Categorifying Fundamental Physics

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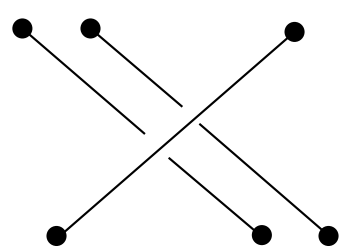
Despite the incredible progress over most of the 20th century, and a continuing flow of new *observational* discoveries — neutrino oscillations, dark matter, dark energy, and evidence for inflation — *theoretical* research in fundamental physics seems to be in a 'stuck' phase. So, now more than ever, it seems important to re-examine basic assumptions and seek fundamentally new ideas.

Work along these lines is inherently risky: many different directions must be explored, since while few will lead to important new insights, it is hard to know in advance which these will be. For this reason, I have ceased for now to work on loop quantum gravity, and begun to rethink basic mathematical structures in physics. The unifying idea behind this multi-pronged project is 'categorification', or in simple terms: *giving up the naive concept of equality.*

While equations play an utterly fundamental role in physics, and this is unlikely to change, equations between elements of a set often arise as a shorthand for something deeper: isomorphisms between objects in a category. For example, the equation $1 + 2 = 2 + 1$ summarizes the isomorphism between the sets $X \cup Y$ and $Y \cup X$, where X has 1 elements and Y has 2. In physics, such isomorphisms are closely related to the passage of time. For example, we can demonstrate that $1 + 2 = 2 + 1$ by switching two sets of particles:



If we think of this as a physical process, the isomorphism between the initial set $X \cup Y$ and the final set $Y \cup X$ consists of *the worldlines traced out by the particles with the passage of time.*

In topology, a diagram as above is called a 'braid', and a category that allows isomorphisms of this type is called a 'braided monoidal category'. Quite generally, sets with commutative operations (like addition of numbers) turn out to arise from braided monoidal categories. Indeed, a vast amount of familiar mathematics is turning out to be just a watered-down version of *categorified* mathematics — some already known, but much just beginning to be glimpsed.

In applications to physics, the process of categorification tends to go hand-in-hand with 'boosting the dimension by 1'. The picture above already illustrates this: the particles' worldlines are 1-dimensional, while the particles themselves are 0-dimensional points. A more impressive example is how certain commutative algebras yielding 2d topological field theories can be categorified to give braided monoidal categories, which in turn yield 3d topological field theories [1, 2, 3]. Yet another example, which I have recently developed with various collaborators, is 'higher gauge theory' [4].

Gauge theory has been remarkably successful in describing the interaction of *point particles* with various forces. But the mathematics of gauge theory — Lie groups [5], Lie algebras [6], bundles [7] and connections [8] — can all be categorified. The resulting 'higher gauge theory' turns out to describe the interaction of *strings* with background fields, including the somewhat mysterious B field [10]. Here it is the dimension, not of the ambient spacetime, but of the matter, that is getting boosted: worldlines become worldsheets. Further categorification clarifies the behavior of higher-dimensional branes [11].

I have now begun projects with three graduate students to categorify other ideas from fundamental physics. Christopher Walker is working on the algebra of quantum theory, Chris Rogers is categorifying classical mechanics and geometric quantization, and John Huerta is using category-theoretic methods to study the octonions and exceptional Lie groups and their role in particle physics.

These students are very talented; what they need most of all is time. They need time for research, and time to contribute to the online dissemination of work in progress — via blogging, online lecture notes and videos — that is a key part of my research style [15]. Sharing our successes and our mistakes will let the public see how research is done, and let our colleagues see the work as a whole, including ideas that are left out of the final polished work.

At a modest expense, the Foundational Questions Institute could free these students from a portion of their teaching duties and give them more time to work on these projects.

Why FQXi? Funding for this sort of multi-faceted fundamental research tends to fall between the cracks, since it sits between mathematics and physics. My work that touches on computer science *has* received funding, as explained in the last section of this narrative. But the physics aspect does not fit squarely in any established research program: while it touches on string theory and loop quantum gravity, it is not really either of these.

On the mathematics side, the NSF has divided mathematics into specialties, and my work crosses all the boundaries between these. In particular, the mathematicians at the NSF are loath to fund research that smells of category theory. So, I am turning to FQXi for help.

Categorifying the Algebra of Quantum Theory

Christopher Walker and I are working on an approach to quantum theory in which Hilbert spaces are replaced by purely combinatorial structures. The extensive use of the complex numbers in quantum physics is often taken for granted. With the research we are undertaking, we hope to show that by replacing the complex numbers with something more fundamental, we can see deeper into the algebra used in quantum theory.

Combinatorics naturally explains some of the discreteness seen in quantum mechanics. While this sounds superficially plausible, it actually came as a great surprise when James Dolan and I found that Joyal's work on combinatorics [12] gave a new way to understand the quantum harmonic oscillator, or more generally quantum field theory. For example, the "creation" and "annihilation" operators a^* and a, which physically describe the process of adding or removing a particle, can be seen as operations that describe the process of adding or removing an element from a finite set. A fundamental fact about quantum theory is that these operators fail to commute: $aa^* - a^*a = 1$ in units where Planck's constant is 1. Remarkably, this fact has a simple combinatorial interpretation: there is one more way to add an element to a finite set and then remove one, than to remove one and then add one. Furthermore, this interpretation can be used to *categorify* the commutation relation $aa^* - a^*a = 1$: that is, to see it as the summary of an isomorphism.

Starting from simple ideas like these, Dolan and I were able to develop a purely combinatorial theory of Feynman diagrams, which does not refer to the complex numbers [13]. This work was unable to handle the 'free' time evolution of the unperturbed oscillator, which involves complex phases. However, my student Jeffrey Morton [14] later extended our ideas to give a full-fledged categorification of perturbed quantum harmonic oscillators, using finite sets with elements labelled by phases which rotate at a constant rate. So, for quantum systems of this sort — such as many quantum field theories — the complex numbers play a more limited role than one would at first think: all the truly interesting structure is purely combinatorial.

Further work with James Dolan suggests that the combinatorial approach to quantum mechanics can be extended to give new insights into

q-deformation [15]. Quantum groups [16] are algebraic structures which appear in several approaches to fundamental physics, including string theory [17] and loop quantum gravity [18]. Like groups, they are used to describe symmetries, but they are not really groups. Instead, they are a type of algebra whose multiplication depends on a parameter q. However, in the limit as $q \to 1$ they essentially reduce to groups, so we can think of them as 'q-deformed groups'. Along with groups, many other structures can be q-deformed, including the quantum harmonic oscillator. It is as if a large portion of physics had a dial attached to it, which lets us adjust the value of q. At $q = 1$ we are doing physics as usual, but at other dial settings we are doing something new.

While q-deformation is a mathematically natural procedure, its true meaning remains shrouded in mystery, starting with the parameter q. In the original applications of quantum groups to completely solvable quantum systems, q was a function of Planck's constant ($q = e^{i\hbar}$), hence the name 'quantum group'. In this context, letting $q \to 1$ meant letting quantum mechanics reduce to classical mechanics. However, in quantum gravity it is often fruitful to think of q as a function of the cosmological constant [19, 20]. Then $q \to 1$ means letting the energy density of the vacuum go to zero. Given the profoundly mysterious nature of quantum gravity and the the cosmological constant, this is surely worthy of further study.

The goal of this project is to probe more deeply into the meaning of q-deformation by categorifying quantum groups. My student Aaron Lauda has already succeeded in categorifying quantum groups in recent work with Khovanov [21], but Walker and I are taking a different approach, rooted in the combinatorial ideas sketched above. In this approach, q-deformation arises naturally when we replace the combinatorics of *sets* by the combinatorics of *vector spaces over finite fields*.

The number system of primary interest in quantum physics is $\mathbb{C}$, the complex numbers. But there are also number systems with finitely many elements, called 'finite fields'. There is one of these with q elements, called $\mathbb{F}_q$, whenever q is a power of a prime number. Amazingly, combinatorial formulas involving vector spaces over $\mathbb{F}_q$ reduce to analogous formulas involving sets in the $q \to 1$ limit! For example, just as the number of k-element subsets of an n-element set is counted by the binomial coefficient $\binom{n}{k}$, the number of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ is counted by a certain 'q-binomial coefficient'. This reduces to the ordinary binomial coefficient when we formally take the limit $q \to 1$.

Using ideas of this sort, we have shown that just as the harmonic oscillator can be understood using the combinatorics of finite sets, the q-deformed harmonic oscillator can be understood using the combinatorics of finite-dimensional vector spaces over $\mathbb{F}_q$. More generally, our work in progress is revealing that much of the theory of quantum groups admits a combinatorial interpretation in terms of finite fields, which lets us categorify this theory. For example, it is known that the usual 'Jordan–Schwinger' representation of the group $\mathrm{SU}(n)$ on the Hilbert space of the quantum harmonic oscillator with n degrees of freedom can be q-deformed [22, 23, 24]; we have already seen how to categorify a substantial portion of this setup, and we hope to do more. We are also categorifying representations of other quantum groups, as well as the theory of Hecke algebras.

Of course, the idea of taking q to be the power of a prime number only heightens the mystery of its *physical* significance. To tackle this puzzle we need to take some of our results, feed them back into the physical applications of quantum groups, see how much of this physics can be given a purely combinatorial interpretation — and see what this means.

Categorifying Classical Mechanics and Geometric Quantization

Chris Rogers and I are categorifying classical mechanics and the quantization technique known as geometric quantization. Together with my student Alex Hoffnung, we have already completed some preliminary work which showed, somewhat to our surprise, that categorifying classical particle mechanics leads naturally to classical string theory [25]. We anticipate that similarly, the quantum string will arise naturally from categorifying the quantum particle. Indeed, there is already plenty of evidence for this [10]. However, we wish to more firmly link the classical and quantum aspects of this story using geometric quantization.

Geometric quantization is a procedure for quantizing a classical system in a purely geometric manner without the coordinate dependence present in canonical quantization [26]. This procedure involves treating the classical phase space as a manifold X equipped with a 'symplectic structure': that is, a closed nondegenerate 2-form, usually called ω. Under favorable conditions, we can construct a $\mathrm{U}(1)$ bundle over X with a connection whose curvature is the symplectic structure. Associated to this $\mathrm{U}(1)$ bundle is a complex line bundle over X, and quantum states are certain special sections of this line bundle. Singling out the physically acceptable quantum states requires that the phase space be given some extra structure. Classical observables that generate symmetries preserving this extra structure can be easily quantized; others are more problematic. This approach is appealing, since it allows one to analyze the ambiguities that arise from quantization in situations where no special symmetries are present to specify a preferred coordinate system.

What happens when we categorify geometric quantization? Of course, to answer this we must first know how to categorify classical mechanics! A strong clue comes from the theory of gerbes [27], which is a a categorified version of the theory of U(1) bundles. As we just hinted, the curvature of a connection on a U(1) bundle is a 2-form. Similarly, the curvature of a connection on a gerbe is a 3-form: we see here the phenomenon of 'dimension boosting' so typical of categorification. This suggests that when we categorify classical mechanics, the usual symplectic structure on phase space will be replaced by a 3-form — and when we categorify geometric quantization, the U(1) bundle on phase space will be replaced by a gerbe.

The question now becomes: what sort of physical system has a phase space equipped with a *3-form* instead of the usual 2-form? Amusingly, the answer to this puzzle goes back to the work of DeDonder [28] and Weyl [29] in the 1930s, which later gave rise to an interesting subject called 'multisymplectic geometry' [30, 31, 32]. Recently this has been taken up by Rovelli [33, 34]. It turns out that associated to any 2-dimensional classical field theory there is a finite-dimensional 'extended phase space' equipped with a closed nondegenerate 3-form. Such theories include classical bosonic string theory. So, the 'dimension boosting' that occurs when we categorify classical mechanics is closely related to the obvious dimension boosting that happens when we move from point particles to strings!

To see this in a bit more detail, it is instructive to first recall some details from the theory of classical point particles. In this theory we may start with the space of possible positions of the point particle, the configuration space M. Then, the phase space of the particle, that is the space of possible positions and momenta, is the cotangent bundle of M, denoted T^*M. This comes equipped with a symplectic structure ω. Observables are smooth real-valued functions on X. Every such observable H gives rise to a vector field vector field v_H on T^*M such that

$$dH(u) = \omega(v_H, u)$$

for every vector field v. In particular, if H is the Hamiltonian of our particle, the flow generated by v_H is the solution to Hamilton's equations of motion. The 'Poisson bracket' of observables G and H says how fast G changes if we use H as our Hamiltonian: it is given by $\{H, G\} = v_H(G)$. The Poisson bracket is the classical analogue of the commutator bracket in quantum mechanics. In particular, it makes the observables into a Lie algebra.

Following the multisymplectic approach to field theory, we can boost the dimension of everything in this familiar story. Starting from any manifold M called the 'extended configuration space', we can create an 'extended phase space' X equipped with a nondegenerate closed 3-form ω. This time X is not the cotangent bundle of M, but rather its second exterior power: $X = \Lambda^2(T^*M)$. Similarly, we take as observables, not functions on X, but certain 1-forms on X which we call 'Hamiltonian'. A 1-form H is Hamiltonian if there exists a vector field v_H on X such that

$$dH(u, u') = \omega(v_H, u, u')$$

for all vector fields u and u'. We write this formula merely so the reader can see the close analogy to the ordinary classical mechanics, recalled above.

Carrying this analogy further, we can define the Poisson bracket $\{H, G\}$ of Hamiltonian 1-forms, and show that this describes the rate of change of G when we use H as the Hamiltonian. But at this point, we observe a novel phenomenon. The Poisson bracket does *not* make the Hamiltonian 1-forms into a Lie algebra, because the bracket operation fails to be antisymmetric: $\{H, G\} \neq -\{G, H\}$. Luckily, the Hamiltonian 1-forms turned out to be part of a larger structure: a category equipped with a Lie bracket which is antisymmetric *up to isomorphism*. Such a thing is called a 'Lie 2-algebra' [6, 35]. So, we have categorified classical mechanics, replacing the Lie algebra of observables by a Lie 2-algebra!

But what do these new ideas mean for physics? We can illustrate them with the example of a classical bosonic string. The dynamics of such a string can be neatly described in terms of our setup if we take the extended configuration space M to be the cartesian product of the string worldsheet and the target space. In particular, the components of the energy-momentum tensor for the string are *objects* in the Lie 2-algebra of observables: that is, Hamiltonian 1-forms. These generate the expected dynamics of the string. However, a category has not only objects but also morphisms! A morphism *between* Hamiltonian 1-forms turns out to be a kind of 'gauge symmetry between observables': that is, a way of changing observables that does not affect the dynamics they generate.

This work, which we are still writing up [25], suggests that the multisymplectic approach to the classical mechanics of strings really amounts to a *categorification* of traditional mechanics of point particles. We hope that ultimately categorified classical mechanics will go beyond traditional string theory, but for now this example makes an excellent test case. Therefore we plan to continue this project by categorifying the theory of geometric quantization and applying it to the bosonic string.

This presents significant challenges. However, there are more clues to guide us. For example, we expect that just as geometric quantization gives rise to a Hilbert space of quantum states, the categorified version will give rise to a categorified version of a Hilbert space, called a '2-Hilbert space' [36]. This might seem like a wild guess were there not already evidence for it in the work of Freed [37]. The role of gerbes is also already familiar in string theory, and particularly well-understood in the case of the Wess–Zumino–Witten model [38]. So, I believe that categorifying geometric quantization is a goal within reach, which will shed new light on fundamental physics.

GUTs, Octonions, and Exceptional Lie Groups

John Huerta and I are studying the octonions and exceptional Lie groups and their role in fundamental physics: especially grand unified theories (GUTs) and superstring theory, but also non-mainstream theories such as those of Dixon [39] and Lisi [40]. Categorification is not involved in this aspect of the project, but categories are. To briefly recall:

- Grand unified theories (GUTs) are theories of physics that seek to unify all fundamental forces except gravity [41]. They involve enlarging the gauge group of the Standard Model, and therefore make heavy use of group representation theory. The simplest of these are the SU(5) theory and the SO(10) theory, both named after the gauge groups used in these theories. Both predict proton decay at a rate later found to be unacceptably high, but both — and especially the latter — do a nice job of fitting a single generation of fermions into a neat pattern. A less 'unified' but also interesting theory is the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ theory due to Pati and Salam, which treats lepton number on an equal footing with the three colors of quark, and has left-right symmetry. Many others have been considered as well.

- The octonions are a number system discovered by John Graves in 1843 shortly after his friend Hamilton discovered the quaternions [42]. They are the largest of the four 'normed division algebras', which are number systems satisfying the rule $|ab| = |a||b|$. The most familiar normed division algebras are the 1-dimensional real numbers ($\mathbb{R}$) and the 2-dimensional complex numbers ($\mathbb{C}$), but the quaternions ($\mathbb{H}$) are a 4-dimensional normed division algebra in which the commutative law fails ($ab \neq ba$), and the octonions ($\mathbb{O}$) are an 8-dimensional normed division algebra in which the associative law also fails ($a(bc) \neq (ab)c$).

The real significance of the octonions remains mysterious. However, their existence explains why superstring theory lives in 10 dimensions! A certain relation between spinors and vectors is required to write down a classical superstring theory; this relation can only happen when the space of directions perpendicular to the string worldsheet forms a normed division algebra [43, 44]. So, classical superstring theories exist only in spacetimes of dimensions

$$1+2=3,\ 2+2=4,\ 4+2=6,\ \text{and } 8+2=10.$$

Of these, it appears only the 10-dimensional one admits a consistent quantization.

The octonions also show up in other theories of physics. For example, Dixon [39] has proposed an extension of the Standard Model based on all four normed division algebras combined into a single algebra $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. While this theory has problematic features, it shows that the curious collection of forces and particles in the Standard Model follow certain patterns related to normed division algebras.

- The exceptional Lie groups were first discovered in the late 1800s when Killing and Cartan classified the so-called 'simple' Lie groups — the ones now used in truly unified GUTs. Almost all the simple Lie groups fit into nice families, but there were five exceptions, now called $\mathrm{F}_4, \mathrm{G}_2, \mathrm{E}_6, \mathrm{E}_7$ and E_8.

  They remain very puzzling, especially the largest of them, E_8, which has not yet been exhibited as the symmetry group of anything less complicated than *itself*. However, all five exceptional Lie groups can be constructed using the octonions. Furthermore, these groups show up in theories of physics including certain supergravity theories [45] and heterotic string theory [46]. Recently Lisi [40] proposed a theory based on E_8, which ignited a storm of controversy. While there are serious problems with his theory, it takes advantage of some intriguing patterns relating exceptional Lie groups to the forces and particles of the Standard Model.

In short, there is a tangled web of obscure clues hinting at some possible role for the octonions and exceptional Lie groups in fundamental physics. However, the clues seem to point in different directions. Furthermore, none of the physical theories mentioned has made predictions confirmed by experiment, and all suffer internal problems. So, the direction forward is not clear.

Luckily, there is no shortage of patterns that might shed some light on the situation. For example, John Huerta and I have already worked out how the SU(5), SO(10) and $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ GUTs fit together in a single story. The relevant facts are all in the physics literature, but not vividly spelled out.

Namely: SO(10) is the group of rotations of 10-dimensional space. The fermions in one generation of the Standard Model form the spinor representation of this rotation group (or actually its double cover). If we look at the subgroup of SO(10) preserving a complex inner product and volume form on 10-dimensional space, we get SU(5). If we look at the subgroup preserving a splitting of 10-dimensional space into 4+6 dimensions, we get $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$.

This is very suggestive. First of all, not only these GUTs *but also superstring theory* makes use of 10-dimensional space, its splitting into 4+6 dimensions, and the subgroup of SO(10) preserving a complex inner product and volume form [47]. In fact, this subgroup plays a key role in the definition of a 'Calabi–Yau manifold'.

Secondly, Dixon's theory based on $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ is also set in 10-dimensional space, and also makes use of a splitting of 10 dimensions into 4+6. Indeed, while superstring theory and Dixon's theory are drastically different, the octonions enter in a similar way — and in both, the splitting of 10 dimensions into 4+6 can be seen as arising from how the complex numbers sit inside the octonions.

We believe these facts are puzzling and insufficiently recognized. Technical details aside, they imply that *superficially different approaches to extending the Standard Model secretly rely on the same mathematical structures.* So, in our first paper on this subject, we plan to clarify this phenomenon — and, just as importantly, explain what we find in the simplest possible terms, so more physicists and mathematicians can ponder it [48].

In future work, we want to describe the octonions and exceptional Lie groups using the work of Cvitanovic [49]. This work, rooted in category theory, describes the identities satisfied by group representations using diagrams called spin networks, avoiding tedious index manipulations. We hope this technology can help us find a better explanation of how the groups F_4, E_6, E_7 and E_8 are built from the rotation groups in dimensions

$$1 + 8 = 9,\ 2 + 8 = 10,\ 4 + 8 = 12, \text{ and } 8 + 8 = 16$$

together with the spinor representations of these groups [42]. In each case the construction is fairly simple, but checking that it actually works currently requires some brute-force computations involving Fierz identities.

Cvitanovic's diagrammatic approach to Fierz identities may simplify the calculation and ultimately lead us to a more conceptual construction of these four exceptional Lie groups. The case of E_6 is interesting because it is built from SO(10) and its spinor representation — precisely the key ingredients in the SO(10) GUT! The case of E_8 is also interesting, since this construction, building it from SO(16) and its spinor representation, is the simplest known for this most mysterious of Lie groups [46].

Categorifying the Theory of Computation

This final project, involving my students Alex Hoffnung and Mike Stay, is *not* part of the current grant proposal, because I have been able to obtain funding for it from the NSF program on 'Quantum Information and Revolutionary Computing'. I mention it briefly here just to round out the overall picture of what my students are doing.

Hoffnung, Stay and I are investigating the relationship between quantum and classical computation. The goal is to more deeply understand how classical and quantum processes create, delete, and transfer information. A key ingredent in our approach is to exploit the analogy between physical processes and processes of computation — an analogy that can be made precise with the help of category theory.

As the first step of this project, Mike Stay and I spelled out this analogy in great detail [50]. Categories have already been succesfully applied to computer science in a long line of work that goes back to the work of Lambek [51, 52]. A category has objects and also morphisms going between objects. In applications to computer science, the objects often represent *data types*, while the morphisms represent *programs* taking data of a given type as input, and returning data of some other type as output. The data types turn out to be analogous to *types of particles*, while the programs are analogous to *Feynman diagrams* with a given collection of particles coming in and another collection going out. While the analogy is already strong for ordinary 'classical' computation, it is even stronger for quantum computation. Abramsky, Coecke [53] and others have recently taken advantage of this to develop a diagrammatic notation for quantum programs. In the other direction, I have argued that this analogy renders some of the famously puzzling features of quantum mechanics less mysterious [54].

The next step of this project is to categorify what has been done so far. The reason is that so far, in the above applications of category theory to computer science, two programs are considered equal if they are related by a sequence of steps of computation. So, a program that first computes 2+1 and then prints out 3 is counted as *equal* to one that simply prints out 3. This is clearly inadequate for describing the actual process of computation. It would be better to say there is a *morphism between* these programs, namely actual step of computation.

However, recall that in the applications of category theory to computer science that we have been discussing, objects correspond to data types and morphisms correspond to programs. So, when we start thinking about steps of computation as morphisms *between* programs, we are contemplating — like it or not — some sort of structure that has *morphisms between morphisms*.

Luckily, such structures have been intensively studied for decades: they are called 2-categories [55]. Indeed, categorification is like a crank one can turn over and over: if we take the concept of set and turn this crank once, we get the concept of category, but if we turn the crank a second time, we get the concept of 2-category!

So, it is actually very natural to take the existing applications of category theory to computer science, and replace the categories everywhere by 2-categories, to better understand the process of computation. This is what we are doing. Currently Stay is focusing classical computation, which requires work on 'cartesian closed 2-categories', while Hoffnung plans to tackle quantum computation, which requires 'symmetric monoidal closed 2-categories'. The classical case has already been studied a bit by Seely [56] and Hilken [57], but the quantum case seems to be wide open.

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