learn about it here:

- Wikipedia, Spherical trigonometry.

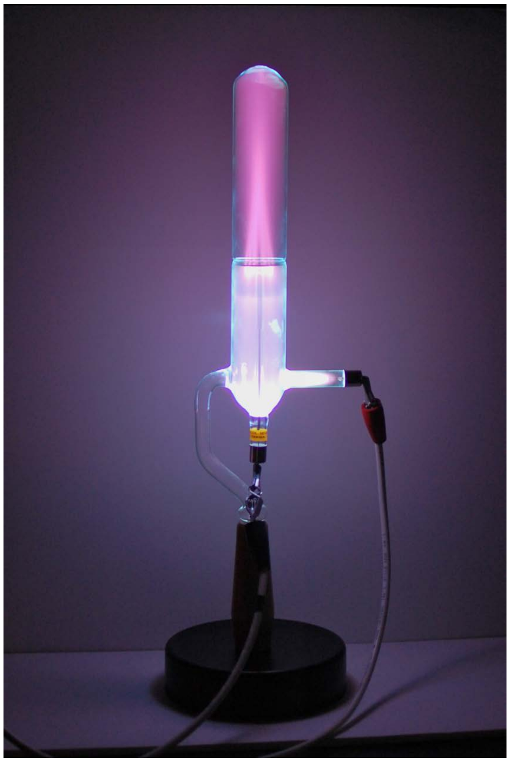
April 11, 2020

If you read math papers it pays to keep this in mind:

Most mathematicians are not writing for people. They're writing for God the Mathematician. And they're hoping God will give them a pat on the back and say "yes, that's exactly how I think about it".

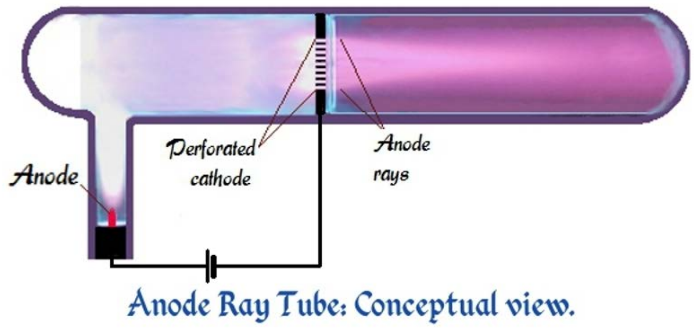
April 12, 2020

THE GOLDEN AGE OF STEAMPUNK PHYSICS



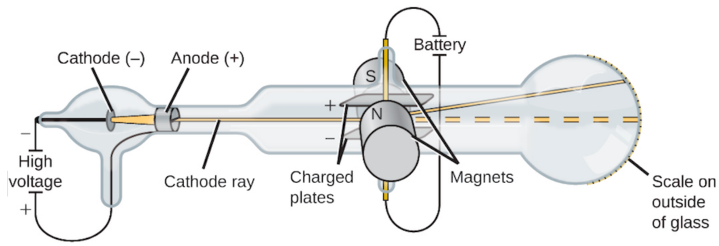
Back in 1886, you didn't need an enormous particle accelerator to discover new particles. You could build a gadget like this and see faint rays emanating from the positively charged metal tip.

They called them 'canal rays'. They also called them 'anode rays', since a positively charged metal tip is called an 'anode'.



We now know these anode rays are atoms that have some electrons stripped off, also known as 'positively charged ions'. So, they come in different kinds!

But back in 1886 when Goldstein discovered them, it wasn't clear whether canal rays were particles or just 'rays' of some mysterious sort. 'Cathode rays', now known to be electrons, had already been discovered in 1876.



X-rays (now known to be energetic photons) came later, in 1895.

Lots of rays! And there were also 'N-rays', now known to be a mistake.

I love the complicated story of how people studied these various 'rays' and discovered that atoms were electrons orbiting atomic nuclei made of protons and neutrons... and that light itself is made of photons.

These were the glory days of physics — the wild west.

To learn more about these stories, I recommend the start of this:

- Emilio Segre, *From X-Rays to Quarks: Modern Physicists and Their Discoveries*, W. H. Freeman, San Francisco, 1990.

But there should be some fun books or papers that focus on the study of 'rays' from 1869 to 1915. Do you know one? Here's the best I've found so far:

- Karl Wien, 100 years of ion beams: Willy Wien's canal rays, *Brazilian Journal of Physics* **29** (1999), 401–414.

This notes that anode rays were also called 'positive light'.

April 13, 2020

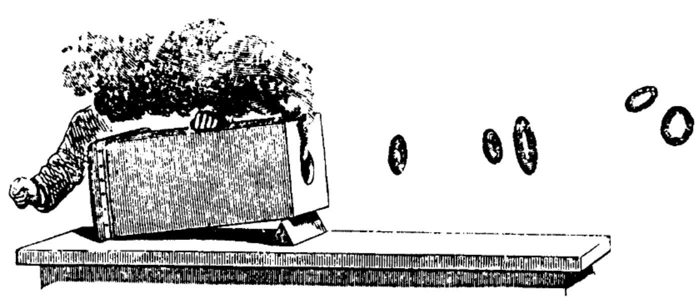


Fig. 1. Tait's smoke ring apparatus. Source: Tait 1876, p. 292.

In the late 1800s, when physicists were trying to understand how stable atoms of different kinds could exist, Tait's experiments with smoke rings seemed quite exciting. They're quite stable. So people thought: maybe atoms are vortices in the 'aether' — the substance filling all space, whose vibrations were supposed to explain electromagnetism!

To explain different kinds of atoms, Tait suggested they were *knotted* or *linked* vortex rings. He classified knots to see if this could explain the atoms we see. And thus knot theory was born!

Later Kelvin became fascinated by the theory of vortex atoms. He begain studying vortex rings, using ideas developed by Helmholtz starting around 1858.

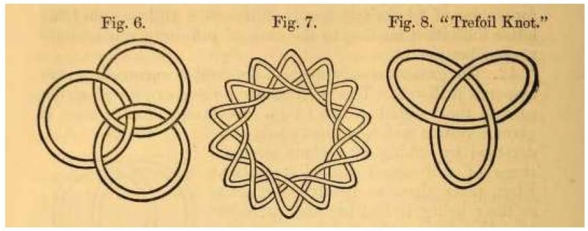


Figure 2.1: Different types of vortex rings, from Kelvin's 1880 paper "Vortex Statics". This is how Kelvin thought different elements were possible in the vortex theory of the atom. Every element would be represented by a particular type of configuration. (Kelvin 1880a, p. 104)

Kelvin (= Thomson) argued that vortex rings were a better theory than the main alternative: atoms as point particles or infinitely rigid balls. He started calculating the vibrational modes of vortex rings, hoping they could explain atomic spectra!

> After a correspondence with Tait, Thomson publicly announced his visionary hypothesis of vortex atoms on February 18, 1867, in a lecture before the Royal Society of Edinburgh. Here, he enthusiastically elaborated on the points mentioned in his letter to Helmholtz. He strongly criticised "the monstrous assumption of infinitely strong and infinitely rigid pieces of matter" that even "the greatest modern chemists" took for granted. As he pointed out, this classical conception of the atom was able to explain the properties of matter only by attributing them to the atom itself. To the mind of Thomson, this was not an explanation at all. As a much preferable alternative he considered that "the only true atoms" were the vortex rings and filaments described by Helmholtz's theory. Among the properties of the vortices that appealed to him were their definite modes of vibration, which not only presented "an intensely interesting problem of pure mathematics" but also suggested an explanation of the spectra produced by chemical elements.

For a while, in the late 1800s, you could get a job at a British university studying vortex atoms. The subject was never very popular outside Great Britain.

The "discovery" of the vortex atom, itself a product of mathematical hydrodynamics, triggered a series of mathematical investigations directly or indirectly related to Thomson's theory. Over the next three decades, the original vortex theory mutated into a large number of mathematical models that, in many cases, had only little connection to claims of physical reality. Although interest in vortex motion was not limited to Great Britain, it was only in this country that researchers sought to develop the theory into a vortex model of ether and matter, or both. During the last quarter of the century, more than a dozen, mainly Cambridge-trained British mathematicians and physicists were busy with developing the insights originally obtained by Helmholtz and Thomson. The growing vortex programme, backed up by the immense authority of Thomson, became a career possibility for many British physicists (Pauly 1975). The most active scientists in the field, apart from Thomson himself, were William Hicks and Joseph J. Thomson, but they were not alone. To the same research tradition in vortex hydrodynamics, or topics related to it, must be counted Augustus Love, Micaih Hill, Horace Lamb, Thomas Lewis, Charles Chree, Alfred Greenhill, Henry Pocklington, C. V. Coates, Arthur Leahy, Alfred Basset, Richard Hargreaves, and Horatio Carslaw, and possibly a few others. As to Tait, Lodge and FitzGerald, they stood somewhat outside this group.

Over time the models became more complex, but they never fit the behavior of real-world atoms with any precision. Eventually Kelvin gave up on vortex atoms... but he didn't admit it publicly until much later.

In an 1896 talk FitzGerald, another expert on vortex atoms, recognized the problems — but argued that it was "almost impossible" to falsify the theory, because it was so flexible.

To put it in a nutshell, "the" vortex atom theory was so rich and flexible, and so undetermined, that it was practically beyond falsification. Of course, for the very same reason it was also unverifiable. The mathematical richness of the vortex theory might be considered a blessing, but it was a curse as well. It made FitzGerald believe that it was "almost impossible" that the universe would not be explainable in vortex terms (see section 4). The generalised vortex theory that he dreamt of could in principle explain everything, and therefore also the properties of the one and only universe. But could it also explain why the numerous other conceivable states of the universe, all of them describable within the theory's framework, do *not* exist? The theory could (again in principle) explain the mass of an atom of chlorine, but had chlorine had any other atomic weight the theory could account for that as well. In short, the theory explained too much – and therefore too little.

Much of the magic of the vortex atom programme rested on its claim of being a unified and all-encompassing theory, indeed a theory of everything. Never, since the days of Descartes, had there been such an ambitious and fundamental theory of physics. It is scarcely surprising that much of the unification rhetoric of the vortex programme can be found also in later theories of a similar grand scope. The successor, in a sense, of the vortex atom theory was the generalised electron theory or so-called electromagnetic world picture based on the theories of Larmor, Lorentz, Max Abraham and others.

The theory of vortex atoms was never quite disproved. But eventually people lost interest in them — thanks in part to rise of Maxwell's equations (which led to other theories of atoms), and later perhaps in part to the discovery of electrons, and "canal rays", and other clues that would eventually help us unravel the mystery of atoms.

All the quotes above are from this wonderful article:

- Helge Kragh, *The Vortex Atom: A Victorian Theory of Everything*, *Centaurus* **44** (2002), 32–114.

Read it and you'll be transported to a bygone age... with some lessons for the present, perhaps.

Jim Baggott also recommends this:

- Helge Kragh, *Higher Speculations: Grand Theories and Failed Revolutions in Physics*, Oxford U. Press, Oxford, 2011.

April 14, 2020

A term of length 4,523,659,424,929

A. R. D. Mathias

Universidad de los Andes, Santa Fé de Bogotá and Humboldt Universität zu Berlin

Abstract Bourbaki suggest that their definition of the number 1 runs to some tens of thousands of symbols. We show that that is a considerable under-estimate, the true number of symbols being that in the title, not counting 1,179,618,517,981 disambiguatory links.

The French mathematicians who went under the pseudonym Nicolas Bourbaki did a lot of good things - but not so much in the foundations of mathematics. Adrian Mathias, a student of John Conway, showed their definition of "1" would be incredibly long, written out in full.

One reason is that their definition of the number 1 is complicated in the first place. Here it is. I don't understand it. Do you?

Bourbaki's abbreviated definition of 1

Chapters I and II of Bourbaki's *Théorie des Ensembles* were published in 1954, and Chapter III in 1956. Among the primitive signs used was a reverse C, standing presumably for "couple", to denote the ordered pair of two objects. Being typographically unable to reproduce that symbol, we use instead the symbol •. With that change, the footnote on page 55 of Chapter III reads

Bien entendu, il ne faut pas confondre le terme mathématique désigné (chap. I, §1, nº 1) par le symbole "1" et le mot "un" du langage ordinaire. Le terme désigneé par "1" est égal, en vertu de la définition donnée ci-dessus, au terme désigné par le symbole

$$\tau_Z((\exists u)(\exists U)(u = (U, \{\varnothing\}, Z) \text{ et } U \subset \{\varnothing\} \times Z \text{ et } (\forall x)((x \in \{\varnothing\}) \implies (\exists y)((x, y) \in U))$$
$$\text{et } (\forall x)(\forall y)(\forall y')(((x, y) \in U \text{ et } (x, y') \in U) \implies (y = y')) \text{ et } (\forall y)((y \in Z) \implies (\exists x)((x, y) \in U)))).$$

Une estimation grossière montre que le terme ainsi désigné est un assemblage de plusieurs dizaines de milliers de signes (chacun de ces signes étant l'un des signes τ, $\Box$, $\vee$, $\neg$, $=$, $\in$, •).

But worse, they don't take , "there exists", as primitive. Instead they *define* it—in a truly wretched way.

They use a version of Hilbert's "choice operator". For any formula $\Phi(x)$ they define a quantity that's a choice of x making $\Phi(x)$ true if such an choice exists, and just anything otherwise. Then they define $x\Phi(x)$ to mean Φ holds for this choice.

Quantifiers are introduced as follows:

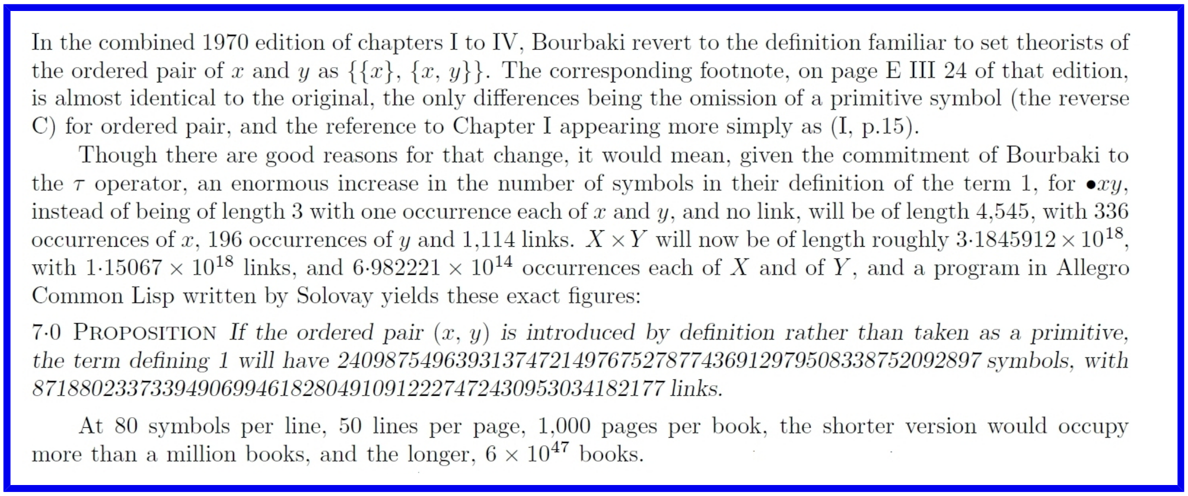
B·5 Definition $(\exists x)R$ is $(\tau_x(R) \mid x)R$;

B·6 Definition $(\forall x)R$ is $\neg(\exists x)\neg R$.

Thus in this formalism quantifiers are not primitive. Informally, the idea is to choose at the outset, for any formula $\Phi(x)$ a witness, some a such that $\Phi(a)$; call it $\tau_x\Phi$. If there is no such witness, let $\tau_x\Phi$ be anything you like, say the empty set.

This builds the axiom of choice into the definition of and . Worse, their implementation of this idea leads to huge formulas.

And in the 1970 edition, things got much worse!



In the combined 1970 edition of chapters I to IV, Bourbaki revert to the definition familiar to set theorists of the ordered pair of x and y as $\{\{x\}, \{x, y\}\}$. The corresponding footnote, on page E III 24 of that edition, is almost identical to the original, the only differences being the omission of a primitive symbol (the reverse C) for ordered pair, and the reference to Chapter I appearing more simply as (I, p.15).

Though there are good reasons for that change, it would mean, given the commitment of Bourbaki to the τ operator, an enormous increase in the number of symbols in their definition of the term 1, for $\bullet xy$, instead of being of length 3 with one occurrence each of x and y, and no link, will be of length 4,545, with 336 occurrences of x, 196 occurrences of y and 1,114 links. $X \times Y$ will now be of length roughly $3{\cdot}1845912 \times 10^{18}$, with $1{\cdot}15067 \times 10^{18}$ links, and $6{\cdot}982221 \times 10^{14}$ occurrences each of X and of Y, and a program in Allegro Common Lisp written by Solovay yields these exact figures:

7·0 Proposition *If the ordered pair* (x, y) *is introduced by definition rather than taken as a primitive, the term defining 1 will have 2409875496393137472149767527877436912979508338752092897 symbols, with 871880233733949069946182804910912227472430953034182177 links.*

At 80 symbols per line, 50 lines per page, 1,000 pages per book, the shorter version would occupy more than a million books, and the longer, 6×10^{47} books.

You can read Mathias' paper here:

- Adrian R. D. Mathias, A term of length 4,523,659,424,929 *Synthese* **133** (2002), 75–86.

For my own overview, see:

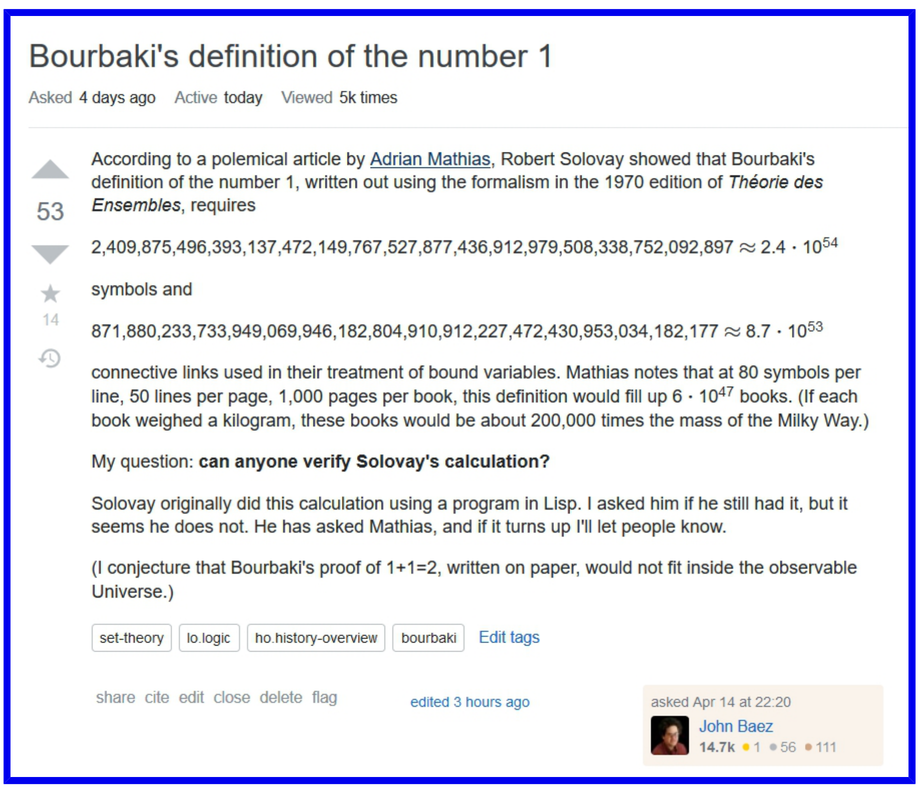
- John Baez, Bigness (Part 1), *Azimuth*, April 13, 2020.

April 16, 2020

Bourbaki's final perfected definition of the number 1, printed out on paper, would be 200,000 times as massive as the Milky Way.

At least that's what a calculation by the logician Robert Solovay showed. But the details of that calculation are lost. So I asked around. I asked Robert Solovay, who is retired now, and he said he would redo the calculation.

I asked on MathOverflow, and was surprised to find my question harshly attacked. I was accused of "ranting". Someone said the style of my question was "awful".



Bourbaki's definition of the number 1

Asked 4 days ago Active today Viewed 5k times

53

According to a polemical article by Adrian Mathias, Robert Solovay showed that Bourbaki's definition of the number 1, written out using the formalism in the 1970 edition of *Théorie des Ensembles*, requires

$2{,}409{,}875{,}496{,}393{,}137{,}472{,}149{,}767{,}527{,}877{,}436{,}912{,}979{,}508{,}338{,}752{,}092{,}897 \approx 2.4 \cdot 10^{54}$

symbols and

$871{,}880{,}233{,}733{,}949{,}069{,}946{,}182{,}804{,}910{,}912{,}227{,}472{,}430{,}953{,}034{,}182{,}177 \approx 8.7 \cdot 10^{53}$

connective links used in their treatment of bound variables. Mathias notes that at 80 symbols per line, 50 lines per page, 1,000 pages per book, this definition would fill up $6 \cdot 10^{47}$ books. (If each book weighed a kilogram, these books would be about 200,000 times the mass of the Milky Way.)

My question: **can anyone verify Solovay's calculation?**

Solovay originally did this calculation using a program in Lisp. I asked him if he still had it, but it seems he does not. He has asked Mathias, and if it turns up I'll let people know.

(I conjecture that Bourbaki's proof of 1+1=2, written on paper, would not fit inside the observable Universe.)

set-theory lo.logic ho.history-overview bourbaki Edit tags

share cite edit close delete flag edited 3 hours ago asked Apr 14 at 22:20 John Baez 14.7k ●1 ●56 ●111

Maybe they thought I was attacking Bourbaki. That's not my real goal here. I'm thinking of writing a book about large numbers, so I'm doing a bit of research.

Admittedly, I added the remark saying Mathias' paper is "polemical" after Todd Trimble, a moderator at MathOverflow, recommended doing some such thing.

Later Solovay said it would be hard to redo his calculation — and if he did he'd probably get a different answer, because there are different ways to make the definition precise.

But here's some good news. José Grimm redid the calculation. He did it *twice*, and got two different answers, both bigger than Solovay's. According to these results Bourbaki's definition of "1", written on paper, may be 400 billion times heavier than the Milky Way.

42

These calculations have been carried out by José Grimm; see [1] as well as [2]. According to one version of the formalism in the original Bourbaki, Grimm gets

$$16420314314806459564661629306079999627642979365493156625 \approx 1.6 \times 10^{55}$$

(see page 517 of [1, version 10]). The discrepancy with Solovay's number is probably due to some subtle difference of interpretation of some detail. Note that the English translation of Bourbaki introduces some "small" changes and Grimm gets a rather different value:

$$5733067044017980337582376403672241161543539419681476659296689 \approx 5.7 \times 10^{60}$$

EDIT: As suggested in the comments, here are the full citations for Grimm's papers.

[1] José Grimm. Implementation of Bourbaki's Elements of Mathematics in Coq: Part Two; Ordered Sets, Cardinals, Integers. [Research Report] RR-7150, Inria Sophia Antipolis; INRIA. 2018, pp.826. inria-00440786v10. doi:10.6092/issn.1972-5787/4771

[2] Grimm, J. (2010). Implementation of Bourbaki's Elements of Mathematics in Coq: Part One, Theory of Sets. Journal of Formalized Reasoning, 3(1), 79-126. doi:10.6092/issn.1972-5787/1899

share cite edit follow flag

edited 2 days ago
John Baez
14.7k • 1 • 56 • 111

answered Apr 15 at 15:19
Timothy Chow
46.9k • 15 • 205 • 366

I'm now quite convinced that a full proof of 1 + 1 = 2 in Bourbaki's formalism, written on paper, would require more atoms than available in the observable Universe.

Of course, they weren't aiming for efficiency.

April 17, 2020

Some news:

- On MathOverflow, Alex Nelson reports the results of his own calculations, which give a result *perfectly matching Robert Solovay's:* namely, Bourbaki's 1970 definition of the number 1 requires

$$2,409,875,496,393,137,472,149,767,527,877,436,912,979,508,338,752,092,897 \approx 2.4 \times 10^{54}$$

  symbols. He writes:

  > I was able to reproduce Mathias's [actually Solovay's] results with some Haskell code with some specific details about how many symbols are needed in each term. (As a sanity check, I verified I recovered the same results term-by-term when the ordered product was primitive.)
  >
  > - Size of 1 = 2,409,875,496,393,137,472,149,767,527,877,436,912,979,508,338,752,092,897
  > - Size of term A = 15,756,227
  > - Size of term B = 10,006,221,599,868,316,846
  > - Size of term C = 59,308,566,315

- Size of term D = 364,936,653,508,895,574,881
- Size of term E = 101,217,516,631

One thing worth noting is that, well, this seems dishonest. I mean, there are a lot of double negations which are not simplified, which bloats the size quite a bit (an additional 1.863×10^{53} symbols or so). I wouldn't be surprised if there were other simplifications which would cut down the bloat further...not that we'd get anything less than 10^{50} or so.

If you'd like to check the number of links, I can do that too.

- Alex Nelson also computed the length of the expression "1 + 1 = 2", confirming my impression that the proof of this statement (which would be even longer) couldn't be written on paper in the observable Universe. The statement "1 + 1 = 2" in Bourbaki's 1970 setup uses about 10^{76} characters, and there seem to be at most 10^{82} atoms in the observable Universe, allowing only a million atoms per symbol — and most of them are hydrogen.

  He writes:

  > **Addendum.** The relation "1+1=2" can be computed, and found to have a length of
  >
  > 22, 411, 322, 875, 029, 037, 193, 545, 441, 224, 646, 148, 573, 589, 725, 893, 763, 139, 344, 694, 162, 029, 240, 084, 343, 041
  >
  > or approximately
  >
  > $$2.24113228750290371 \times 10^{76}.$$
  >
  > This is using the definitions in Bourbaki of cardinal addition a + b using the disjoint sum of the indexed family $f\colon \mathrm{Card}(2) \to \{a, b\}$ considered as a graph. It's *really convoluted*, but the details can be found in Bourbaki's *Theory of Sets* Chapter II sections 3.4, 4.1, and 4.8 as well as Proposition 5 (in chapter III, section 3.3); this all works with the Kuratowski ordered pair, not a primitive $\cdot AB$ ordered pair.
  >
  > For what it's worth, computing the size of 1 was nearly instantaneous, whereas computing the size of "1+1=2" took about 7 minutes and 30 seconds.

- I asked Robert Solovay if he still had the original program he used to compute the length of Bourbaki's definition of "1" (using their 1970 definition of ordered pairs). He didn't, but Adrian Mathias did, and Solovay has allowed me to release it. So, here are three documents:

  - calcusol.pdf. This is a document entitled "The Bourbaki Constant 1". It begins:

    > This is a private document [for the eyes of RMS and ARDM only] which extends ARDM's computation of the length of the Bourbaki rendition of "The ineffable name of 1" to the case when the Kuratowski ordered pair is employed. My plan is to write programs in Allegro Common Lisp to compute the relevant numbers.
    >
    > I program using the style of "literate programming" introduced by Knuth. However the Web and Tangle introduced by Knuth [which have been refined to CWEb and CTangle by Levy] are limited to languages closely linked to C or Pascal. So I prefer to use a more flexible literate programming language which permits fairly arbitrary target languages. Currently, I use Nuweb which is available on the TEX archives in the directory /web/nuweb.
    >
    > One of the nice things about literate programming is that one can write programs in the natural psychological order, but arrange that the output files have the order needed for the target programming language. We will exploit this heavily in what follows.

  - solfact.txt. This is a short document including the output of the program described in the previous document. Here it is, in its entirety:

```
From solovay@math.berkeley.edu Wed Nov 11 13:39:46 1998
Date: Tue, 10 Nov 1998 23:43:04 -0800 (PST)
From: "Robert M. Solovay"
To: amathias@rasputin.uniandes.edu.co
Cc: solovay@math.berkeley.edu
Subject: Results
```

```
Adrian,

    Here is the printout of my calculation of the length of the
Bourbaki term for 1. If we do the original definition, I get approx.

4.524 * 10^{12}

    If we use the Kuratowski ordered pair, I get approx.

2.41 * 10^{54}

    This is big, but not nearly as big as the 2 * 10^{73} that you
claim. This is certainly related to the smaller estimate that I have
for the size of the Kuratowski ordered pair.

    I omitted some trivial lines from this printout where I "gave
the wrong commands to the genie".


USER(1): (setq p 0) ;;;[Doing the original Bourbaki definition where
                    ;;; ordered pair is a basic undefined notion.]
0

USER(3): (load "compute.cl")
; Loading ./compute.cl
T
USER(4): J_length
4523659424929
USER(5): (log J_length 10)
12.65549
USER(6): J_links
1179618517981
USER(7): (log J_links 10)
12.071742
USER(8): (setq p 1) ;;; Now use Kuratowski ordered pair

1

USER(10): (load "compute.cl")
; Loading ./compute.cl
T
USER(11): J_length
2409875496393137472149767527877436912979508338752092897
USER(12): (log J_length 10)
54.381996
USER(13): J_links
871880233733949069946182804910912227472430953034182177
USER(14): (log J_links 10)
53.940456
```

- solpair.pdf. This is a short note on some of Bourbaki's definitions including the definition of ordered pair.

April 18, 2020

This is a yellow-bellied three-toed skink.

Near the coast of eastern Australia, it lays eggs. But up in the mountains, the same species gives birth to live young! Intermediate populations lay eggs that take only a short time to hatch.

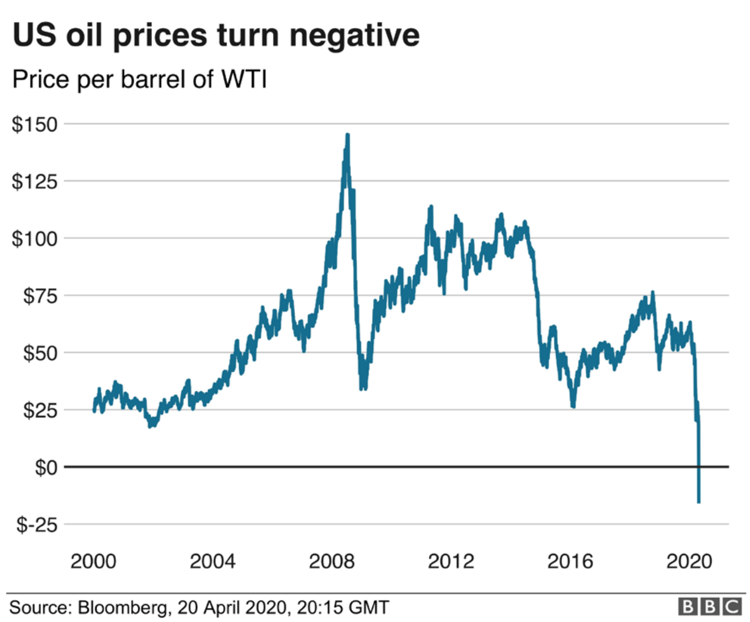
Even more surprisingly, [Dr. Camilla Whittington](#) found a yellow-bellied three-toed skink that lay three eggs and then weeks later, give birth to a live baby from the same pregnancy!

(Here, alas, she is holding a different species of skink.)

The yellow-bellied three-toed skink may give us clues about how and *why* some animals transitioned from egg-laying (ovipary) to live birth (vivipary). I wish I knew more details!

Its Latin name is *Saiphos equalis*. It's the only species of its genus.

April 20, 2020



I predict that next week the price of oil will hit negative infinity, then start coming down from positive infinity, then take a left turn and develop a positive imaginary part.

April 24, 2020

Funny how it works. Learning condensed matter physics led me to the '10-fold way' and 'super division algebras'.

That made me want to learn more about division algebras over fields other than the real numbers!

Now I'm studying generalizations of the quaternions.

QUATERNION ALGEBRAS

For any field F and any $a, b \in F$, we define the quaternion algebra $Q_{a,b}$ to be the vector space

$$Q_{a,b} = \{t1 + xi + yj + zk\colon\ t, x, y, z \in F\}$$

with the multiplication obeying

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji.$$

If $ax^2 + by^2 = 1$ has a solution in F, then $Q_{a,b}$ is isomorphic to the algebra of 2×2 matrices with entries in F. Otherwise $Q_{a,b}$ is a division algebra: if the product of two elements is zero, one must be zero.

Every 4-dimensional division algebra is a quaternion algebra!

Here's a great little introduction to quaternion algebras:

- Thomas R. Shemanske, Perspectives on the Albert-Brauer-Hasse-Noether Theorem for quaternion algebras.

It proves the stuff in the box above.

April 26, 2020

When they have trouble understanding a theorem, ordinary mathematicians ask: "What's an example of this?"

Category theorists ask: "What's this an example of?"

I'm in that situation myself trying to learn about division algebras and how they're connected to Galois theory. Gille and Szamuely's book *Central Simple Algebras and Galois Cohomology* is a great introduction.

But one of the key ideas, 'Galois descent', was explained in a way that was hard for me to understand.

It was hard because I sensed a beautiful general construction buried under distracting details. Like a skier buried under an avalanche, I wanted to dig it out.

I started digging, and soon saw the outlines of the body. We have a field k and a Galois extension K. We have the category of algebras over k, $\mathrm{Alg}(k)$, and the category of algebras over K, $\mathrm{Alg}(K)$. There is a functor

$$F\colon \mathrm{Alg}(k) \to \mathrm{Alg}(K)$$

which is a left adjoint.

We fix A $\mathrm{Alg}(K)$. We want to classify, up to isomorphism, all a $\mathrm{Alg}(k)$ such that $F(a)$ A. This is the problem!

The answer is: the set of isomorphism classes of such a is

$$H^1(\mathrm{Gal}(K|k), \mathrm{Aut}(A))$$

This is the first cohomology of the Galois group $\mathrm{Gal}(K|k)$ with coefficients in the group $\mathrm{Aut}(A)$, on which it acts.

I began abstracting away some of the details. My first attempt is here:

- John Baez, Crossed homomorphisms, *The n-Category Café*, April 24, 2020.

It felt awkward and clumsy, but I knew I was making progress. I made a bit more progress in the comments to this article.

But last night, I found Qiaochu Yuan wrote a series of articles tackling exactly this problem: finding a clean categorical understanding of Galois descent! He was tuned into exactly my wavelength.

This is where the series starts:

- Qiaochu Yuan, The puzzle of Galois descent, *Annoying Precision*, November 8, 2015.

The second made the role of the Galois group clear:

- Qiaochu Yuan, Group actions on categories, *Annoying Precision*, November 9, 2015.

$\mathrm{Gal}(K|k)$ doesn't act on any one object of $\mathrm{Alg}(K)$, since Galois transformations aren't K-linear. It acts on the whole category $\mathrm{Alg}(K)$!

The third in his series corrects a mistake. The fourth shows that objects of $\mathrm{Alg}(k)$ are the same as *homotopy fixed points* of the action of $\mathrm{Gal}(K|k)$ on $\mathrm{Alg}(K)$:

- Qiaochu Yuan, Fixed points of group actions on categories, *Annoying Precision*, November 11, 2015.

I've loved homotopy fixed points of group actions on categories for years! They're such a nice generalization of the ordinary concept of 'fixed point'.

With this background, Qiaochu is able to state the problem of Galois descent in a clear and general way, which handles lots of other problems besides than the one I've been talking about:

- Qiaochu Yuan, Stating Galois descent, *Annoying Precision*, November 16, 2015.

Then, finally, he explains how group cohomology gets into the game: why the set of isomorphism classes of $a \quad \mathrm{Alg}(k)$ such that $F(a) \quad \mathrm{Alg}(K)$ is

$$H^1(\mathrm{Gal}(K|k), \mathrm{Aut}(A))$$

So I am happy!!! But there's more to understand...

For one, group cohomology has a strong connection to *topology*. I explained this in the case of H^1 in a comment on the *n-Category Café*. Since Galois extensions of fields are analogous to covering spaces in topology, this should give us extra insight into Galois descent!

So I have some more fun thinking to do, despite the enormous boost provided by Qiaochu Yuan.

By the way, I'm sure some experts in algebraic geometry already have the categorical/topological perspective I'm seeking. This is not new research yet: this is study.

For my May 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

home

For my April 2020 diary, go here.

Diary — May 2020

John Baez

May 1, 2020

This spring I learned that most of the weeds in our yard are edible — and delicious. We don't have weeds. We have salad!

Growing your own arugula: it's not rocket science.

When it rains, which happens only in the winter and spring, we get a lot of Australian rocket (*Sisymbrium erysimoides*), wild lettuce (*Lactuca virosa*), sow thistle (*Sonchus arvensis*), prickly sow thistle (*Sonchus asper*) and wild turnip, a kind of mustard (*Brassica rapa sylvestris*).

May 2, 2020

There are three finite-dimensional associative division algebras over the real numbers: finite-dimensional real vector spaces equipped with an associative product where you can divide by anything nonzero!

- The reals themselves, R
- The complex numbers, C

Processing math: 100%

The quaternions, H

R and H form the Brauer group of the reals. Let me explain!

You can see this by taking tensor products of R, C, and H. A tensor product of two real vector spaces is a new real vector space whose dimension is the product of their dimensions. You can make the tensor product of algebras over the real numbers into another algebra over the reals!

Let's try it. Here are some boring examples:

- R   R   R
- R   C   C
- R   H   H

As you can see, tensoring with R doesn't do anything. So, it will be the identity element of the Brauer group.

But let's do some more interesting examples....

- C   R   C
- C   C   C   C
- C   H   C[2]

Here C   C is the algebra of pairs of complex numbers, while C[2] is the algebra of 2 × 2 complex matrices!

Note how C 'eats' everything else and makes it complex. It's not in the Brauer group!

- H   R   H
- H   C   C[2]
- H   H   R[4]

When you tensor two copies of the quaternions you get the algebra R[4] of 4Ã—4 real matrices. In short:

- real   real = real
- real   quaternionic = quaternionic
- quaternionic   quaternionic = real

but

- complex   real = complex
- complex   complex = complex
- complex   quaternionic = complex

So when we tensor 'real', 'quaternionic' and 'complex' it's like multiplying 1, -1 and 0. 0 eats everything, but 1 and -1 form a group. This is the Brauer group of the real numbers!

In short, the Brauer group of the real numbers has two elements. The complex numbers is not allowed into this group because it eats everything you tensor it with. Turns out that's because it's not just a division algebra, it's a *field*: it's commutative!

You can do this 'Brauer group' game starting with any field. There are associative division algebras over this field. You can tensor them, and figure out a multiplication table. Some will have inverses, and they're in the Brauer group!

The Brauer group of the rational numbers, Q, is a *lot* more interesting than the Brauer group of R.

Brauer, Hasse and Noether teamed up and figured out the Brauer group of any 'algebraic number field', meaning Q with some algebraic numbers like $\sqrt{-2}$ and $\sqrt[3]{7}$ thrown in.

The Brauer group $\mathrm{Br}(K)$ of an algebraic number field K fits into an exact sequence constructed by Hasse:

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in S} \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$

where S is the set of all places of K and the right arrow is the sum of the local invariants. The **Albert–Brauer–Hasse–Noether theorem** says the left arrow is injective.

Here's a really fun way to learn more about this:

- Paul Roquette, The Brauer-Hasse-Noether theorem in historical perspective.

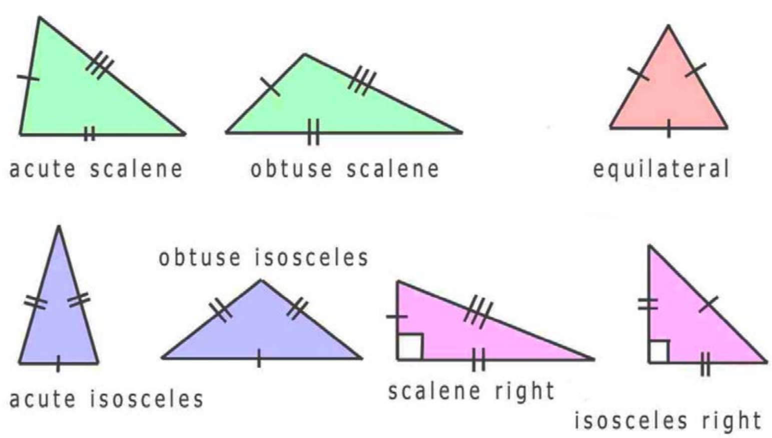
It explains the math as well as the history. (The American mathematician Albert, known for his work on octonions, also did work on this theorem.)

It's fun to read the first letter from Noether to Brauer on this subject in 1927. She is quite dominant! But it makes sense: Brauer was younger, and he had sent her his thesis for comments just earlier that year.

NOETHER TO BRAUER

Dear Mr. Brauer! I am very glad that now you have also recognized the connection between representation theory and the theory of noncommutative rings, the "algebras", and the connection between the Schur index and division algebras.

May 4, 2020



There's a category of triangles! Objects are triangles in the plane. Morphisms are ways of translating, rotating and/or reflecting the plane to carry one triangle to another.

Triangles with symmetries — isosceles and equilateral — have morphisms to themselves.

This category is a 'groupoid': all morphisms have inverses.

In fact it's a 'Lie groupoid': there's a smooth manifold of objects, a smooth manifold of morphisms, and composition is a smooth function. (There's a bit more to the definition, but that's most of it.)

Any Lie groupoid gives a 'differentiable stack'. I won't define those, but the advantage of working with stacks is that the morphisms are more flexible. Only when you move on up to differentiable stacks are you combining groupoids & manifolds in the best way!

Differentiable stacks tend to be good when you've got a space of things with symmetries — like the space of all triangles in the plane. As a thing moves around in this space, its amount of symmetry can suddenly increase, like when a scalene triangle become isosceles.

The 'moduli space' — the space of *isomorphism classes* of things — will have singularities at the points where those things have more symmetry. But the differentiable stack will still be nice there, because you're not modding out by those symmetries.

Stacks are still scary to most mathematicians. The Stacks Project aims at becoming a complete reference on stacks as used in *algebraic* geometry:

- The Stacks Project.

But differentiable stacks are something you've already met in school, without knowing it! For a detailed introduction to stacks, with a lot about the stack of triangles, try this:

- Kai Behrend, Introduction to algebraic stacks, December 17, 2012.

QUATERNION ALGEBRAS

For any field F and any $a, b \in F$, we define the quaternion algebra $Q_{a,b}$ to be the vector space

$$Q_{a,b} = \{t1 + xi + yj + zk\colon\ t, x, y, z \in F\}$$

with the multiplication obeying

$$i^2 = a,\ \ j^2 = b,\ \ ij = k = -ji.$$

If $ax^2 + by^2 = 1$ has a solution in F, then $Q_{a,b}$ is isomorphic to the algebra of 2×2 matrices with entries in F. Otherwise $Q_{a,b}$ is a division algebra: if the product of two elements is zero, one must be zero.

Every 4-dimensional division algebra is a quaternion algebra!

May 7, 2020

Like many scientists I have a grudging admiration for the *Star Trek* franchise: grudging because the science is so often silly, and could often have been improved easily without spoiling the stories; admiration because they've created a hopeful vision of the future, some fun stories, and some enduringly interesting characters. In *Discovery* we heard about the Logic Extremists, a dissident faction of Vulcans who wanted to leave the Federation. But we didn't learn much about their core beliefs! They seemed rather similar to the Vulcan Isolationists, who came about a hundred years later. There seemed to be an interesting untold story lurking behind the name.

So, I went to T'Karath and spent a couple of weeks poring through the historical documents on this movement. Here's a quick sketch of what I found.

In the first half of the 22nd century, the central government had become corrupt, with Romulan operatives infiltrating the Vulcan High Command. Some Vulcans, the Syrannites, attempted to reinstate and develop the original teachings of the Vulcan philosopher Surak. But around 2140, another small group decided that Surak had not developed logic with sufficient thoroughness. They argued that all deductive reasoning should be formalized, all inductive reasoning should be Bayesian with explicit probabilities on hypotheses, and all decision-making should maximize utility.

The Pure Logic movement, as they called themselves, moved to Xir'tan and set up a commune there. They began a program of formal concept analysis so that all words would have precise definitions. Before each meal they bowed, seemingly in prayer, but actually to optimize their activities to come. Children were schooled in an even more disciplined way than usual: less high-tech than the skill domes of the 2200s, but with an intense focus on logic, semiotics, probability, and statistics.

Conflicts erupted in 2200 between what we would call Jaynesian-Bayesians and hardcore subjective Bayesians. The former advocated entropy-maximizing priors. The latter argued that no prior counts as 'right' without further assumptions, so one can start with any prior.

As the Pure Logic movement became established, they spread and set up communes the main continent, especially in Gol, Xial and Raal. They started influencing the political establishment, first locally and then at the federal level.

As this happened, factions with radical positions gradually gained influence. Especially important were the subjective Bayesians who argued that ethics could not be logically derived, so that instead of maximizing utility, a rational agent was free to maximize any chosen quantity. Their motto was remarkably similar to a saying credited to Hume:

From an "is" one cannot derive an "ought".

Going further, the most extreme subjective Bayesians adopted spreading the Pure Logic movement as their only goal. All decisions were to be evaluated based on how much they furthered the spread of logical thinking. They took a vow to this effect, and pressed this vow on other citizens as a prerequisite for holding office of any sort. Their opponents dubbed them "Logic Extremists".

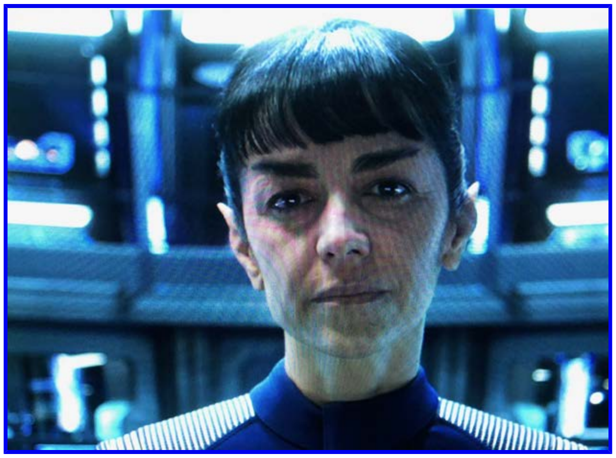
In 2226, in a hard-fought political struggle, these extremists triumphed and completely pushed the Jaynesian-Bayesians and moderate subjective Bayesians out of power. Two years later V'arak took control: a charismatic leader who asserted with 100% prior probability that the Federation was trying to subvert Vulcan culture and stop the spread of the Pure Logic movement.

Any attempt to reason with V'arak and his supporters, or compromise with them, was interpreted as further evidence of an increasingly elaborate Federation conspiracy. Most Vulcans repudiated this stance, and as the Logic Extremists' public support shrank they turned to terrorism.

The violence came to a head around 2256, when V'latak (shown below) attempted to assassinate Sarek before the peace talks on Cancri IV, saying:

My sacrifice will be a rallying cry to those who value logic above all.
Vulcans will soon recognize and withdraw from the failed experiment
known as the Federation.

At this point support for the Logic Extremists rapidly dropped, though Patar still managed to infiltrate Section 31.



However, the most interesting aspect of the Logic Extremists are their early theoretical writings — especially those of Avarak, and Patar's father Tesov. They were an extremely bold attempt to plan a society based purely on logic. I hope they're translated soon.

May 14, 2020

I've been thinking about Morita equivalence.

The basic idea is this: any ring has a category of modules. If two rings have equivalent categories of modules, they're 'Morita equivalent'. So we take the attitude: "the main reason to care about rings is their modules".

Isomorphic rings are Morita equivalent, but the fun part is that *non*isomorphic rings can be Morita equivalent! For example, the ring of $n \times n$ matrices with entries in a ring R is Morita equivalent to R, since a module of this matrix ring must look like M^n for some module M of R.

Lots of properties of rings are invariant under Morita equivalence. For example, if two rings are Morita equivalent, and one of them is 'simple' (has no nontrivial ideals), then so is the other.

But the property of being commutative isn't invariant under Morita equivalence. Can you see why? (I gave you a hint.)

Here's the cool part. There's a bicategory with rings as objects, bimodules as morphisms, and bimodule homomorphisms as 2-morphisms. There's a nontrivial theorem saying that equivalence in this bicategory is Morita equivalence!

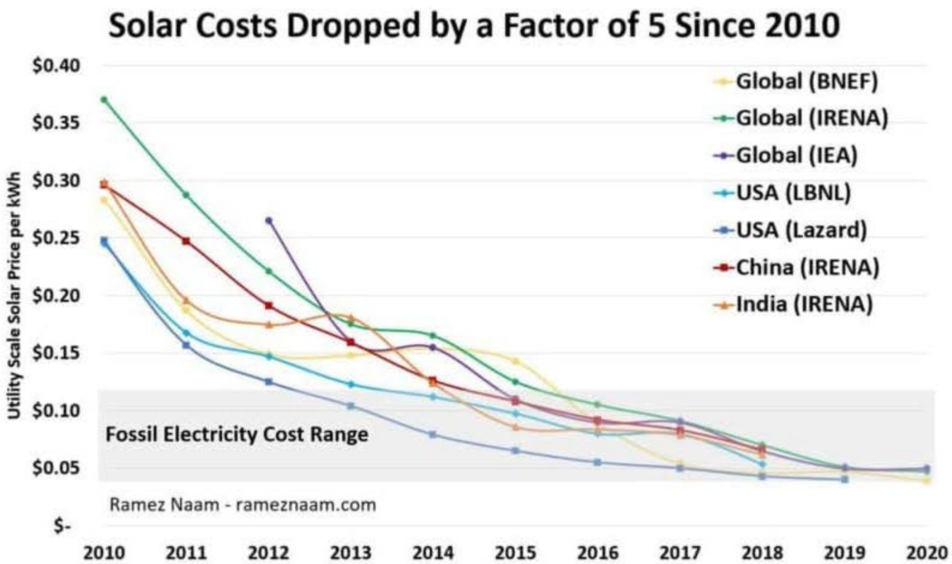
It takes a while to understand what this means, and prove it.

The point is that given rings R and S, an (R, S)-bimodule can be thought of as a funny sort of 'morphism' from R to S. We can tensor an (R, S)-bimodule and an (S, T)-bimodule over S and get an (R, T)-bimodule. This is how we compose these funny morphisms.

So, I'm claiming two rings R and S are Morita equivalent iff there's an (R, S)-bimodule M and an (S, R)-bimodule N such that $M \quad {}_S N \quad R$ as an (R, R)-bimodule, and $N \quad {}_R M$ as an (S, S)-bimodule.

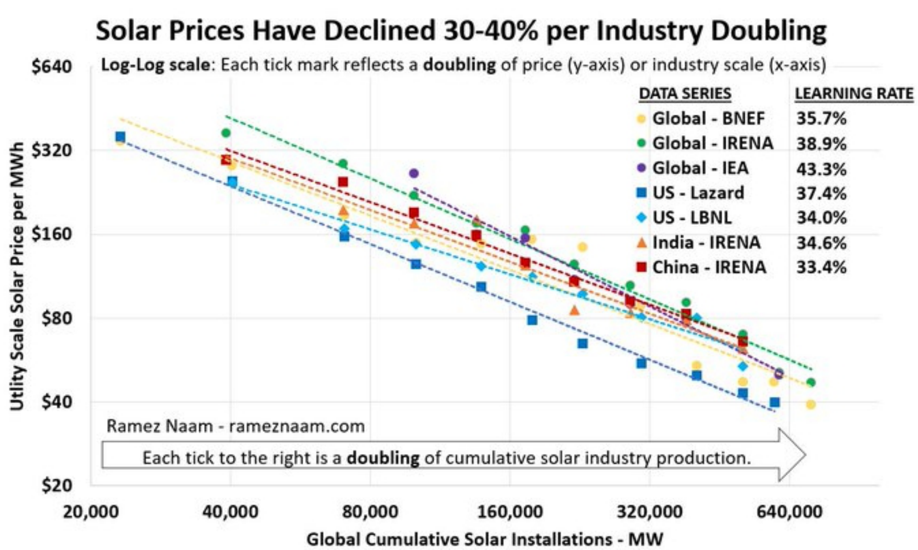
I'm teaching an online 'class' about this on the Category Theory Community Server.

May 15, 2020



Good news: the price of solar power is dropping *very* quickly. This chart by Ramez Naam shows how it's going.

We are now 50-100 years ahead of the International Energy Agency's predictions in 2010. It turns out they were completely clueless.



Naam uses Wrightâ€™s Law to analyze the data. This rule of thumb says that each doubling of the total production of some technology leads to a fixed percentage decline in its price. Solar prices seem to be dropping 30-40% per doubling!

The world will change dramatically as the price of solar continues to plunge. We need this!

Read Ramez Naam's blog for more:

Ramez Naam, Solar's future is insanely cheap (2020).

May 16, 2020

In 3 dimensions a rotating object rotates around an axis. But this way of thinking about it leaves you unprepared for what happens in other dimensions!

You can already see that in 2 dimensions rotations don't happen 'around an axis' - not an axis *in the plane*, anyway!

The right way to think about rotations in 3d is that they're 'in a plane'. That is, there's a plane where points on this plane go round and round... and the point that stands still is the center of rotation.

The 'axis' is just a line at right angles to this plane.

Rotations in 2d are also 'in a plane' — but now this plane is the whole of 2d space.

What about 4d? Now rotations take place in *two* planes. There are two 2d planes at right angles, where points go round and round staying in these planes, maybe at different speeds!

What about 5d? Again rotations take place in *two* planes. There are two 2d planes at right angles where points go round and round... but now there's also a line at right angles to both these planes, where points stay fixed! We're breaking up the dimensions this way:

$$5 = 2 + 2 + 1$$

I hope you get the pattern. In $2n$ dimensions, there are n 2-dimensional planes at right angles where points go round and round, staying in these planes... possibly at different rates. In $2n + 1$ dimensions that's still true, but there is also a fixed axis.

For this reason, rotations in even-dimensional spaces are very different from rotations in odd-dimensional spaces! This shows up all over math and physics. For example, the 'Dynkin diagrams' for rotation groups look very different depending on whether the dimension is even or odd.

The situation gets even subtler when we think about 'spinors' — the gadgets sort of like vectors that describe spin-1/2 particles like the electron.

The math of spinors depends a lot on the dimension of spacetime, not just mod 2, but mod 8.

Now we've gone from something that's obvious if you think about it hard to something that's *far* from obvious. Why should spinning particles care about the dimension of space modulo 8? This is something I've studied over and over again, learning a bit more each time.

I wrote a quick intro to how spinors work in different dimensions here, back when I was first learning how supersymmetry is connected to the octonions:

- This Week's Finds in Mathematical Physics, Week 93, October 27, 1996.

That could be a good place to get started. Maybe too hard... maybe too sketchy... but short.

Someday I'm gonna write a book about this stuff. Only a book can go slowly from the most obvious facts to the most non-obvious, gradually making more and more things seem 'obvious'.

There's a lot to say about rotations in different dimensions!

May 17, 2020

NEGATIVE PROBABILITIES?

Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money. — Paul Dirac

Dirac took negative-energy electrons seriously. He realized a *missing* negative-energy electron would act like a positively charged particle with a *positive* amount of energy: a 'positron'. Then people actually found positrons. (They'd already seen them but couldn't believe it.)

Could he be right about taking negative probabilities seriously?

Negative numbers were invented by Venetian bankers. They started writing numbers in red to symbolize debts — hence the phrase "being in the red". Bankers couldnâ€™t get so rich as they do if negative money didnâ€™t exist.

But can you owe someone a probability?

NEGATIVE PROBABILITIES!

It is usual to suppose that, since the probabilities of events must be positive, a theory which gives negative numbers for such quantities must be absurd. I should show here how negative probabilities might be interpreted.

— Richard Feynman

In 1987 Feynman wrote an essay explaining how negative probabilities could be used. Read it! He explains things well:

- Richard P. Feynman, Negative probability, in *Quantum Implications: Essays in Honour of David Bohm*, eds. F. David Peat and Basil Hiley, Routledge & Kegan Paul Ltd, London, 1987, pp. 235–248.

The idea is that negative probabilities are only allowed in intermediate steps of a calculation, not the final results.

A nice example is the heat equation. It describes how the probability of finding a particle somewhere in a box spreads out in Brownian motion. We can solve it using Fourier series. The individual terms in the Fourier series can give negative probabilities!

Another example is the 'half-coin'. Say you make a bet where you get $1 if a coin comes up heads and $0 if it comes up tails. Say you want this bet to be the same as making two bets involving two separate 'half-coins'. You can do it with negative probabilities! Details here:

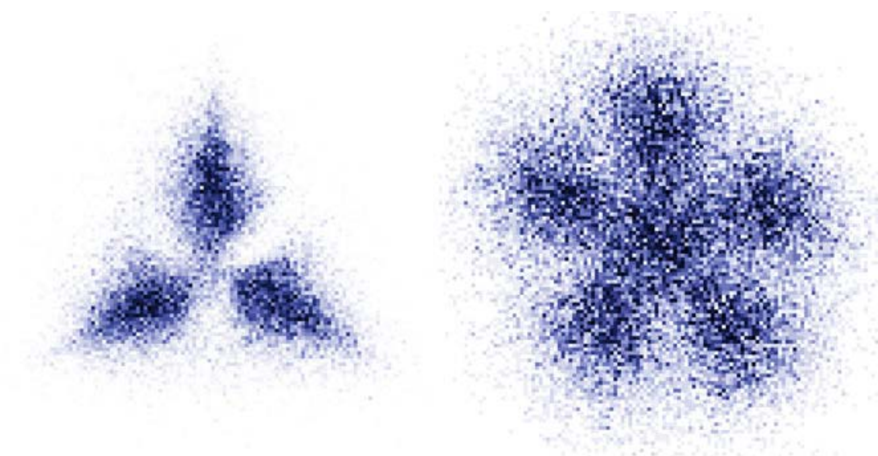
- John Baez, Negative probabilities, *Azimuth*, July 19, 2013.

When I wrote the above stuff on Twitter, Nassim Taleb responded:

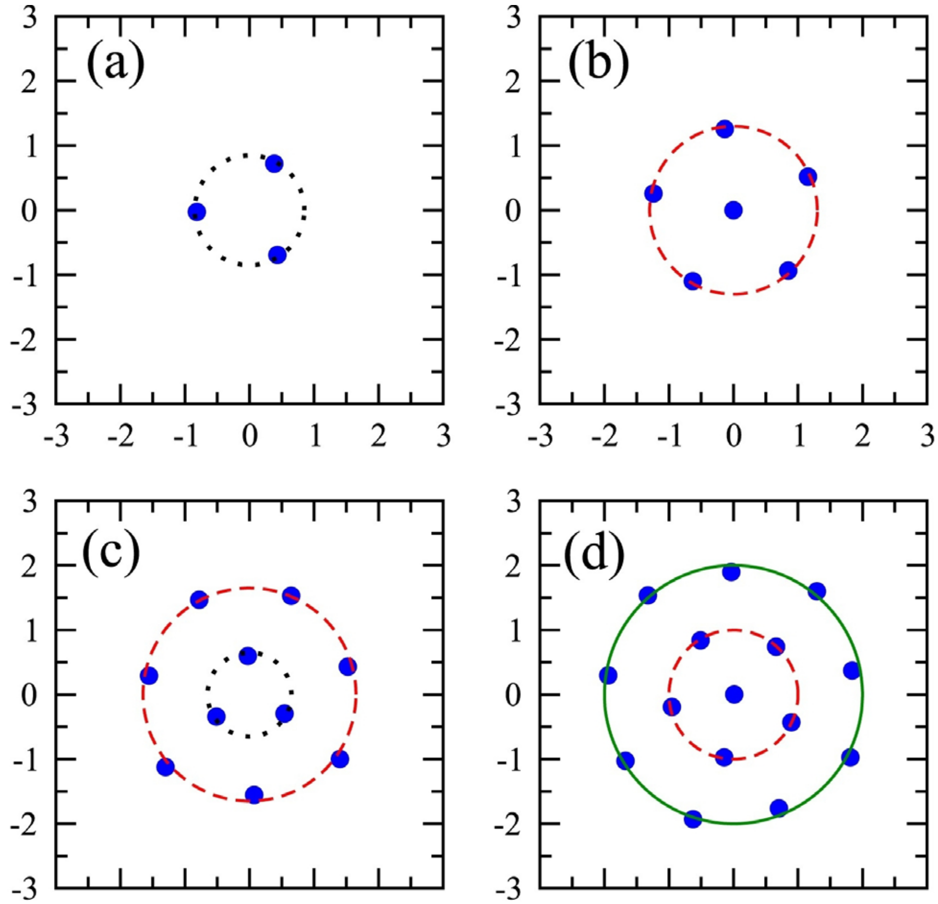
> We use negative probabilities in quant finance since probability is just a KERNEL inside an integral, rarely a "real thing" outside of binary payoffs. So long as *no arbitrage* is satisfied. Only thing that matters is (by scaling) â^«densities=1. Similar to negative prices in oil.
>
> To visualize the intricacy, see my comments #RWRI on negative prices via arbitrage/squeezes.

I guess it's not surprising that that financiers are already using negative probabilities, given that bankers first invented negative numbers!

May 20, 2020



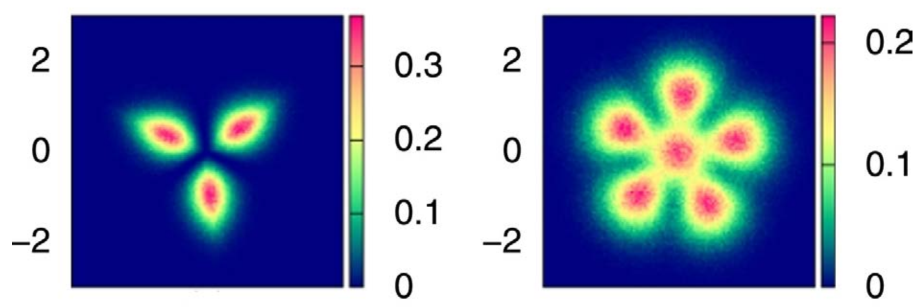
Physicists have just seen 'Pauli crystals'! They're formed when a group of atoms, trapped in a potential well, repel each other only by the Pauli exclusion principle — the rule saying that two fermions can't be in the same state. They are very fragile and tiny.



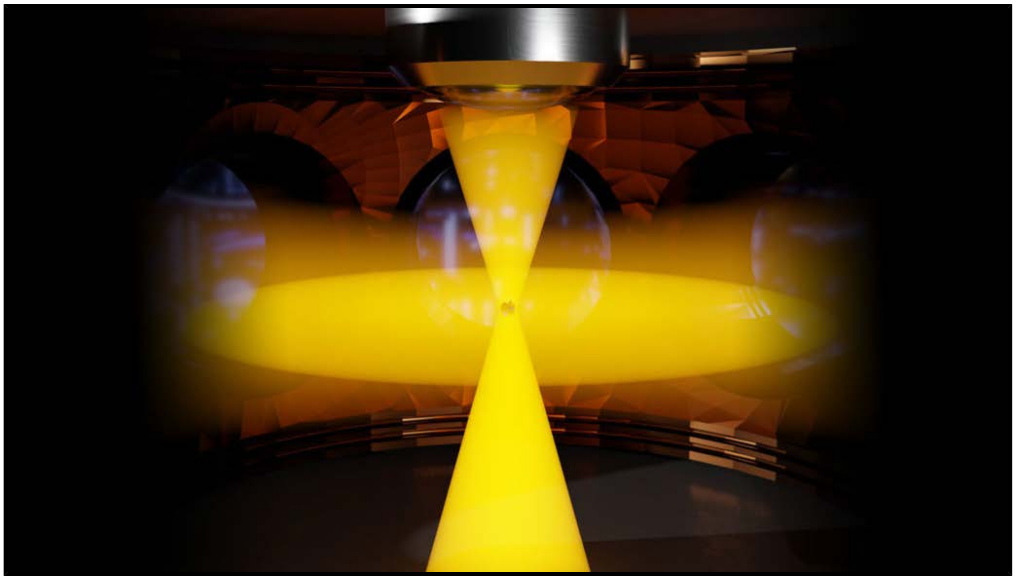
For noninteracting fermions in the plane, trapped in a harmonic oscillator potential, you get nice Pauli crystals with 1, 3, 6, 10, 15, ... atoms. These act like 'closed shells' in chemistry. These numbers show up because they're the triangular numbers 1, 1+2, 1+2+3, 1+2+3+4, etc.

Seeing a Pauli crystal is hard! You need to image all the atoms at once, and they keep wiggling around due to quantum fluctuations, so you have to take repeated images to get a good picture... and you have to keep rotating these images to get them to line up.

A team of physicists trapped lithium-6 atoms in a laser beam to create Pauli crystals. These atoms have 3 protons, 3 neutrons and 3 electrons; since the total 9 is odd they are fermions.



The experiment looks really cool!



Here's the paper on Pauli crystals:

- Marvin Holten, Luca Bayha, Keerthan Subramanian, Carl Heintze, Philipp M. Preiss and Selim Jochim, Observation of Pauli crystals.

It showed up in the 'quantum gases' section of the arXiv, which alas I'll never have a paper in. And here's a nice popular article:

- Bob Yirka, Team in Germany observes Pauli crystals for the first time, May 19, 2020.

May 28, 2020

I'm finally ready to think about Isbell duality.

'Dualities' are important because they show you two different-looking things are secretly two views of the same thing — or at least closely linked.

I'll sketch the idea of Isbell duality; you can see if you're ready for it. 🤪

Let C be a category. Let C^{op} be its opposite category. Let Set be the category of sets.

Let $[C^{op}, \mathrm{Set}]$ be the category of all functors from C^{op} to Set. Such functors are called **presheaves** on C, and the **Yoneda embedding** is a functor

$$y : C \to [C^{op}, \mathrm{Set}].$$

Let $[C, \mathrm{Set}]^{op}$ be the opposite of the category of all functors from C to Set. These are less famous; they're called **copresheaves**. There's a **co-Yoneda embedding**

$$z : C \to [C, \mathrm{Set}]^{op}.$$

The category of presheaves $[C^{op}, \mathrm{Set}]$ is the free category with all colimits on C. It also has all limits, but its universal property is that any functor $C \to D$ where D has all colimits extends uniquely to a functor $[C^{op}, \mathrm{Set}] \to D$ that preserves colimits.

Dually, the category of copresheaves $[C, \mathrm{Set}]^{op}$ is the free category with all limits on C. It also has all colimits, but its universal property is that any functor $C \to D$ where D has all limits extends uniquely to a functor $[C, \mathrm{Set}]^{\mathrm{op}}$â †′D that preserves limits.

Now the fun starts. Take the co-Yoneda embedding

$$z : C \to [C, \mathrm{Set}]^{\mathrm{op}}.$$

Since $[C, \mathrm{Set}]^{\mathrm{op}}$ has all colimits, this functor extends uniquely to a functor

$$[C^{\mathrm{op}}, \mathrm{Set}] \to [C, \mathrm{Set}]^{\mathrm{op}}.$$

that preserves colimits.

Dually, take the Yoneda embedding

$$y : C \to [C^{\mathrm{op}}, \mathrm{Set}].$$

Since $[C^{\mathrm{op}}, \mathrm{Set}]$ has all limits, this functor extends uniquely to a functor

$$[C, \mathrm{Set}]^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{Set}]$$

that preserves limits.

So now we have functors sending presheaves to copresheaves:

$$[C^{\mathrm{op}}, \mathrm{Set}] \to [C, \mathrm{Set}]^{\mathrm{op}}$$

and copresheaves to presheaves:

$$[C, \mathrm{Set}]^{\mathrm{op}} \to [C^{\mathrm{op}}, \mathrm{Set}]$$

Isbell duality says these are adjoint functors!

Isbell duality seems to be the "mother of all dualities"... but I haven't stated it in its most general form.

Yesterday in the ACT@UCR seminar that I'm running, Simon Willerton explained how the Legendre transform arises from Isbell duality! Great talk!

Simon Willerton also showed how Dedekind's construction of the real line using cuts — and many other things! — come from the general form of Isbell duality.

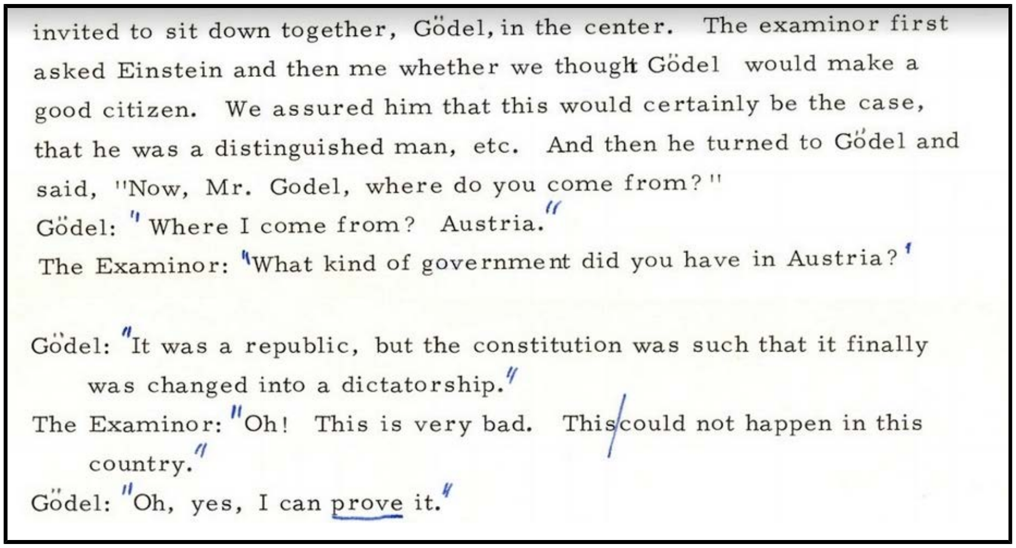
Here you can see his slides, his paper, and a blog article he wrote on this stuff:

- The Legendre transform: a category theoretic perspective, *Azimuth*, May 27, 2020.

May 30, 2020

In 1946, Gödel wanted to become a U.S. citizen. He took his friend Einstein along.

Unfortunately Gödel had spent time studying the U.S. constitution and claimed to have found an "inner contradiction" that made it possible for someone to become a dictator in a completely legal way. Einstein had tried to persuade him to just shut up about it... but then everything went awry.



invited to sit down together, Gödel, in the center. The examinor first asked Einstein and then me whether we thought Gödel would make a good citizen. We assured him that this would certainly be the case, that he was a distinguished man, etc. And then he turned to Gödel and said, "Now, Mr. Godel, where do you come from?"

Gödel: "Where I come from? Austria."

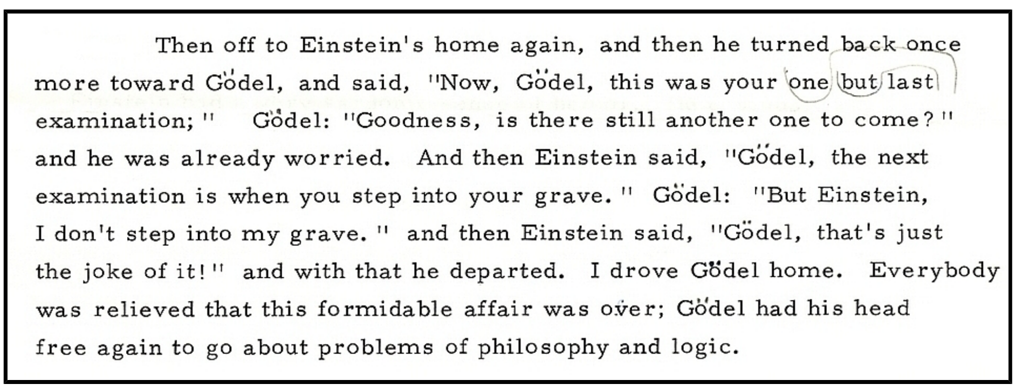
The Examinor: "What kind of government did you have in Austria?"

Gödel: "It was a republic, but the constitution was such that it finally was changed into a dictatorship."

The Examinor: "Oh! This is very bad. This could not happen in this country."

Gödel: "Oh, yes, I can prove it."

Throughout the whole thing Einstein was joking around as usual.

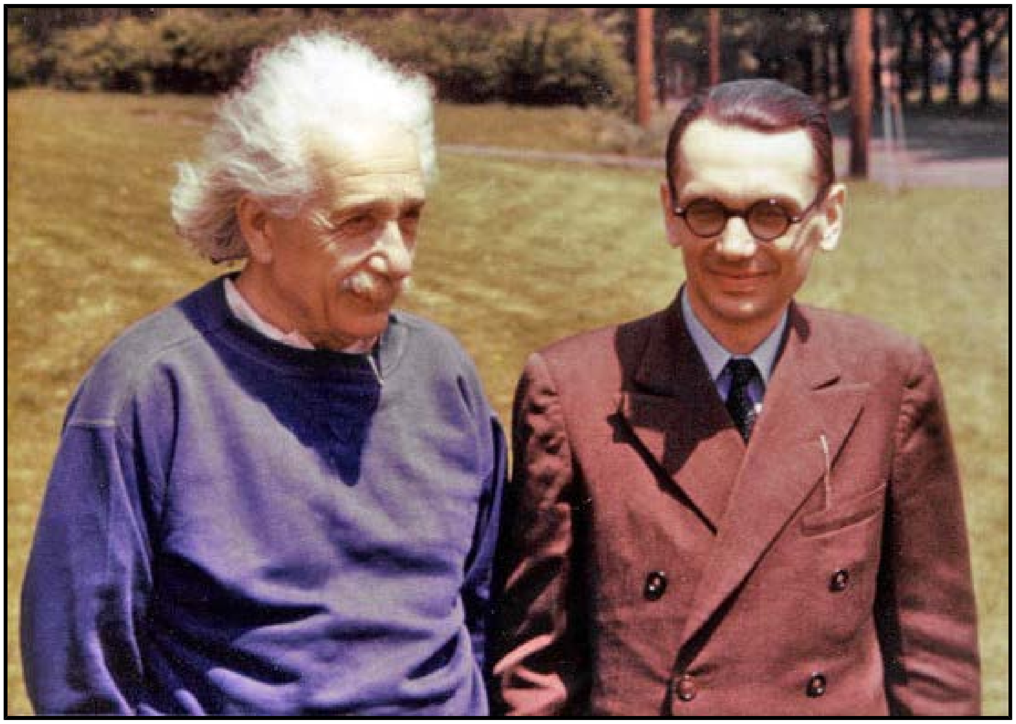


Then off to Einstein's home again, and then he turned back once more toward Gödel, and said, "Now, Gödel, this was your one but last examination;" Gödel: "Goodness, is there still another one to come?" and he was already worried. And then Einstein said, "Gödel, the next examination is when you step into your grave." Gödel: "But Einstein, I don't step into my grave." and then Einstein said, "Gödel, that's just the joke of it!" and with that he departed. I drove Gödel home. Everybody was relieved that this formidable affair was over; Gödel had his head free again to go about problems of philosophy and logic.

The story appears in a letter — fragments of which are shown above — written by Oskar Morgenstern, who also accompanied Gödel to his citizenship hearing. This letter was lost for a while, but now you can read it here:

- Jeffrey Kegler, Kurt Gödel: a contradiction in the U.S. constitution?

In the end Gödel got his citizenship and became best friends with Einstein. They were often seen walking around together.



Gödel learned general relativity and found a solution of Einstein's equations that has closed timelike loops: in this universe your future is your past. It's called the Gödel universe.

For my June 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

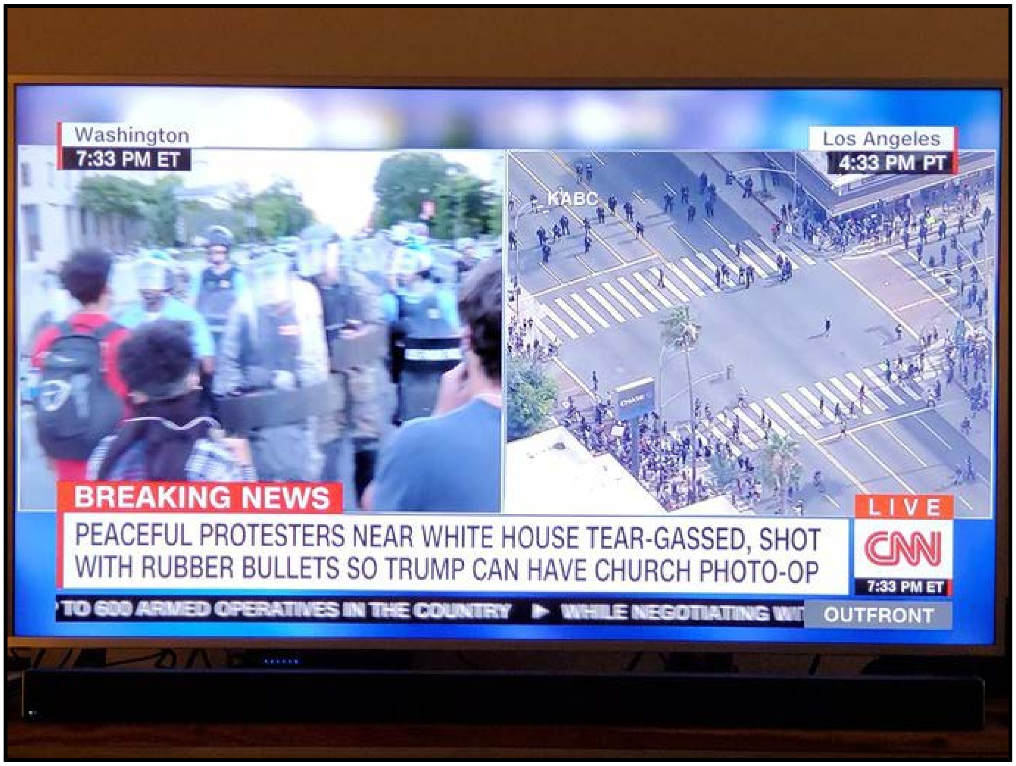
home

For my April 2020 diary, go here.

Diary — June 2020

John Baez

June 1, 2020



June 2, 2020

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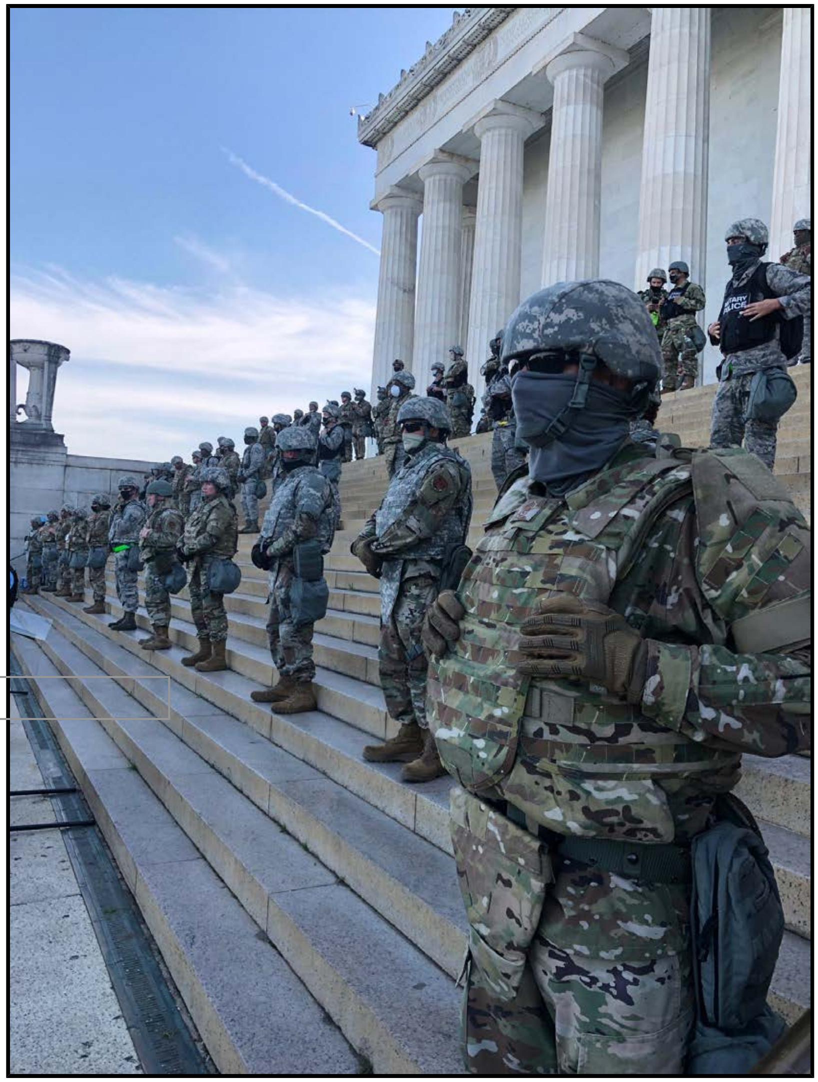


These are paramilitary troops surrounding the Lincoln Memorial in Washington D.C. during protests of George Floyd's murder. In explaining this strategy Defense Secretary Esper said to U.S. governors:

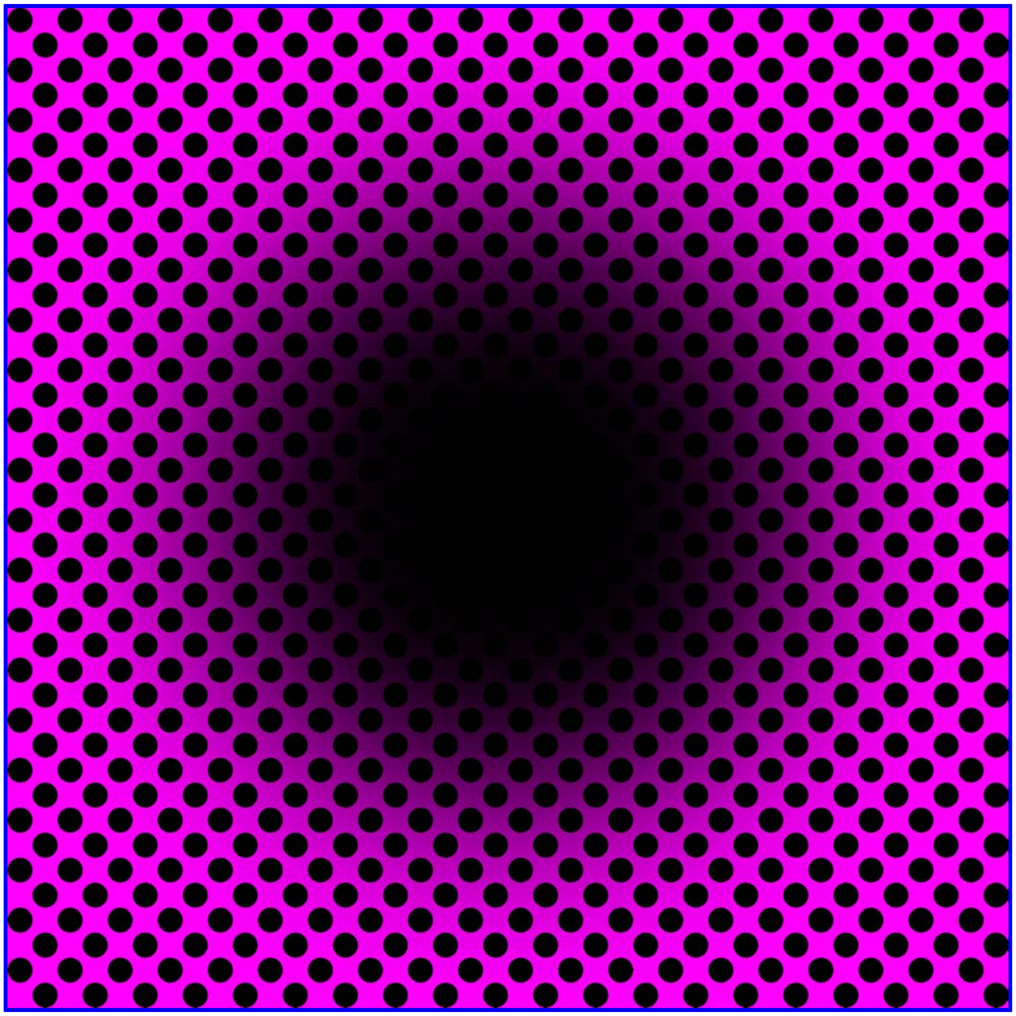
“We need to dominate the battle space.”

Lincoln had a different opinion:

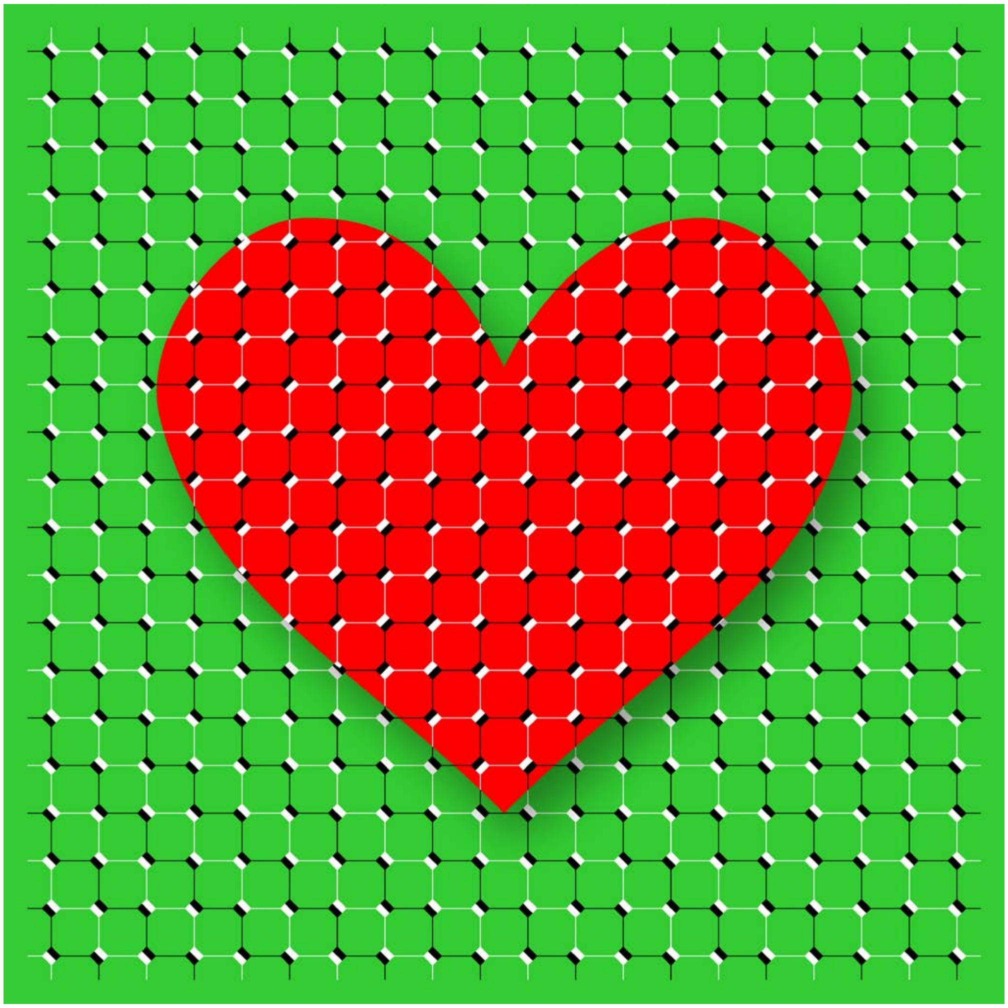
“Those who deny freedom to others, deserve it not for themselves.”

June 6, 2020

Here's a wonderful illusion by Akiyoshi Kitaoka: the black hole appears to expand, though it does not.



This heart seems to shift, especially if you move your head a little:



June 8, 2020

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Whoops! You fools can't do *anything* right!

The page on which this statement has been printed has been intentionally left devoid of substantive content, such that the present statement is the only text printed thereon.

Better.

The second one came from here:

- Glen Wright, FX Coudert, Martin Bentley, Sylvain Deville and Graham Steel, This Study is Intentionally Left Blank - a systematic literature review of blank pages in academic publishing.

June 10, 2020

Here *Quanta* explains 'group representations', where you map elements of a group to matrices, so that group multiplication becomes matrix multiplication:

- Kevin Hartnett, The 'useless' perspective that transformed mathematics, *Quanta*, June 9, 2020.

Let's dig into the history! Did the group theorist Burnside really think group representations were useless?

In 1897 William Burnside wrote the first book in English on finite groups. In the preface, he explained that "in the present state of our knowledge" there weren't theorems about groups that could best be proved by representing them as linear transformations (matrices). He wrote:

> Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book that professes to leave all applications to one side, a considerable space is devoted to substitution groups [permutation groups], but other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could most directly be obtained by the consideration of groups of linear transformations.

So, Burnside didn't actually say group representations were useless! His book was a "pure" study of finite groups, which "professes to leave all applications to one side".

But months after this book was published, he discovered the work of Georg Frobenius. Frobenius was a master of group representations! Burnside started using them in his own work on finite groups, and by the time he wrote the second edition of his book in 1911, he'd changed his tune completely:

> Very considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for

> omitting any account of it no longer holds good. In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear transformations.

Are there results about finite groups that we only know how to prove using their representations on vector spaces? Yes! For example, the 'odd order theorem'. A group is 'solvable' if — roughly — it's built from abelian pieces. The odd order theorem says that group with an odd number of elements is solvable!

A group G is called **solvable** if it has subgroups $1 = G_0 < G_1 < \cdots < G_k = G$ such that G_{j-1} is a normal subgroup of G_j, and G_j/G_{j-1} is an abelian group, for $j = 1, 2, \ldots, k$.

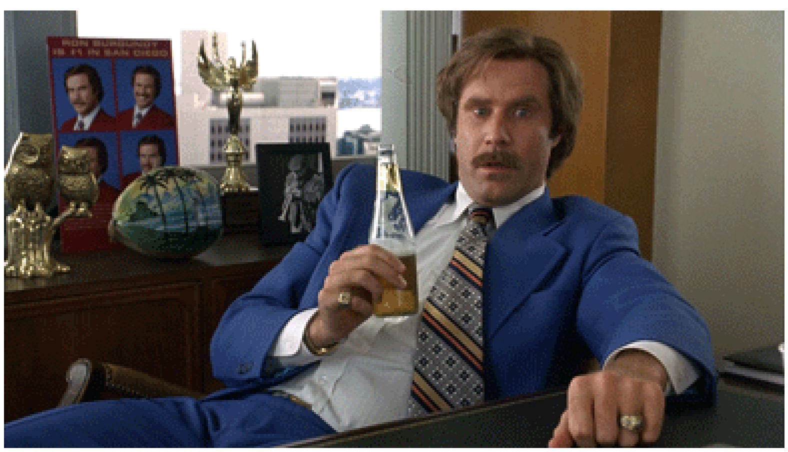
But be careful:

In 1904 Burnside showed that every group of order $p^a q^b$ is solvable if p and q are prime. To do it he used group representations. But then, in 1972, Helmut Bender found a proof that avoids linear algebra!

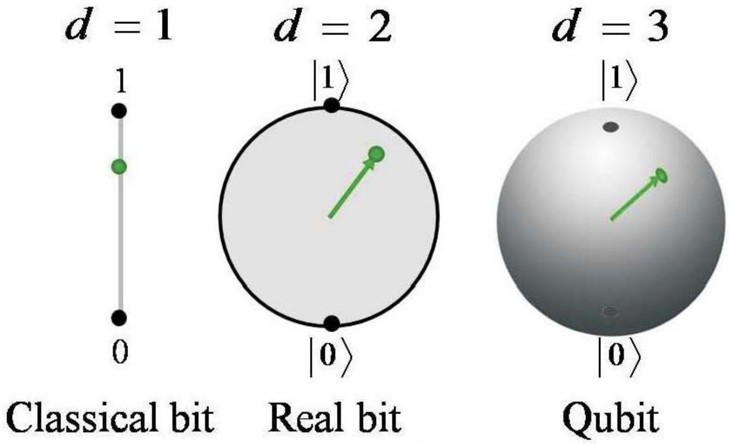
It's an interesting challenge to state *precisely* how group representations help us understand finite groups. For more on this, read my imaginary dialog between Burnside and Frobenius in 'week252'. Skip past the stuff about astronomy (if you can).

June 12, 2020

Ralf Wüsthofen claims he has a simple one-page proof of Goldbach's conjecture which "immediately leads to a proof of the inconsistency of arithmetic."



June 14, 2020

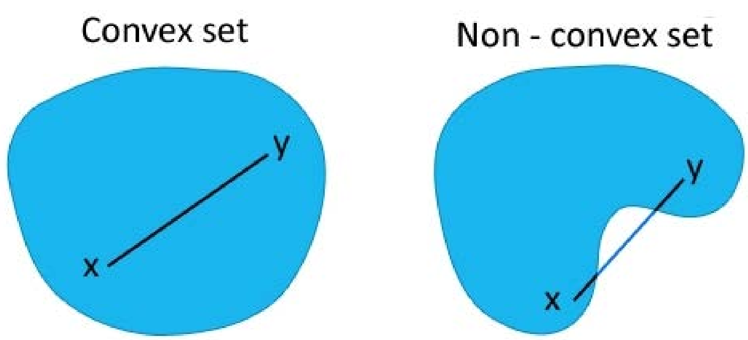


A classical bit is 0 or 1, but if we allow probabilities we can get anything in between. A qubit takes values on the unit sphere, but if we allow probabilistic 'mixed states' we can get anything in a 3-dimensional ball.

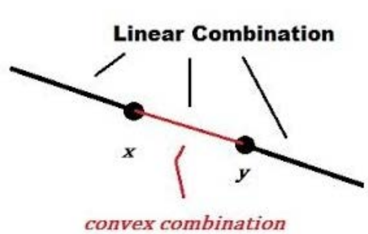
Then there's 'real quantum mechanics', where we use real numbers instead of complex numbers. This gives a 2d disk of mixed states.

All these are examples of 'generalized probabilistic theories' — a broad class of theories that people study to better understand what makes quantum mechanics and classical probability theory so special.

These theories use the math of convex sets.

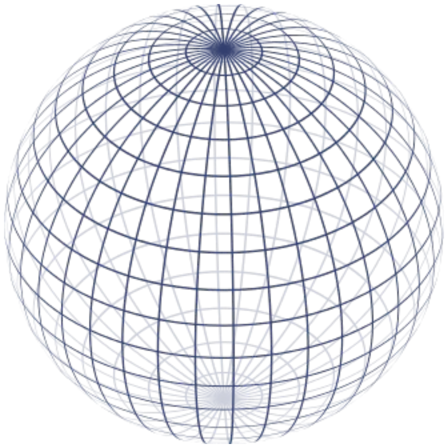


A generalized probabilistic theory describes a system with a convex set of 'states'. If x and y are states, so is $px + (1 - p)y$ where $0 \le p \le 1$ is any probability. This is the result of preparing the system in the state x with probability p and the state y with probability $1 - p$.

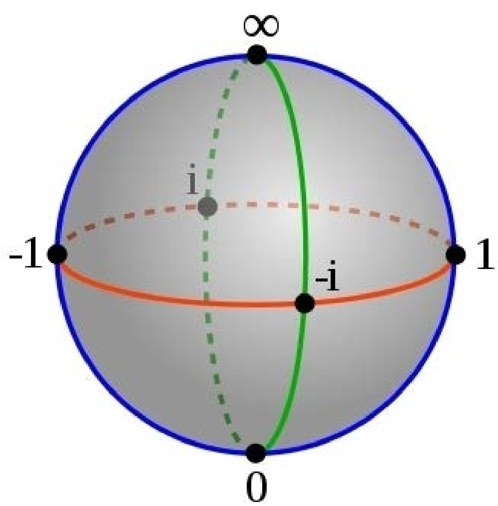


In ordinary complex quantum mechanics, the states of a qubit form a 3-dimensional ball. States on the surface are 'pure states': they're not convex combinations of other states. States in the interior are 'mixed'.

In the center is the state of maximum entropy.



The pure states of a qubit form a sphere. Physicists call it the 'Bloch sphere ', but mathematicians call it the 'Riemann sphere' or CP^1. There are many ways to think about this, and I love them all. One way: take the complex numbers together with a 'point at infinity'.



What if we use real numbers instead? In 'real quantum mechanics', the pure states of a 'real qubit' form a circle, also called RP^1. It's the circle containing the real numbers in the above picture, like $0, 1, -1$, along with the point at infinity.

The mixed states of the real qubit form the 2d disk bounded by this circle.

Any 'formally real Jordan algebra' gives a generalized probabilistic theory. For example: the Jordan algebra of 2×2 self-adjoint complex matrices gives the theory of a qubit.

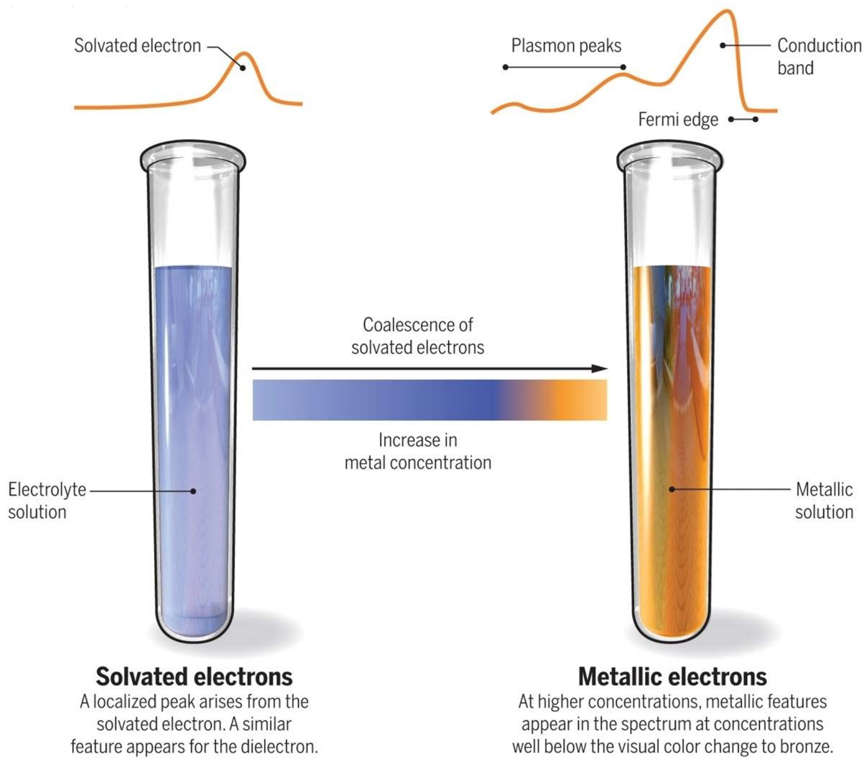
But you could use self-adjoint 2×2 real matrices instead, and get the real qubit!

For more, try my gentle introduction to formally real Jordan algebras and their connection to projective geometry and quantum mechanics:

- Octonionic projective geometry.

Now I'm writing a paper on Jordan algebras and Noether's theorem, thinking about these things a bit more deeply and learning a lot.

June 18, 2020



When you dissolve more and more sodium in ammonia, it changes from blue to bronze. And it starts conducting electricity just like a metal! Electrons and pairs of electrons called 'dielectrons' start moving freely through the solution.

I love condensed matter physics! I hadn't even known dielectrons were a thing: normally electrons repel. But in a sodium-ammonia solution, electrons pair up with opposite spins. Here's a new simulation of a dielectron, lasting 2.75 picoseconds:

The wavefunction of a loose electron in ammonia is smeared out over ~12 ammonia molecules, much bigger than one in water, which spreads out over ~5 water molecules.

This allows the formation of dielectrons, which are slightly bigger still.

As you add more sodium to ammonia, you get more of these loose electrons and dielectrons, and gradually a 'conduction band' arises — meaning that if they have the right momentum, these particles can move freely for long distances, like electrons in a metal!

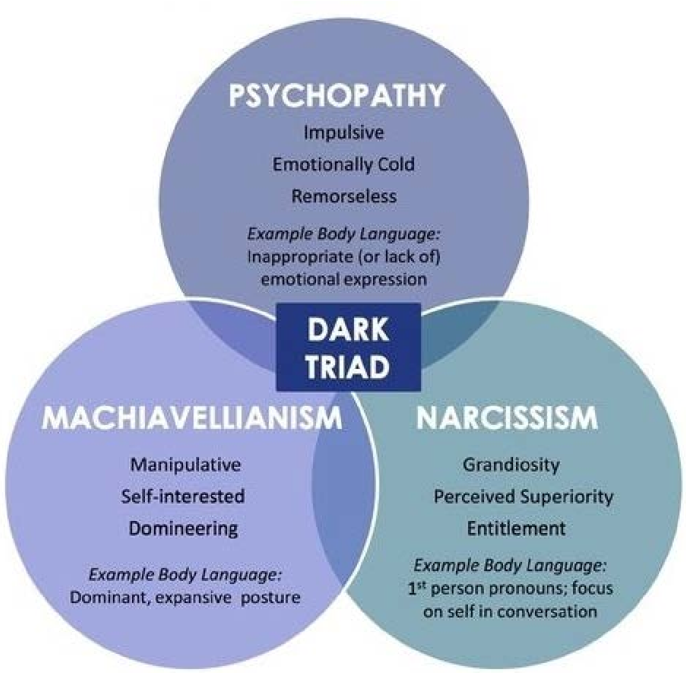
I read about this in a new paper:

- Tillmann Buttersack, Philip E. Mason, Ryan S. McMullen, H. Christian Schewe, Tomas Martinek, Krystof Brezina1, Martin Crhan1, Axel Gomez, Dennis Hein, Garlef Wartner, Robert Seidel, Hebatallah Ali, Stephan Thürmer, Ondrej Marsalek, Bernd Winter, Stephen E. Bradforth and Pavel Jungwirth, [Photoelectron spectra of alkali metal–ammonia microjets: from blue electrolyte to bronze metal](), *Science* **368** (2020), 1086–1091.

June 21, 2020

Can we understand evil using evolutionary biology, or perhaps game theory?

Psychologists have latched onto 3 personality traits, the 'dark triad', and tried to measure how much they are genetically inherited.

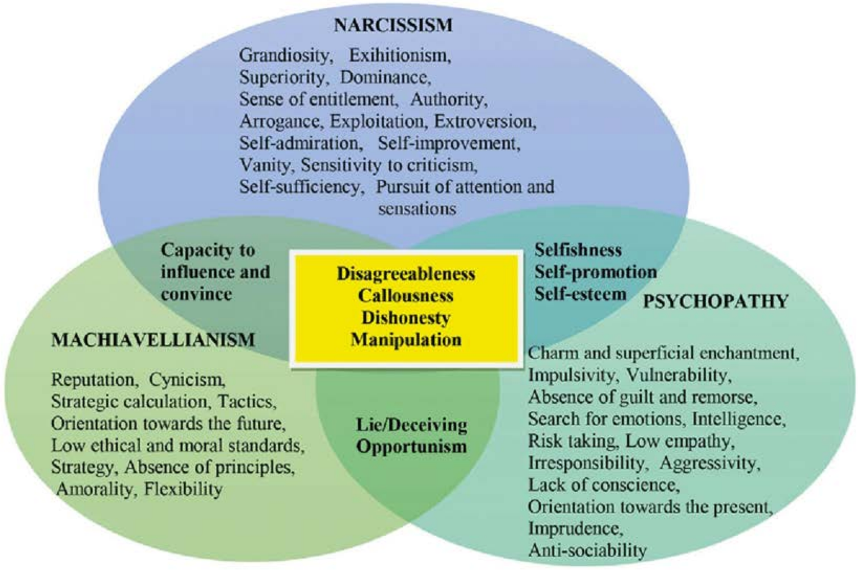


How are these traits defined?

- Narcissism involves grandiosity, pride, and egotism.
- Machiavellianism involves coldness, cynicism, and manipulation and exploitation of others.
- Psychopathy involves impulsivity, thrill-seeking, lack of remorse, and aggressiveness.

But they have much in common. People with any of these traits tend to be callous, manipulative, dishonest and disagreeable.

They also tend to be less compassionate, agreeable, empathetic, satisfied with their lives — and less likely to believe they and other people are good.



Psychologists have tried to determine how much these traits are genetically inherited, by comparing identical vs. fraternal twins. (If a trait has a genetic component, it should be shared more often by identical twins than fraternal twins, even if these twins are raised the same way.)

Roughly speaking: psychopathy is 65% inherited, narcissism 60%, but Machiavellianism just 30%. (Technicaly, these are h^2 figures.)

So, some have tried to develop evolutionary explanations of the dark triad traits, and why they persist despite their disadvantages. (Sub-clinical and clinical psychopaths form about 1% and 0.2% of the population, respectively; I know no estimates for the others.)

One theory is these traits are connected to a 'fast life strategy' with an emphasis on mating over parenting. People with these traits tend to have more sexual partners, but are less likely to form stable relationships. They tend to dress more attractively. They are also more likely to commit rape:

- Peter K. Jonason, Mary Girgis and Josephine Milne-Holme, The exploitive mating strategy of the dark triad traits: tests of rape-enabling attitudes, *Archives of Sexual Behavior* **46** (2017), 697–706.

Dark triad traits are also advantageous in certain situations, called 'dark niches'. Narcissists are known to do well in job interviews and first dates. Machiavellianism works well in politics and finance. Psychopathy works well in street gangs.

Recently a fourth trait has been considered: 'everyday sadism'. For example, those bullies and internet trolls who enjoy making people suffer.

Separating out evil into distinct traits is a challenging job!

Table 1. Key Features of the Dark Tetrad of Personalities

Feature	Narcissism	Machiavellianism	Psychopathy	Sadism
Callousness	++	++	++	++
Impulsivity	+		++	
Manipulation	+	++	++	
Criminality		Only white-collar	++	
Grandiosity	++		+	
Enjoyment of cruelty				++

Note: A double plus sign indicates high levels of a given trait (top quintile) relative to the average population-wide level. A single plus sign indicates slightly elevated levels (top tertile). A blank entry indicates average levels of a trait.

But to fight evil, we must understand it!

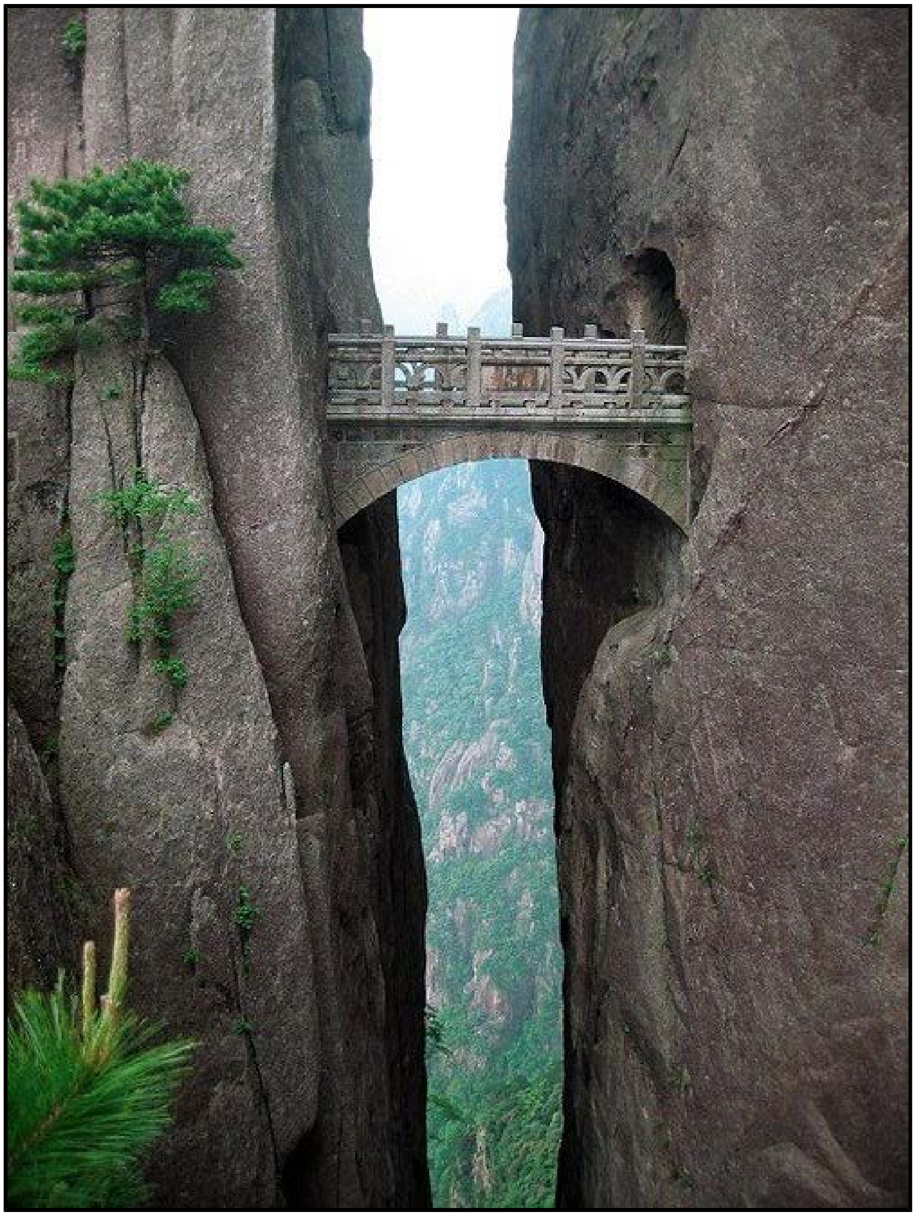
Here's an easy way to learn more:

- Delroy Paulhus, Toward a taxonomy of dark personalities, *Current Directions in Psychological Science* **23** (2014), 421–426.

Also:

- Wikipedia, Dark triad.

June 22, 2020



This is the Bridge of Immortals, in Huangshan, in southern Anhui province. Like a hard theorem in mathematics that bridges two fields of mathematics, the view from the top is your reward for the difficult climb.

June 24, 2020

Definition of a Lie algebra

A **Lie algebra** is a vector space $\mathfrak{g}$ together with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ obeying the following identities:

- **Alternativity**:

$$[x,x] = 0$$

- The **Jacobi identity**:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

Bilinearity and alternativity together imply **anticommutativity**: $[x,y] = -[y,x]$.

A Lie algebra is one of the more scary gadgets one meets as one first starts learning algebra, because the Jacobi identity looks unfamiliar. But it's secretly the product rule:

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]]$$

is like

$$D(fg) = D(f)g + fD(g)$$

The easiest way to get a Lie algebra is to take square matrices, with the usual matrix multiplication, and define

$$[x,y] = xy - yx$$

Then it's obvious that $[x,x] = 0$, and the Jacobi identity follows from this 'product rule'

$$[x,yz] = [x,y]z + y[x,z]$$

which is easy to check.

The inventors of quantum mechanics noticed that you can treat functions as infinite-sized square matrices, and then the derivative Df of a function f can be written

$$Df = [\partial, f] = \partial f - f\partial$$

for some matrix ∂, so the usual product rule follows from the product rule for matrices.

So, there's a Lie algebra where functions and the matrix ∂ coexist, and these days it's called the 'Heisenberg algebra'. Basically it turns a bunch of calculus into the algebra of infinite square matrices!

There's an infinite amount to say about this, but I want to talk about something else. There are also gadgets with *trilinear* operations: operations that need 3 inputs, that are linear in each input. These are a bit scary at first, since the most familiar operations in math just take 2 inputs.

An example is a 'Lie triple system'.

A **Lie triple system** is a vector space L with a trilinear map $[\cdot,\cdot,\cdot]\colon L \times L \times L \to L$ obeying the following identities:

$$[u,v,w] = -[v,u,w]$$
$$[u,v,w] + [w,u,v] + [v,w,u] = 0$$
$$[u,v,[w,x,y]] = [[u,v,w],x,y] + [w,[u,v,x],y] + [w,x,[u,v,y]]$$

Any Lie algebra gives a Lie triple system if we define $[u, v, w] = [[u, v], w]$ The first rule in the definition of Lie triple system then follows from the antisymmetry of the Lie bracket, $[u, v] = -[u, v]$. The second rule is just the Jacobi identity in disguise. And the third, most complicated rule says each operation $[u, v, -]$ acts like 'differentiation', since it obeys a product rule similar to

$$D(wxy) = D(w)xy + wD(x)y + wxD(y)$$

Namely:

$$[u, v, [w, x, y]] = [[u, v, w], x, y] + [w, [u, v, x], y] + [w, x, [u, v, y]]$$

It's not so hard to remember if you keep this in mind.

But the really interesting Lie triple systems come from $Z/2$-graded Lie algebras. These are Lie algebras split into an 'even part' g_0 and an "odd part" g_1 with

$$[g_0, g_0] \quad g_0$$

$$[g_0, g_1] \quad g_1$$

$$[g_1, g_0] \quad g_1$$

$$[g_1, g_1] \quad g_0$$

Think "even plus even is even", etc.

In case you know about Lie superalgebras: $Z/2$-graded Lie algebras *are not those!* They're just plain old Lie algebras chopped into an even and odd part obeying the rules above.

These rules imply the bracket of three odd things is odd:

$$[[\mathrm{g}_1, \mathrm{g}_1], \mathrm{g}_1] \quad \mathrm{g}_1$$

So, if we take a $\mathrm{Z}/2$-graded Lie algebra and only keep the odd part g_0, it becomes a Lie triple system if we define

$$[x, y, z] = [[x, y], z]$$

All this becomes a lot more exciting when you see how $\mathrm{Z}/2$-graded Lie algebras show up in geometry.

If you take a Lie group, its tangent space at the identity is a Lie algebra. But more generally, the tangent space at any point of a 'symmetric space' is a Lie triple system.

For example, the tangent space at any point of a sphere is a Lie triple system!

There's a Lie group $\mathrm{SO}(n)$ of rotations in n dimensions. Its tangent space at the identity is a Lie algebra called $\mathrm{so}(n)$.

$\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is an n-sphere, which is a symmetric space.

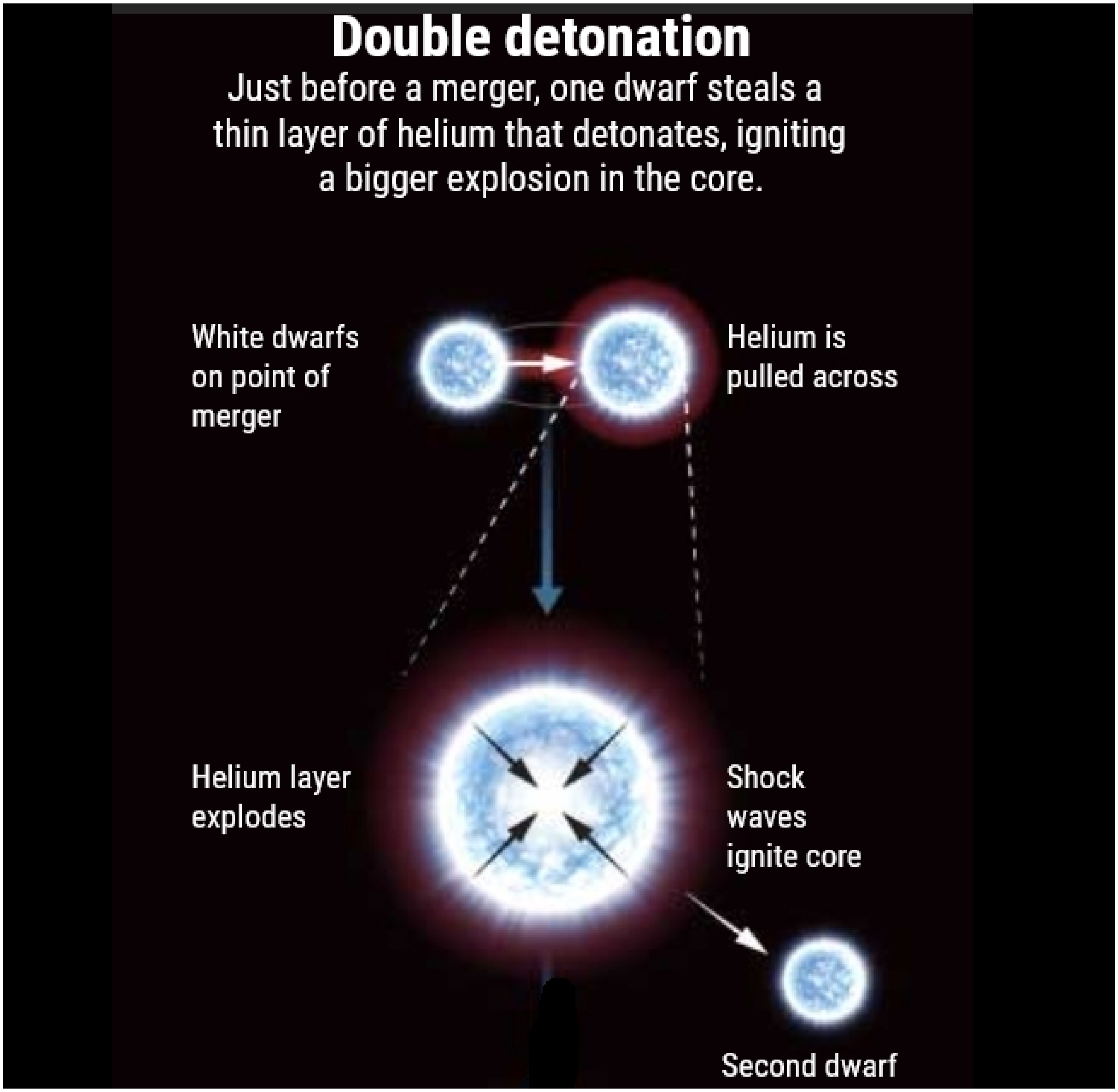
$\mathrm{so}(n)/\mathrm{so}(n-1)$ is the tangent space of the n-sphere at the north pole, which is a Lie triple system!

In other words, we can make $\mathrm{so}(n)$ into a $\mathrm{Z}/2$-graded Lie algebra by splitting it into two parts:

- even part: the infinitesimal rotations $\mathrm{so}(n-1)$ that fix some point on the n-sphere
- odd part: the tangent space of that point on the n-sphere, which looks like $\mathrm{so}(n)/\mathrm{so}(n-1) \quad \mathrm{R}^n$.

The odd part is a Lie triple system!

June 25, 2020



In 2018, three white dwarf stars were seen zipping across the Milky Way at over 1000 kilometers per second — thousands of times faster than a speeding bullet, fast enough to escape the galaxy. It was one of several clues that we were wrong about what causes many supernovae.

Type II supernovae happen when a big star collapses. But type Ia supernovae happen when a white dwarf explodes for some reason. We thought most of did this when they stole gas from a red giant companion. But no!

It now seems most type Ia supernovae happen when two white dwarfs spiral into each other and nearly collide. This is called — get ready for it! — a "dynamically driven double-degenerate double detonation".

Astronomers estimate about half a billion white dwarf binaries have merged in the Milky Way since its formation! If one-sixth of these mergers led to a type Ia supernova, there would be one supernova every 200 years — which is what we see.

And if type Ia supernovae work differently than we thought, that affects our understanding of the expansion of the Universe, since they're used as a 'standard candle' to measure the distance of faraway galaxies. Maybe we've got some things a bit wrong.

For more, read this:

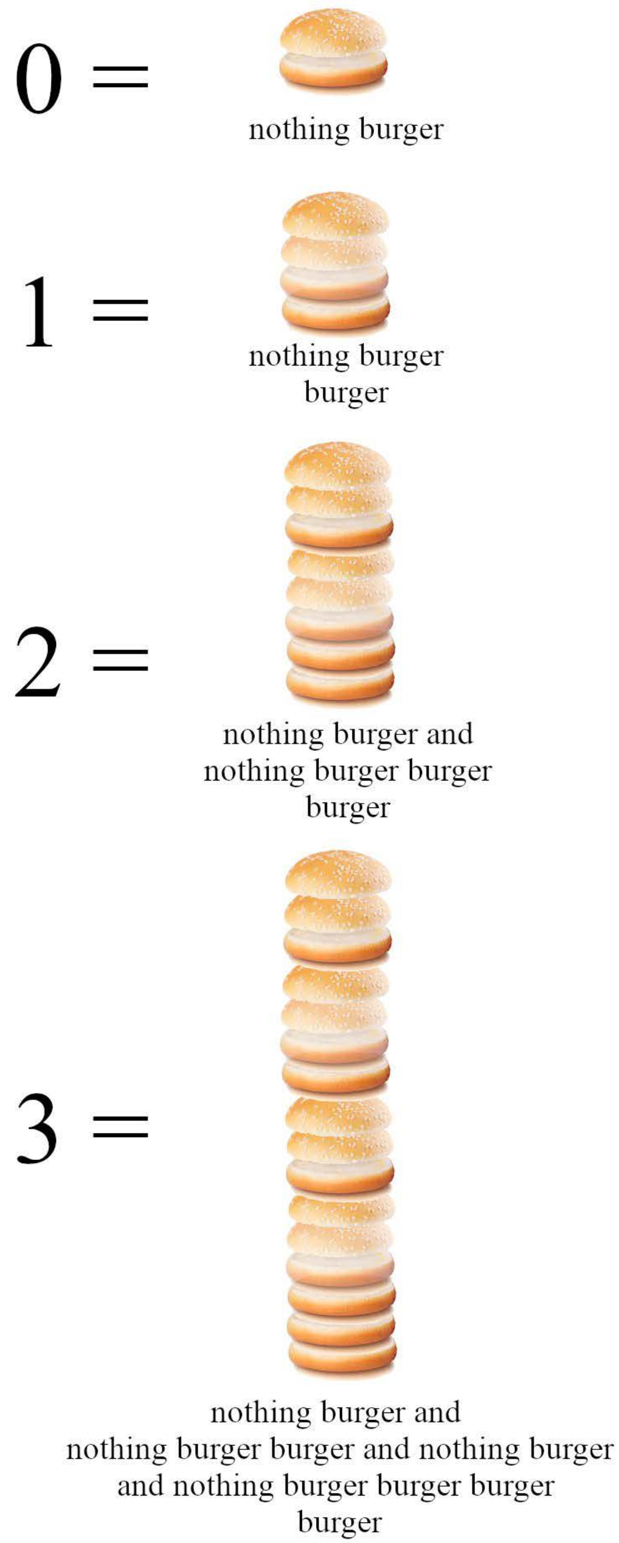
- Daniel Clery, The galaxy's brightest explosions go nuclear with an unexpected trigger: pairs of dead stars, *Science*, June 4, 2020.

June 26, 2020

Mathematicians sometimes disagree about arbitrary conventions, like whether 0 is a natural number.

Luckily we have ways of settling these disputes.

June 27, 2020



Tired of nothing burgers? Try Keith Peterson's new *nothing burger burger!* It's twice as satisfying, and still vegetarian.

Or if that's not enough for you, try the nothing burger and nothing burger burger burger.

No matter how hungry you are, ZFC has a burger for you.

June 28, 2020

> A **Jordan algebra** is a not necessarily associative algebra over a field whose multiplication satisfies these rules:
>
> 1. $x \circ y = y \circ x$ (**commutative law**)
> 2. $(x \circ x) \circ (x \circ y) = x \circ ((x \circ x) \circ y)$ (**Jordan identity**)
>
> These imply that a Jordan algebra is **power-associative**, meaning that $x^n = x \circ \cdots \circ x$ is independent of how we parenthesize this expression. They also imply that $x^m \circ (x^n \circ y) = x^n \circ (x^m \circ y)$ for all positive integers *m* and *n*. Thus, we may equivalently define a Jordan algebra to be a commutative, power-associative algebra such that for any element x, the operations of multiplying by powers x^n all commute.

Jordan algebras are fascinating to me — but annoying. The usual definition uses the 'Jordan identity', which looks completely random. With help from David Madore, I found an equivalent definition that says more about what's good about Jordan algebras.

While they're weird, Jordan algebras come from physics. Jordan invented his algebras to study quantum mechanics.

In quantum mechanics, self-adjoint $n \times n$ complex matrices are 'observables'. The product of observables is not an observable but

$$x \quad y = (xy + yx)/2$$

is, and this gives a Jordan algebra.

There's also a Jordan algebra of self-adjoint $n \times n$ real matrices, which works the same way. And there's also a Jordan algebra of self-adjoint $n \times n$ quaternionic matrices! So, we can compare real and quaternionic quantum mechanics to ordinary complex quantum mechanics using Jordan algebras.

Working with Wigner and von Neumann, Jordan discovered a weirder fact. Self-adjoint $n \times n$ *octonionic* matrices form a Jordan algebra... but only for $n \leq 3$.

The Jordan algebra of self-adjoint 2×2 octonionic matrices is 10-dimensional and connected to superstring theory.

The Jordan algebra of self-adjoint 3×3 octonionic matrices is 27-dimensional. It's called the 'Albert algebra' or 'exceptional Jordan algebra'. Its connection to physics, if any, remains quite obscure. I've been puzzling over this for years.

But in math, the exceptional Jordan algebra is great! Let me tell you just one reason why.

There's a systematic way to get three Lie algebras from a Jordan algebra: a little one, a medium-sized one and a big one. See the intro here for details:

- Jakov Palmkvist, A generalization of the Kantor–Koecher–Tits construction.

If you apply this construction to the Jordan algebra of 2×2 self-adjoint complex matrices, you get the Lie algebras of these three groups:

1. the group of rotations in 3d space
2. the Lorentz group of 4d Minkowski spacetime
3. the group of conformal transformations of 4d Minkowski spacetime.

Applying the same construction to the Jordan algebra of 2 × 2 self-adjoint octonionic matrices, you get the Lie algebras of these groups:

- the group of rotations in 9d space
- the Lorentz group of 10d Minkowski spacetime
- the group of conformal transformations of 10d Minkowski spacetime.

Then for the really exciting case: applying this construction to the Jordan algebra of 3 × 3 self-adjoint octonionic matrices, you get the Lie algebras of these groups:

- the exceptional group F_4
- the exceptional group E_6
- the exceptional group E_7.

There are five exceptional simple Lie groups, and here we get three. (More precisely we get certain real forms of these groups.)

Where is this going? Nobody knows yet. Jordan started by trying to understand and generalize quantum mechanics. But his algebras are also connected to the geometry of spacetime, and generalizations of that!

These mathematical ideas springing from physics could be clues leading us to better theories. Or, they could be leading us to dead ends. By now, I doubt I'll ever know.

The world offers us many mysteries; only a few will be solved during our lifetime.

To avoid frustration with cosmic questions I spend most of my time trying to solve little puzzles, like:

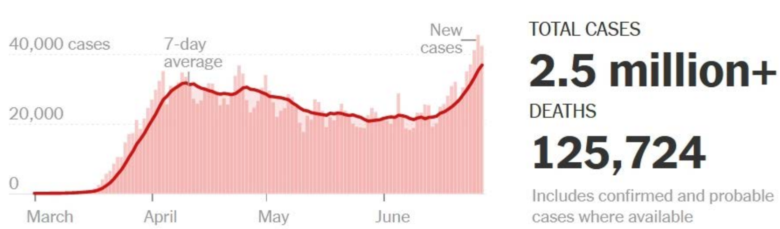
How *exactly* do Jordan algebras generalize quantum mechanics? What makes complex quantum mechanics 'better' than the real or quaternionic versions? Etc.

Tomorrow I'll have a new paper on the arXiv:

- [Getting to the bottom of Noether's theorem](#).

It wound up being largely about Jordan algebras! It has very little new math — but it has a new physical interpretation of some of this math, which leads to some new insights and also new questions.

Meanwhile COVID-19 continues to spiral out of control in many US states:



For my July 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu

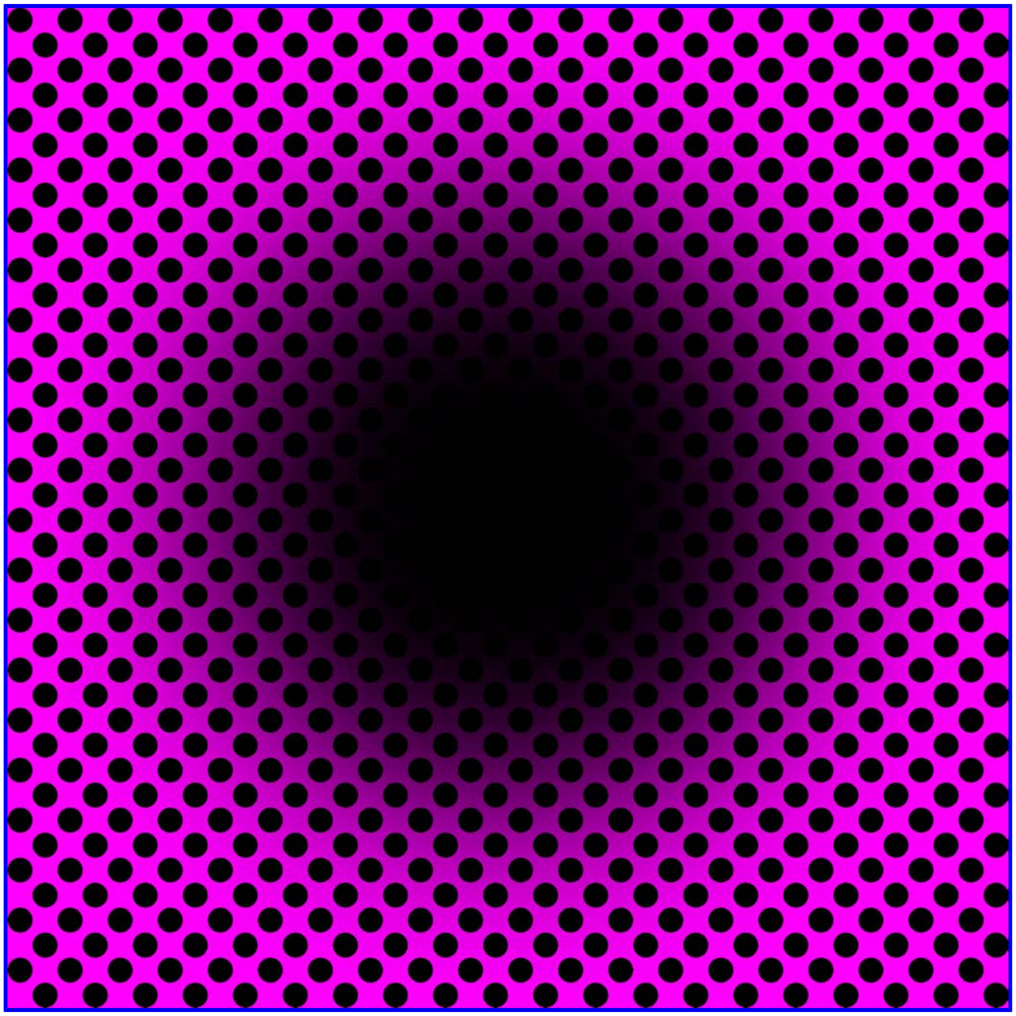
home

[For my June 2020 diary, go here.]

Diary — July 2020

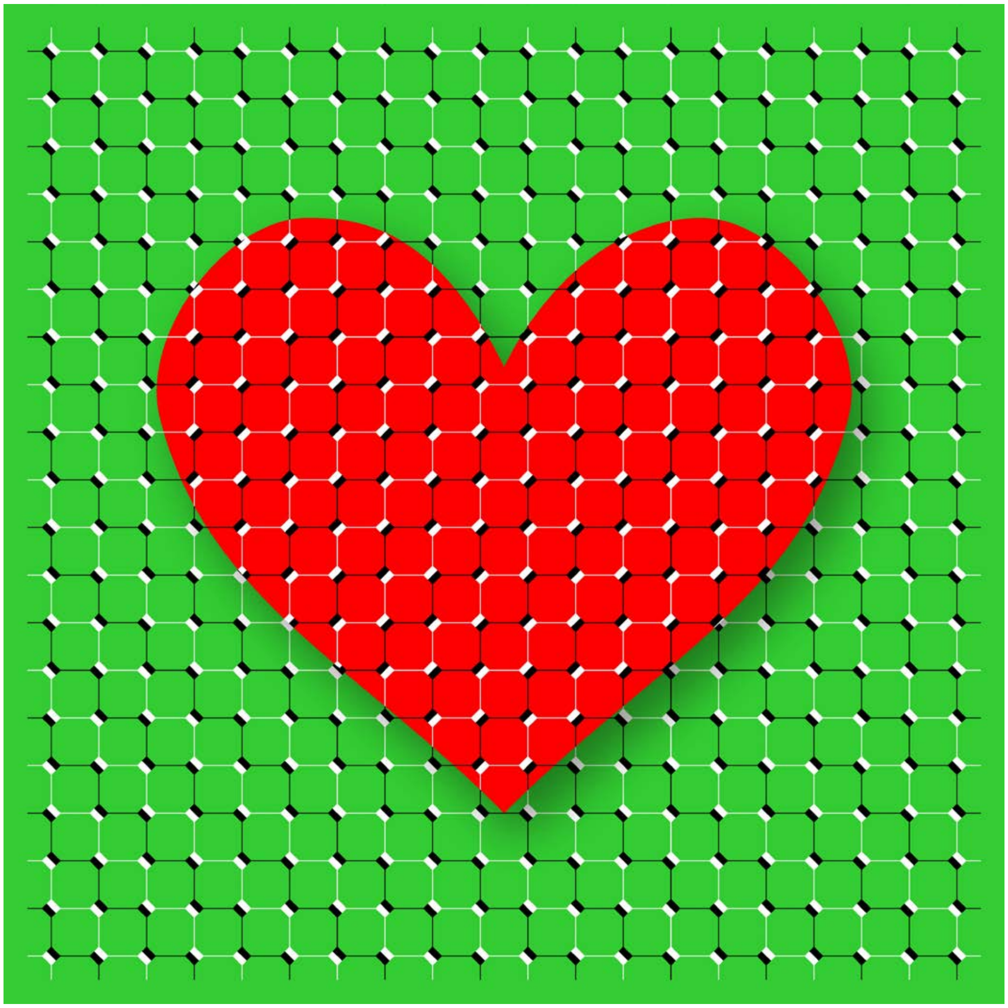
John Baez

July 1, 2020



In this wonderful illusion by Akiyoshi Kitaoka the black hole appears to expand, though it does not.

This heart seems to shift, especially if you move your head a little:



July 2, 2020

I just learned a shocking theorem!

A bounded operator N on a Hilbert space is **normal** iff $NN \quad = N \quad N$. Putnam's theorem says that if M and N are normal and

$$MT = TN$$

for some bounded operator T, then

$$M \quad T = TN$$

It's shocking because it's all just equations but you can't prove it just by fiddling around with equations. Rosenblum gave a nice proof. First note that $MT = TN$ implies

$$M^n T = TN^n$$

for all n. Then use power series to show

$$e^{cM} T = Te^{cN}$$

for all complex c, so

$$cM \quad -cN$$

$$e \quad Te \quad = T \qquad (\dagger)$$

Then the magic starts. Let

$$F(z) = e^{zM} \; Te^{-zN}$$

for complex z. If we can show this is constant we're done, since differentiating with respect to z and setting $z = 0$ gives

$$M \quad T - TN \quad = 0$$

which shows $M \quad T = TN$.

But how can we show it's constant?

Well, 'obviously' we use Liouville's theorem: a bounded analytic function is constant. This is true even for analytic functions taking values in the space of bounded linear operators!

Now

$$F(z) = e^{zM} \; Te^{-zN}$$

is clearly analytic — but why is it bounded as a function of z?

We fiendishly note that

$$\begin{aligned} F(z) &= e^{zM} \; Te^{-zN} \\ &= e^{zM} \; e^{-z \quad M} Te^{z \quad N} e^{-zN} \end{aligned}$$

since $e^{-z \quad M} Te^{z \quad N} = T$ by (†). We then get

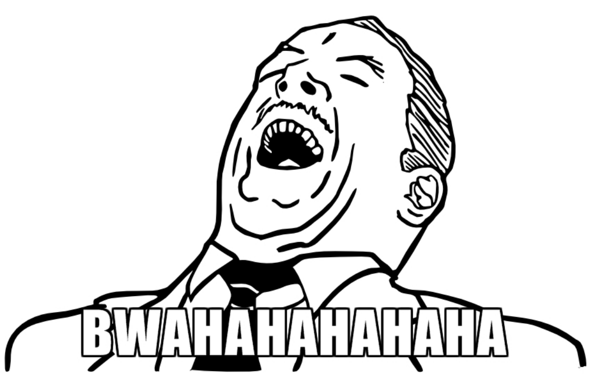
$$F(z) = e^{zM-(zM)} \; Te^{z \quad N-(z \quad N)}$$

since M commutes with M and N commutes with N .

But $zM - (zM)$ and $z \quad N - (z \quad N)$ are **skew-adjoint**: that is, minus their own adjoint. So when you exponentiate them you get unitary operators, which have norm 1! So

$$\|F(z)\| \le \|T\|$$

so $F(z)$ is bounded analytic and we're done.



July 3, 2020

In the Standard Model, the symmetry group of the forces other than gravity is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$: $\mathrm{SU}(3)$ for the strong force and $\mathrm{SU}(2) \times \mathrm{U}(1)$ for weak and electromagnetic forces combined.

Why this group? Can we derive it from beautiful math?

Yes! But I don't know if it helps.

First, the true symmetry group is not $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, it's $\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$: the group of 5×5 unitary matrices with determinant 1 that are block diagonal with a 3×3 block and a 2×2 block.

This group is $\mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1)$ mod a certain Z_6 subgroup, and it explains why quarks have charges $2/3$ and $-1/3$. For details, try Section 5 here:

- John Baez and John Huerta, The algebra of grand unified theories, *Bull. Amer. Math. Soc.* **47** (2010), 483–552.

So how can we get $\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$ to fall out of beautiful pure math?

Take the Jordan algebra of 3×3 self-adjoint octonionic matrices. Take the group of automorphisms that preserve a copy of the 2×2 self-adjoint complex matrices sitting inside it. It's $\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$.

This is intriguing because we know the Jordan algebras that can describe observables in finite quantum systems. They come in 4 infinite families, the most famous being the $n \times n$ self-adjoint complex matrices.

Then there's one exception: 3×3 self-adjoint octonionic matrices! This is called the 'exceptional Jordan algebra'.

The 2×2 self-adjoint complex matrices are observables for a 'qubit'. They sit inside the 3×3 self-adjoint octonionic matrices in many ways. The symmetries of this larger Jordan algebra that map the smaller one to itself are the symmetries of the Standard Model!

This was discovered by Michel Dubois-Violette and Ivan Todorov in 2018, and I explained it here:

- Exceptional quantum geometry and the Standard Model, August 27, 2018.

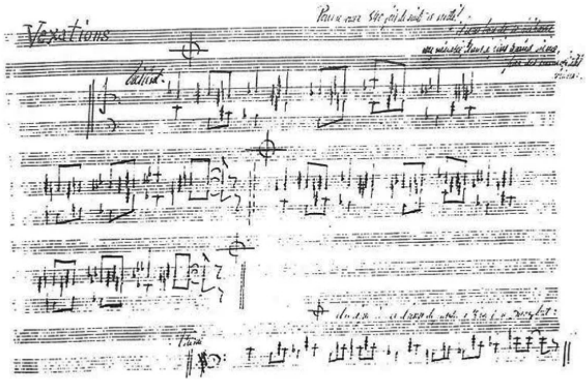
But it's a long way from an observation like this to a theory of particle physics! It could even be a red herring. We don't know. It's frustrating.

Here's a new paper that tries to go further:

- Latham Boyle, The Standard Model, the exceptional Jordan algebra, and triality.

It's not the answer to all our questions... but I'm glad to see someone gnawing on this bone.

July 4, 2020



This is a fun article:

- Sam Sweet, A dangerous and evil piano piece, *The New Yorker*, September 9, 2013.

Here are some excerpts that give the flavor:

> Though Erik Satieâ€™s "Vexations" (1893) consisted of only a half sheet of notation, its recital had previously been deemed impossible, as the French composer had suggested at the top of his original manuscript that the motif be repeated eight hundred and forty times.
>
> Even before repetition, the piano line is unnerving: mild but menacing, exquisite but skewed, modest but exacting. Above the music, Satie included an author's note, as much a warning as direction: â€œIt would be advisable to prepare oneself beforehand...."
>
> The American composer John Cage was the first to insist that staging "Vexations" was not only possible but essential. No one knew what exactly would occur, which is part of what enticed Cage, who had a lust for unknown outcomes.
>
> The performance commenced at 6 P.M. that Monday and continued to the following dayâ€™s lunch hour. To complete the full eight hundred and forty repetitions of "Vexations" took eighteen hours and forty minutes.
>
> The New York Times sent its own relay team of critics to cover the event in its entirety... In the aftermath, some onlookers were bemused; others were agitated. Cage was elated. "I had changed and the world had changed," he later said.
>
> In the years that followed its début, "Vexations" outgrew its status as a curiosity. It became a rite of passage. As performances flourished, its legend intensified... Recitals were part endurance trial, part vision quest.
>
> Even after hundreds of repetitions, players are forced to sight-read from the beginning, as if learning for the first time. Witnesses have reported a similar effect. Listeners that subject themselves to the unnerving melody for several hours still find themselves incapable of humming it.
>
> Those who sit for all eight hundred and forty repetitions tend to agree on a common sequence of reactive stages: fascination morphs into agitation, which gradually morphs into all-encompassing agony. But listeners who withstand that phase enter a state of deep tranquility.
>
> An Australian pianist named Peter Evans abandoned a 1970 solo performance after five hundred and ninety-five repetitions because he claimed he was being overtaken by evil thoughts and noticed strange creatures emerging from the sheet music.

Igor Levit did a live-streamed performance of "Vexations"; a small portion is here:

July 5, 2020

Tarski's theorem on the undefinability of truth says that the concept of truth in any sufficiently powerful system of mathematics is not definable within that system.

Here's a special case. Suppose *N* is a model of Peano arithmetic. Each sentence *x* in Peano arithmetic has a Gödel number *g*(*x*).

Theorem. There is no formula *True*(*n*) in the language of Peano arithmetic that defines the set of Gödel numbers of sentences that hold in the model *N*. That is: there is no formula *True*(*n*) such that *True*(*g*(*x*)) holds in *N* iff *x* holds in *N*.

Roughly speaking, Tarski proved that truth within a system of math can't be defined within that system. Why not? If it could, you could create a statement in that system that means "this statement isn't true".

But there are some loopholes you should know about.

For one, you can define truth within some system of math using a more powerful system. Tarski actually constructed an infinite hierarchy of systems, each more powerful than the ones before, where truth in each system could be defined in all the more powerful ones!

But you can also do this: within Peano arithmetic, you can define truth for sentences that have at most *n* quantifiers!

Sorta like: "Nobody can give you all the money you might ask for, but for any *n* someone can give you up to *n* dollars."

This shocked me at first. Michael Weiss explained it to me on his blog.

- Michael Weiss, Non-standard models of arithmetic 13, *Diagonal Argument*, September 22, 2019.

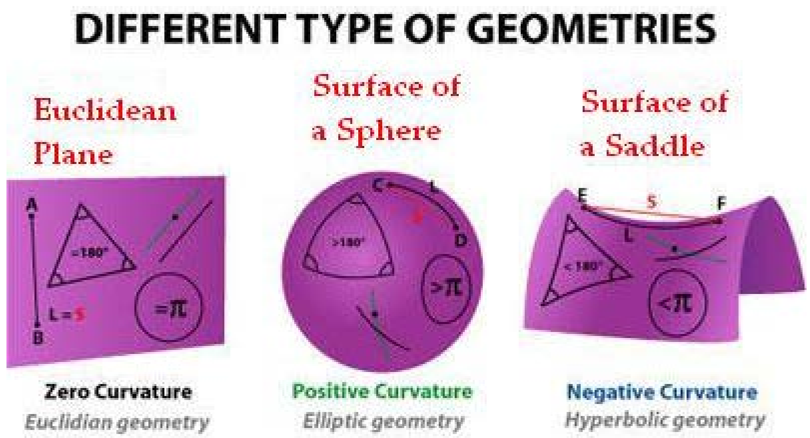
Skip down to where I say "let me think about this a while as I catch my breath."

The reason there's no paradox is that when you try to build the sentence that says "this sentence is false", it has one more quantifier. But Michael explains how you *can* define truth for sentences with at most n quantifiers. It's an inductive construction, based on ideas of Tarski's. For more on his ideas, go here:

- Stanford Encyclopedia of Philosophy, Tarski's truth definitions.

I think the moral is that while you can define mathematical truth in stages, you can never finish.

July 11, 2020



A 'Riemannian manifold' is, roughly speaking, a space in which we can measure lengths and angles. The most symmetrical of these are called 'symmetric spaces'. In 2 dimensions there are 3 kinds, but in higher dimensions there are more.

An 'isometry' of a Riemannian manifold M is a one-one and onto function $f: M \to M$ that preserves distances (and thus, it turns out, angles). Isometries form a group. You should think of this as the group of symmetries of M.

For M to be very symmetrical, we want this group to be big.

The group of isometries of a Riemannian manifold is a manifold in its own right! So it has a dimension.

For example the isometry group of the plane, sphere or hyperbolic plane is 3-dimensional. This is biggest possible for the isometry group of a 2d Riemannian manifold!

For an n-dimensional Riemannian manifold, how big can the dimension of its isometry group be? It turns out the maximum is $n(n+1)/2$. And this happens in just 3 cases:

- n-dimensional Euclidean space,
- the n-dimensional sphere, and
- n-dimensional hyperbolic space.

So, whatever definition of 'highly symmetrical Riemannian manifold' we choose, these 3 cases deserve to be included. (And maybe others if we allow disconnected manifolds, or non-simply connected ones — but we usually don't, in this game.)

Another great bunch of examples come from 'Lie groups': manifolds that are also groups, such that multiplication is a smooth map.

The best Lie groups are the 'compact' ones. These can be made into Riemannian manifolds in such a way that both left and right multiplication by any element is an isometry!

We can completely classify compact Lie groups, and study them endlessly.

So, any decent definition of 'symmetric space' should include Euclidean spaces, spheres, hyperbolic spaces and compact Lie groups — like the rotation groups $SO(n)$, or the unitary groups $U(n)$. And there's a very nice definition that includes all these — and more!

A Riemannian manifold M is a **symmetric space** if it's connected and for each point x there's an isometry $f: M \to M$ called "reflection through x" that maps x to itself and reverses the direction of any tangent vector at x:

$$f(x) = x$$

and

$$df_x = -1$$

For example, take Euclidean space. For any point x, reflection through x maps each point $x + v$ to $x - v$. So it maps x to itself, and reverses directions!

To understand symmetric spaces better, it's good to draw or mentally visualize 'reflection around x' for a sphere.

We can completely classify *compact* symmetric spaces — and spend the rest of our lives happily studying them. Besides the compact Lie groups, there are 7 infinite families and 12 exceptions, which are all connected to the octonions:

- Wikipedia, Symmetric space: classification result.

Symmetric spaces are great if you like geometry, because here's an almost equivalent definition: they are the Riemannian manifolds whose curvature tensor is preserved by parallel translation!

(Some fine print is required for a complete match of definitions.)

Symmetric space are also great if you like algebra! Just as Lie groups can be studied using Lie algebras, symmetric spaces can be studied using '$Z/2$-graded Lie algebras', or equivalently 'Lie triple systems'. I explained that approach here on June 24th.

Even better, the $7 + 3 = 10$ infinite series of compact symmetric spaces (the seven I mentioned plus the three infinite series of compact Lie groups) are fundamental in condensed matter physics! They appear in the something called the '10-fold way', which classifies states of matter:

- The tenfold way.

So, our search for the most symmetrical spaces leads us to a meeting-ground of algebra and geometry that generalizes the theory of Lie groups and Lie algebras and has surprising applications to physics! What more could you want?

Oh yeah, you might want to *learn* this stuff.

Wikipedia is good if you have the math background for it:

- Wikipedia, Symmetric space.

These notes are nice, and they have lots of examples:

- J.-H. Eschenburg, Lecture notes on symmetric spaces.

Then try Sigurdur Helgason's *Differential Geometry, Lie Groups and Symmetric Spaces* — I learned Lie groups from him, and this book of his, when I was a grad student at MIT.

Finally, to really sink into the glorious details of symmetric spaces, I recommend Arthur Besse's *Einstein Manifolds*. Besse is a relative of the famous Nicolas Bourbaki. His book has lots of great tables. It's lots of fun to browse!

July 12, 2020

A topos is a universe in which you can do mathematics, with its own internal logic, which may differ from classical logic.

On the category theory mailing list, Vaughan Pratt once doubted that anyone could really *think* using this internal logic, calling this locker-room boasting.

Steve Vickers replied as follows:

> **Steve Vickers on topos theory:**
>
> **As a parable, I think of toposes as gorillas.... At first they look very fierce and hostile, and the locker-room boasting is all tales of how you overpower the creature and take it back to a zoo to live in a cage — if it's lucky enough not to have been shot first. When it dies you stuff it, mount it in a threatening pose with its teeth bared and display it in a museum to frighten the children. But get to know them in the wild, and gain their trust, then you begin to appreciate their gentleness and can play with them.**
>
> **The gorilla in the cage is the topos in the classical world.**

Here's my very quick intro to topos theory:

- [Topos theory in a nutshell](#).

Here's the full exchange between Pratt and Vickers, which adds useful detail:

- [Learning to love topos theory](#).

July 14, 2020

Duality is a big theme in mathematics. Triality is more exotic. Any vector space has a dual. But a triality can happen only in certain special dimensions!

TRIALITY

Given finite-dimensional real vector spaces V_1, V_2, we may define a **duality** to be a bilinear map

$$f: V_1 \times V_2 \to \mathbb{R}$$

such that if we fix either argument to a nonzero value, the linear functional induced on the other vector space is nonzero. This forces $\dim V_1 = \dim V_2$.

Similarly, given V_1, V_2, V_3, a **triality** is a trilinear map

$$t: V_1 \times V_2 \times V_3 \to \mathbb{R}$$

such that if we fix any two arguments to nonzero values, the linear functional induced on the third vector space is nonzero. This forces $\dim V_1 = \dim V_2 = \dim V_3 = 1, 2, 4$ or 8.

For example, take all three vector spaces to be C, the complex numbers, and define $t(v_1, v_2, v_3) = \mathrm{Re}(v_1 v_2 v_3)$. This is a triality!

The same trick works if we start with the real numbers, the quaternions, or the octonions.

But wait! The octonions aren't associative! So what do I mean by $\mathrm{Re}(v_1 v_2 v_3)$ then? Well, luckily

$$\mathrm{Re}((v_1 v_2) v_3) = \mathrm{Re}(v_1 (v_2 v_3))$$

even when v_1, v_2, v_3 are octonions. The proof that finite-dimensional trialities can happen only in dimensions 1,2,4 or 8 is quite deep. It's easy to show any triality gives a 'division algebra', and I explain that here:

- Spinors and trialities.

But then we need to use a hard topological theorem! It's pretty easy to show that if there's an n-dimensional division algebra, the $(n-1)$-sphere is 'parallelizable': we can find $n-1$ continuous vector fields on this sphere that are linearly independent at each point.

In 1958, Kervaire, Milnor and Bott showed this only happens when $n = 1, 2, 4,$ or 8.

So, trialities are rare. But once you have one, you can do lots of stuff.

Even better, the 'magic square' lets you take *two* trialities and build a Lie algebra. If you take them both to be the octonions, you get E_8.

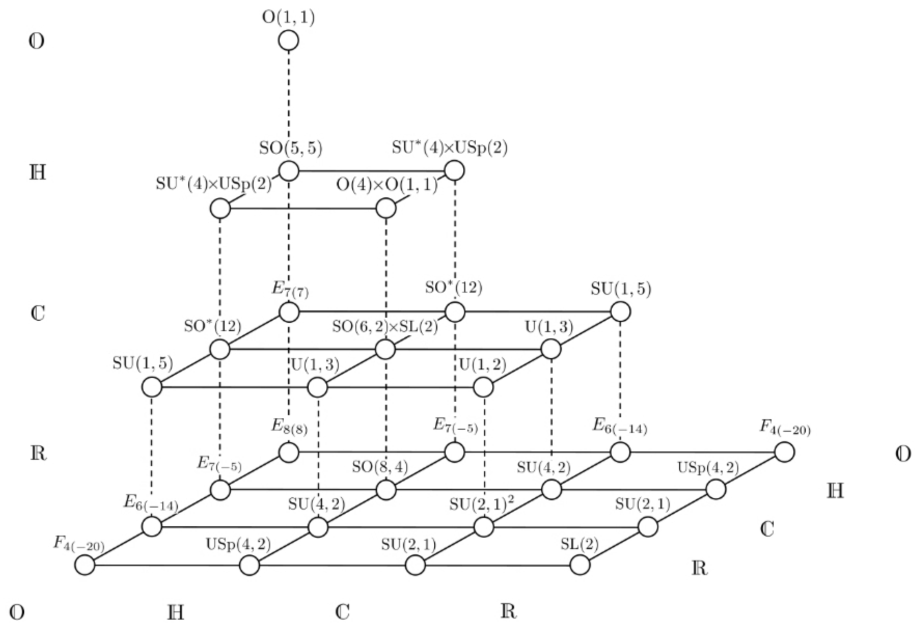
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Details here:

- The magic square.

And what's better than two trialities? Well, duh — *three!!!*

Starting from three trialities — but not just any three — you can build a theory of supergravity. This gives the 'magic pyramid of supergravities':



For more, read this:

- A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes and S. Nagy, A magic pyramid of supergravities.

I don't understand the magic pyramid of supergravities yet, but I'm hoping to learn about it from Mia Hughes' thesis:

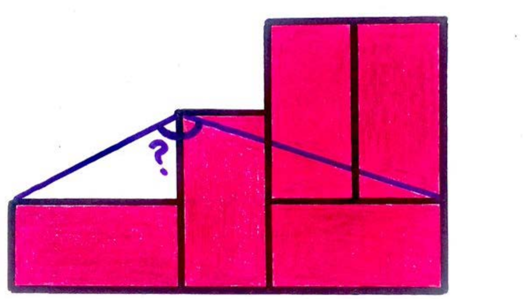
- Mia Hughes, Octonions and supergravity, Ph.D. Thesis, Imperial College London, 2016.

She explains everything in a systematic way that I really dig.

July 15, 2020

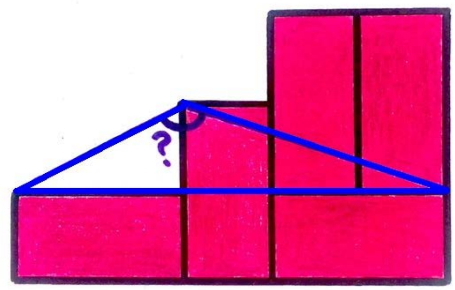
Catriona Shearer creates great geometry puzzles. Here's one:

Five congruent rectangles. Whatâ€™s the angle?



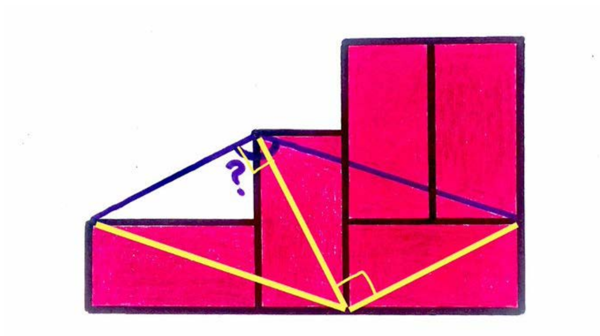
We can solve this in two ways. First, the rectangles must be twice as long as they are wide, so if we chop the mystery angle into two parts as below, we see it's

$$\arctan 2 + \arctan 3.$$



But Vincent Pantaloni chopped the mystery angle into two parts a different way, which makes it clear the angle is

$$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$



Equating these results, we get

$$\arctan 2 + \arctan 3 = \frac{3\pi}{4}.$$

But $\arctan(1) = \pi/4$, so we get

$$\arctan 1 + \arctan 2 + \arctan 3 = \pi.$$

This is nice because $\arctan 2$ and $\arctan 3$ aren't rational multiples of π; they're sort of complicated. For example

$$\arctan 2 = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)}.$$

Someone pointed out another way to show that

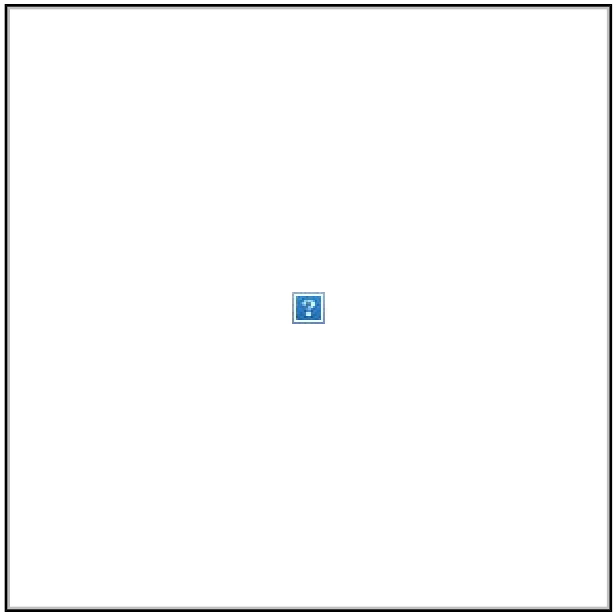
$$\arctan 1 + \arctan 2 + \arctan 3 = \pi.$$

It follows straightaway from

$$(1+i)(1+2i)(1+3i) = -10$$

since angles in the complex plane add when we multiply complex numbers, and -10 is at an angle π from the positive x axis.

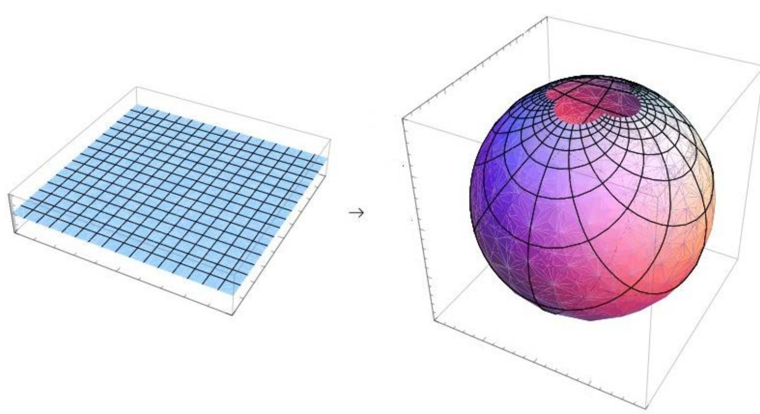
July 16, 2020



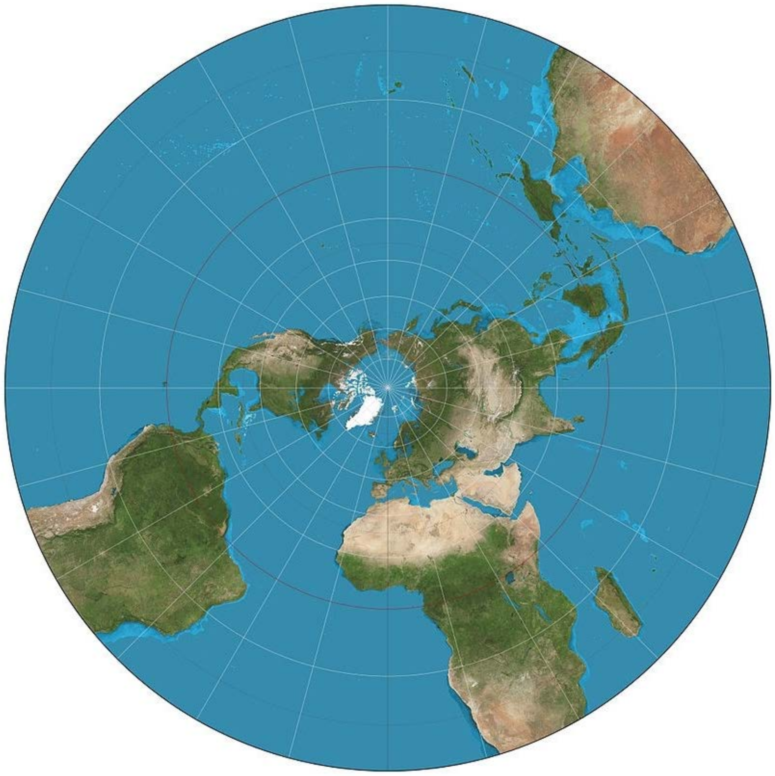
Take a sphere and set it on the plane. You can match up almost every point on the sphere with one on the plane, by drawing lines through the north pole.

There's just one exception: the north pole itself! So, the sphere is like a plane with one extra point added.

The interesting thing about this trick is that *angles* on the plane equal angles on the sphere!



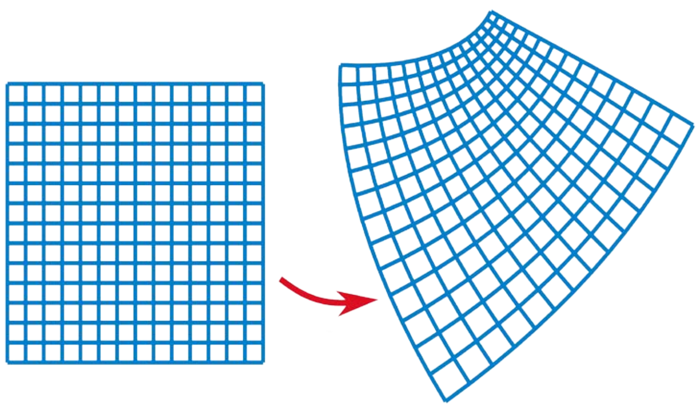
So if you use this trick to draw a map of the Earth, distances get messed up but angles are preserved at each point. Antarctica would stretch on forever:



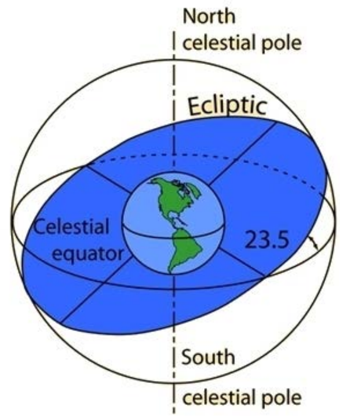
Some useful jargon: an angle-preserving mapping is called 'conformal'.

Mathematicians often call the plane 'the complex numbers', where a point (x, y) is written as the number $x + iy$. Then the sphere is called the 'Riemann sphere': the complex numbers plus one extra point, called ∞. It lets us think of infinity as a number!

And here's a wonderful thing: any differentiable function from the complex numbers to the complex numbers preserves angles — except where its derivative is zero. So it's a conformal mapping!



The Riemann sphere is not some abstract thing, either. It's the sky! More precisely, if you're in outer space and can look in every direction, the 'celestial sphere' you see is the Riemann sphere.



Now suppose you're moving near the speed of light. Thanks to special relativity effects, the constellations will look warped. But all the *angles* will be the same. Your view will be changed by a conformal transformation of the Riemann sphere!

A math book may summarize all this as follows:

$$SO_0(3, 1) \qquad PSL(2, C).$$

In other words: the group of Lorentz transformations is isomorphic to the group of conformal transformations of the Riemann sphere!

So: when reading math, it's often *your* job to bring it to life.

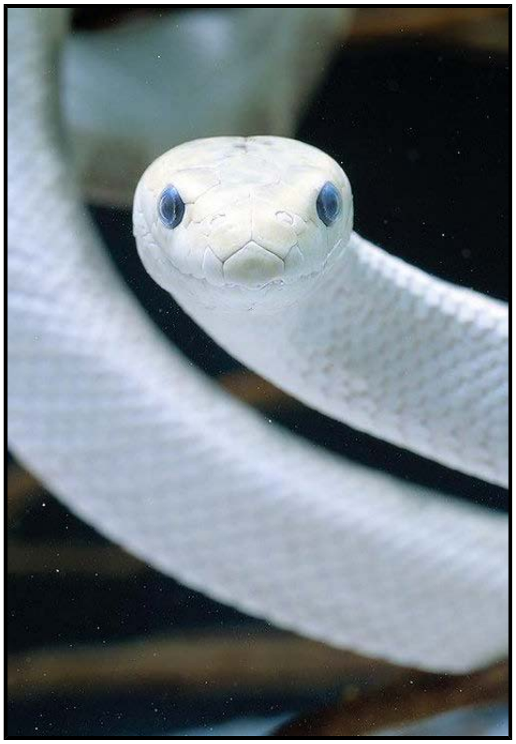
July 18, 2020

There's a lot of depressing news these days. Pictures of animals help me stay happy. Here are some of my faves. First: an insanely cute Cuban flower bat, *Phyllonycteris poeyi*, photographed by Merlin Tuttle.

Second: a devilishly handsome Dracula parrot, *Psittrichas fulgidus*, photographed by Ondrej Prosicky. It lives in New Guinea. It subsists almost entirely on figs. It's also called Pesquet's parrot.

Third: the aptly named 'elegant sea snake', *Hydrophis elegans*.

It's elegant, but it's poisonous.



Fourth, a kitten of a Canada lynx, *Lynx canadensis*.

July 20, 2020

QUATERNIONS

A quaternion combines a scalar and a vector:

$$\mathbf{a} = \underbrace{a}_{\text{scalar part}} + \underbrace{a_1 i + a_2 j + a_3 k}_{\text{vector part, } \vec{a}}$$

Since

$$ij = k = -ji \qquad jk = i = -kj \qquad ki = j = -ik$$
$$i^2 = j^2 = k^2 = -1$$

quaternion multiplication combines all four ways of multiplying scalars and vectors:

$$\mathbf{ab} = (ab - \vec{a} \cdot \vec{b}) + (a\vec{b} + b\vec{a} + \vec{a} \times \vec{b})$$

The beauty of quaternion multiplication is that it combines all ways of multiplying scalars and vectors in a single package, with a concept of absolute value that obeys $|ab| = |a||b|$.

Last week I realized that octonion multiplication works almost the same way — but with *complex* scalars and vectors!

An octonion combines a complex number (or 'scalar') and a complex vector in a single package. You multiply them like quaternions, but with some complex conjugation sprinkled in. We need that to get $|ab| = |a||b|$ for octonions.

QUATERNIONS AND OCTONIONS

A quaternion $a + \vec{a}$ combines a scalar $a \in \mathbb{R}$ and a vector $\vec{a} \in \mathbb{R}^3$. You multiply quaternions like this:

$$(a + \vec{a})(b + \vec{b}) = (ab - \vec{a} \cdot \vec{b}) + (a\vec{b} + b\vec{a} + \vec{a} \times \vec{b})$$

An octonion is a pair $a + \vec{a}$ consisting of a complex scalar $a \in \mathbb{C}$ and a complex vector $\vec{a} \in \mathbb{C}^3$. You multiply octonions like this:

$$(a + \vec{a})(b + \vec{b}) = (ab - \vec{a}^* \cdot \vec{b}) + (a\vec{b} + b^*\vec{a} + (\vec{a} \times \vec{b})^*)$$

Here a^* is the complex conjugate of $a \in \mathbb{C}$, while $\vec{a}^*$ is the vector formed by taking the complex conjugate of each component of $\vec{a} \in \mathbb{C}^3$.

I figured out this formula for octonion multiplication when trying to explain the connection between octonions and the group SU(3), which governs the strong nuclear force. You can see details here:

- Octonions and the Standard Model (part 1).

The octonions are to SU(3) as the quaternions are to SO(3)! The ordinary dot and cross product are invariant under rotations, SO(3), so the automorphism group of the quaternions is SO(3). For octonions we use complex vectors, and dot and cross products adjusted to be invariant under SU(3), so the group of octonion automorphisms fixing *i* is SU(3).

July 22, 2020

In geometry and topology dimensions 0-4 tend to hog the limelight because each one is so radically different than the ones before, and so much amazing stuff happens in these 'low dimensions'. I don't know enough about dimensions 5-7, but... the even part $\mathrm{Cliff}_0(n)$ of the Clifford algebra generated by n anticommuting square roots of -1 follows a cute pattern for n =5,6,7:

- $\mathrm{Cliff}_0(5) = \mathrm{H}[2]$ (2×2 quaternionic matrices)
- $\mathrm{Cliff}_0(6) = \mathrm{C}[4]$ (4×4 complex matrices)
- $\mathrm{Cliff}_0(7) = \mathrm{R}[8]$ (8×8 real matrices)

See it? The dimension of $\mathrm{Cliff}_0(n + 1)$ is always twice that of $\mathrm{Cliff}_0(n)$. But for n = 5,6,7 this happens by making square matrices that are twice as big — so, with 4 times as many entries — having entries in a division algebra whose dimension is half as big:

$$\mathrm{H}[2],\ \mathrm{C}[4],\ \mathrm{R}[8].$$

The obvious representations of these matrix algebras are called 'real spinor representations'. So:

- in 5d space, real spinors are elements of H^2
- in 6d space, they're elements of C^4
- in 7d space, they're elements of R^8

Notice: R^8, C^4, and H^2 are very similar things!

Real spinors in dimensions 5,6,7 form an 8-dimensional real vector space with *extra structure* — and *more* structure as the dimension goes down:

- In 7d it's just a real vector space.
- In 6d it's a complex vector space.
- In 5d it's a 'quaternionic vector space'.

This has nice spinoffs! The double cover of the rotation group SO(n) is called the 'spin group' Spin(n). We can show:

- Spin(6) = SU(4) consists of 4×4 unitary complex matrices with determinant 1.
- Spin(5) = Sp(2) consists of 2×2 unitary quaternionic matrices.

All this sets up a lot cross-talk between geometry and topology in dimensions 5, 6, 7... and, yes, 8, where the octonions become important! 'Calabi-Yau manifolds' are part of this story.

I explain this in a lot more detail in week195 of This Week's Finds:

Summary:

- When M is an 8-dimensional spin manifold with 2 out of these 3 things:
    - a nonzero covariantly constant vector field
    - a nonzero covariantly constant left-handed spinor field
    - a nonzero covariantly constant right-handed spinor field

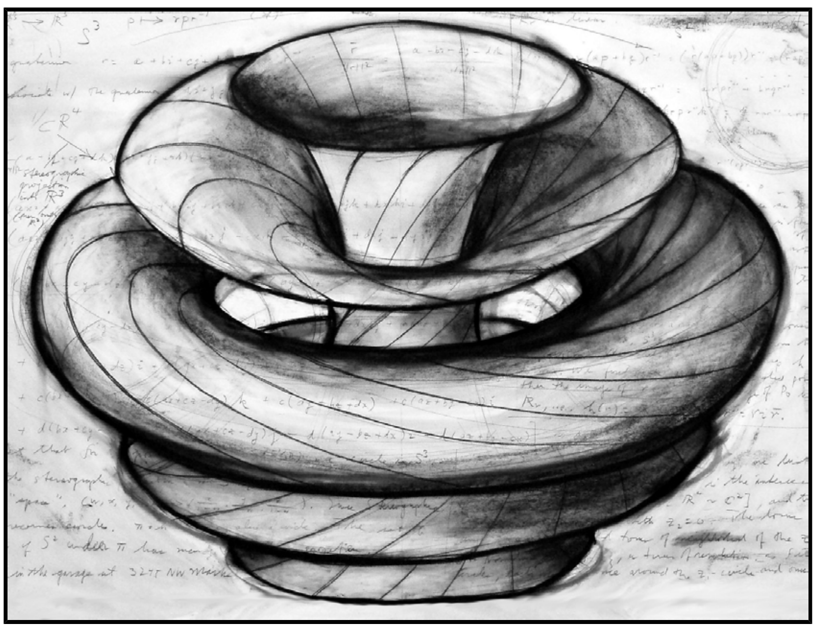
  it automatically gets all three - and its tangent bundle, left-handed spinor bundle and right-handed spinor bundle all become octonion bundles. We call M an octonionic manifold.

- When N is a 7-dimensional spin manifold with a nonzero covariantly constant spinor field, its spinor bundle becomes an octonion bundle, while its tangent bundle becomes an imaginary octonion bundle. We call N a G2 manifold.

- When O is a 6-dimensional spin manifold with a nonzero covariantly constant spinor field, its spinor bundle becomes an octonion bundle, while its tangent bundle plus a trivial line bundle becomes an imaginary octonion bundle. We call O a Calabi-Yau manifold.

- When P is a 5-dimensional spin manifold with a nonzero covariantly constant spinor field, its spinor bundle becomes an octonion bundle, while its tangent bundle plus two trivial line bundles becomes an imaginary octonion bundle. We call O an SU(2) manifold.

July 25, 2020



The 3-sphere S^3 can be seen as R^3 plus a point at infinity. But here London Tsai shows the 'Hopf fibration': S^3 as a bundle of circles over the 2-sphere. Each point in S^3 lies on one circle. The set of all these circles forms a 2-sphere.

S^3 is an S^1 bundle over S^2.

But the 3-sphere S^3 is also a group! It's called SU(2): the group of 2×2 unitary matrices with determinant 1.

So we can see the group SU(2) as an S^1 bundle over S^2. But in fact we can build *many* groups from spheres.

Let's try SU(3). This acts on the unit sphere in C^3. C^3 is 6-dimensional as a real space, so this sphere has dimension one less: it's S . Take your favorite point in here; each element of SU(3) maps it to some other point. Using this we can see SU(3) is a bundle over S^5.

Many elements of SU(3) map your favorite point in S^5 to the same other point. What are they like? They form a copy of SU(2), the subgroup of SU(3) that leaves some unit vector in C^3 fixed.

So SU(3) is an SU(2) bundle over S^5.

But SU(2) is itself a sphere, S^3. So SU(3) is an S^3 bundle over S^5.

In other words, you can slice SU(3) into a bunch of 3-spheres, one for each point on the 5-sphere. It's kind of like a higher-dimensional version of the Hopf fibration shown above.

How about SU(4), the 4×4 unitary matrices with determinant 1? We can copy everything we just did: this group acts on C^4 so it acts on the unit sphere in there, which is S^7. The elements mapping your favorite point to some other form a copy of SU(3).

So, SU(4) is an SU(3) bundle over S^7.

Note: we've seen that SU(4) is an SU(3) bundle over S^7, while SU(3) is an S^3 bundle over S^5.

So SU(4) is a S^3 bundle over an S^5 bundle over S^7.

Maybe you see the pattern now. We can build the groups SU(n) as 'iterated sphere bundles'.

For example, SU(5) is an S^3 bundle over an S^5 bundle over an S^7 bundle over S^9.

As a check, you can compute the dimension of SU(5) in some other way and show that yes, indeed

$$\dim(SU(5)) = 3 + 5 + 7 + 9$$

Even better, the group U(5) of *all* unitary 5×5 matrices is an S^1 bundle over an S^3 bundle over an S^5 bundle over an S^7 bundle over S^9. The S^1 here comes from the choice of determinant.

So: $\dim(U(5)) = 1 + 3 + 5 + 7 + 9 = 5^2$ and this pattern works in general.

It's easy to see that the sum of the first n odd numbers is n^2. But we've found a subtler incarnation of the same fact! We've built $\mathrm{U}(n)$ out of the first n odd-dimensional spheres, as an iterated bundle.

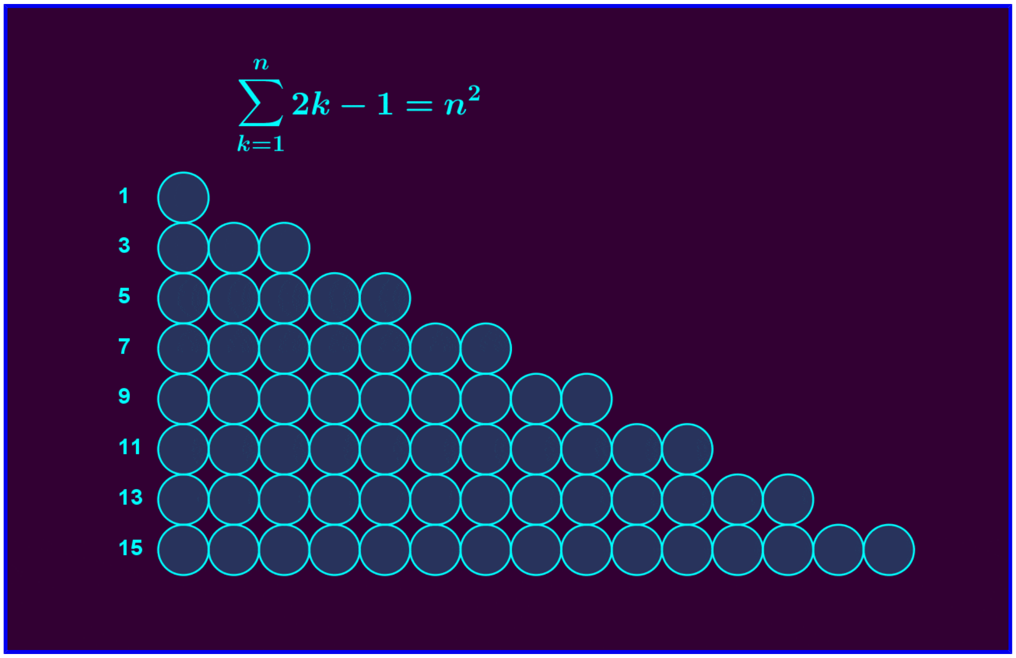
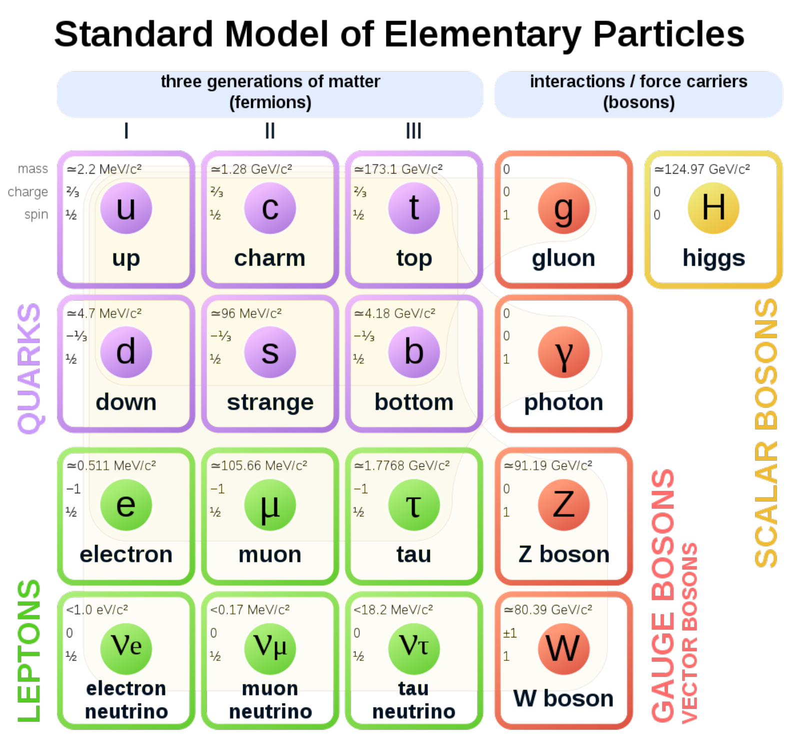


image by [Vincent Pantaloni](#)

Puzzle. Can you describe \mathrm{O}(n), the group of orthogonal n \times n matrices, as an iterated sphere bundle in a similar way?

July 26, 2020



The matter you see is made of up and down quarks, electrons... and then there are electron neutrinos, hard to see. These are the 'first generation' of quarks and leptons.

There are three generations, each with 2 quarks and 2 leptons. Why this pattern?

Short answer: nobody knows.

But we know some stuff.

To get a consistent theory of physics, we need 'anomaly cancellation'. If one generation had just one quark, or a lepton with the wrong charge — and everything else the same — the laws of physics wouldn't work!

Some 'grand unified theories' fit the observed pattern of quarks and leptons quite beautifully. For example, in the $\mathrm{Spin}(10)$ theory all the quarks and leptons in each generation, and their antiparticles, fit into a neat package: an 'irreducible representation' of this group. This theory *forces* there to be a quark of electric charge $+2/3$, a quark of charge $-1/3$, a lepton of charge 0 and a lepton of charge -1 in each generation — which is exactly what we see!

But this theory predicts that protons decay, which we haven't seen (yet?).

A more quirky line of attack, much less well developed, uses octonions. The octonions contain lots of square roots of -1. If you pick one and call it i, the octonions start looking like a quark and a lepton! But only as far as the strong force is concerned.

The strong force has symmetry group $\mathrm{SU}(3)$. Each quark comes in three 'colors': red, green and blue. This is just a colorful way of saying the quark's quantum states, as far as the strong force is concerned, transform according to the usual representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$.

Each lepton, on the other hand, is 'white'. It doesn't feel the strong force at all. As far as the strong force is concerned, its quantum states transform according to the trivial representation of $\mathrm{SU}(3)$ on $\mathbb{C}$.

What does all this have to do with octonions?

Choosing a square root of -1 in the octonions and calling it i makes them into a complex vector space. The group of symmetries of the octonions that preserve i is $\mathrm{SU}(3)$. As a representation of $SU(3)$, the octonions are $\mathbb{C} \oplus \mathbb{C}^3$. Just right for a quark and a lepton!

This is not a theory of physics; this is just a small mathematical observation. It could be a clue. It could also be a coincidence. But it's kind of cute.

To give a clear proof of this fact, I came up with a new construction of the octonions using complex numbers:

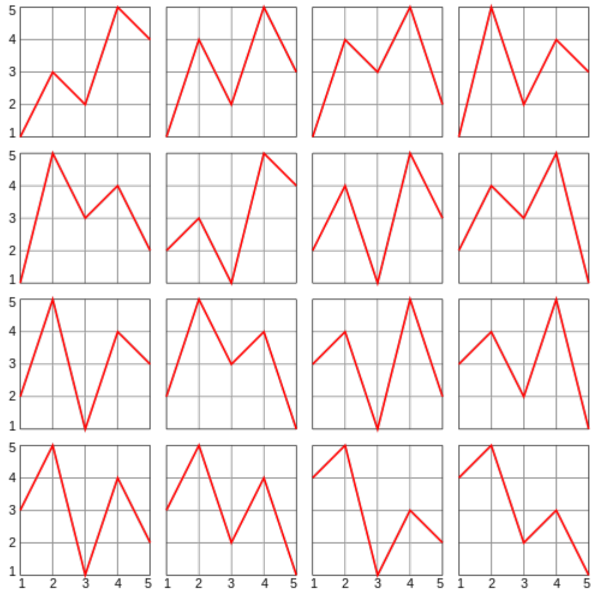
- Octonions and the Standard Model (part 2).

Then I used that here to get the job done:

- Octonions and the Standard Model (part 3)

So, read those if you're curious about this stuff!

July 28, 2020



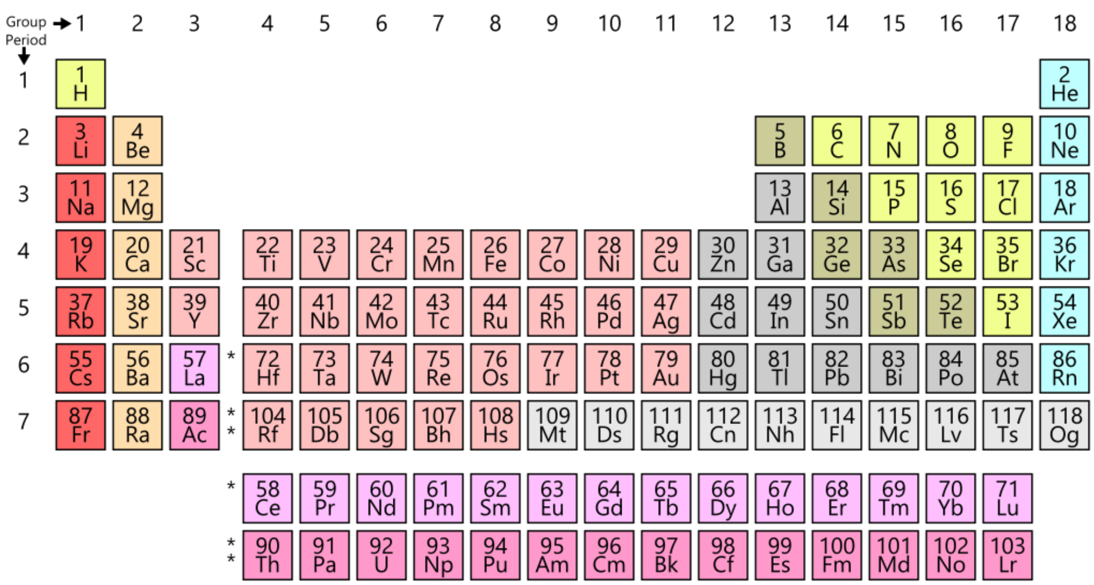
A permutation $\sigma \colon \{1,\dots,n\} \to \{1,...,n\}$ is **alternating** if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \cdots$ The number of alternating permutations of $\{1,\dots,n\}$ is called the nth **zigzag number**, A_n. For example, the picture above shows $A_5 = 16$.

The nth zigzag number equals the nth coefficient of the Taylor series of $\sec x$ or $\tan x$, depending on whether n is even or odd. This remarkable fact is called **André's theorem**. You can see one proof here.

Since $\sec x$ is an even function while $\tan x$ is odd, we can summarize André's theorem by saying $\sum_{n = 0}^\infty \frac{A_n}{n!} x^n = \sec x + \tan x$

Nice! Trig meets zig.

But here's the weird thing. Take an alternating permutation of $\{1,\dots,n\}$ and count the triples $i < j < k$ with $\sigma(i) < \sigma(j) < \sigma(k)$. The maximum possible value of this count is 0 when $n<4$, but then it goes like this: $2, 4, 12, 20, 38, 56, 88, \dots$ Can you spot these numbers in the periodic table?



Yes! These numbers, starting with 4, equal the number of electrons in the alkali earth elements: beryllium, magnesium, calcium, strontium, barium, radium,....

Coincidence? No! I don't understand it yet, but it's explained in the new issue of the *Notices of the American*

Mathematical Society — which like any *good* professional notices, are free to all:

- Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society* **67** (2020), 994–1001.

July 30, 2020

One job of mathematicians is to shield the rest of the human race from insanity by discovering paradoxes and figuring out how to deal with them before anyone else even notices.

July 31, 2020

One of the great things about category theory is how it "eats its own tail". Concepts become so general they subsume themselves!

Let me explain with an example: every Grothendieck topos is equivalent to a category of sheaves on *itself*.

What do all these words mean?

The story starts with complex analysis. Liouville's theorem says every bounded analytic function on the whole complex plane is constant. It follows that every analytic function defined on the whole Riemann sphere is constant.

The *interesting* analytic functions on the Riemann sphere are just *partially defined*: for example, they may have poles at certain points. So we need a rigorous formalism to work with partially defined functions.

That's one reason we need sheaves.

There's a 'sheaf' of analytic functions on the Riemann sphere. Call it $\mathcal{O}$. For any open subset U of the sphere, $\mathcal{O}(U)$ is the set of all analytic functions defined on U.

Note if $V \subseteq U$ we can restrict analytic functions from U to V, so we get a map $\mathcal{O}(U) \to \mathcal{O}(V)$.

Even better, we can tell if a function is analytic on an open set U by looking to see if it's analytic on a bunch of open subsets U_i that cover U. This says that being analytic is a 'local' property.

Technically: if we have a open set U covered by open subsets U_i and analytic functions f_i on the sets U_i that agree when restricted to their intersections $U_i \cap U_j$, there's a unique analytic function on all of U that restricts to each of these f_i. This is a mouthful, but this is the **sheaf condition**: the key idea in the definition of a sheaf.

If you understand this example — the sheaf of analytic functions on the Riemann sphere — you can understand the definition of a sheaf on a topological space.

Roughly: for a topological space X, a 'sheaf' S gives you a set $S(U)$ for any open $U \subseteq X$. There's a 'restriction' map $S(U) \to S(V)$ whenever $V \subseteq U$ is a smaller open set. And a couple of conditions hold — most notably the sheaf condition!

So, sheaves give a rigorous way to study partially defined functions — and more interesting partially defined things — on a topological space. They let us work 'locally' with these entities.

All this was known by the late 1950s. Then Grothendieck came along....

He noticed the open sets of a topological space are the objects of a category. And he showed you could define sheaves on other categories, too!

But to do this you need to choose a 'coverage' (or 'Grothendieck topology') for your category, which says what it means for a bunch of objects to cover another object.

A category with a coverage is called a 'site'. He figured out how to define sheaves on any site. This lets you do math *locally*... but where the concept of 'location' is no longer an open set, but an object in a category!

The category of all sheaves on a site is called a 'Grothendieck topos'. An example would be the category of all sheaves on the Riemann sphere. Or even simpler: the category of all sheaves on a point! This is just the category of sets.

Grothendieck invented this stuff to help prove some conjectures in algebraic geometry. But Grothendieck topoi took on a life of their own — and in fact, they 'eat their own tail', like the mythical ouroboros.

Notice that categories are showing up in two ways so far:

- A site is a category with a coverage.
- A Grothendieck topos is also a category: the category of all sheaves on a site.

So any category theorist worth their salt will wonder: could you make a Grothendieck topos into a site?

And the answer is *yes!* Any Grothendieck topos T has a god-given coverage making it into a site... and the category of sheaves on this site is equivalent to T itself.

So it's equivalent to the category of sheaves on itself!

For details go here:

- nLab, Canonical topology.

For my August 2020 diary, go here.

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baez@math.removethis.ucr.andthis.edu