THREE GENERATIONS IN E_7

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ABSTRACT. Starting from the Standard Model Lie algebra $\mathfrak{g}_{\mathrm{SM}} = \mathfrak{u}(3) \oplus \mathfrak{u}(2) \oplus \mathbb{C}$ sitting inside the complex Lie algebra $\mathfrak{e}_7$, we show how to decompose $\mathfrak{e}_7$ into the direct sum of a Lie subalgebra containing $\mathfrak{g}_{\mathrm{SM}}$ and three 32-dimensional subspaces, each of which forms the same representation of $\mathfrak{g}_{\mathrm{SM}}$ as one generation of fermions and their antiparticles. The setting is due to Nasmith, and much of the mathematics is that underlying the E_7 generation unification of Kugo and Yanagida. New features include the derivation, in which as many results as possible rely only on the embedding $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$, and also the description of each 32-dimensional subspace as a copy of the exterior algebra $\Lambda\mathbb{C}^5$.

1. INTRODUCTION

The complex exceptional Lie algebra $\mathfrak{e}_7$ is large enough to contain the complexified Standard Model Lie algebra

$$\mathfrak{g}_{\mathrm{SM}} \;=\; \mathbb{C} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_3$$

with a great deal of room to spare, since $\dim \mathfrak{e}_7 = 133$. The question is whether this room can be filled in an interesting way. Nasmith [10, 11] argued that it can, showing that $\mathfrak{g}_{\mathrm{SM}}$ acts, via the Lie bracket of $\mathfrak{e}_7$, on three 32-dimensional subspaces of $\mathfrak{e}_7$ in a manner that precisely matches its representation on three generations of Standard Model fermions and their antiparticles—including right-handed neutrinos and their antiparticles.

What follows is a somewhat new account of this story. Starting from a suitable embedding of $\mathfrak{g}_{\mathrm{SM}}$ in $\mathfrak{e}_7$, we construct a Lie subalgebra

$$\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_6 \oplus \mathbb{C}^2 \subset \mathfrak{e}_7$$

and a vector space isomorphism

$$\mathfrak{e}_7 \;\cong\; (\mathfrak{sl}_6 \oplus \mathbb{C}^2) \oplus V$$

where the space V has dimension 3×32. The subalgebra $\mathfrak{g}_{\mathrm{SM}}$ acts on V, via the $\mathfrak{e}_7$ Lie bracket, precisely as it does on three generations of Standard Model fermions and their antiparticles.

Like Nasmith, we are not proposing a theory of physics. We are only observing a fascinating mathematical pattern that might (or might not) be of some use in physics.

We tell our story in the language of complex Lie algebras rather than the language of compact Lie groups because it is more convenient for our calculations. However, everything we say can be translated into a story about the compact real form of E_7 and the Standard Model gauge group

$$G_{\mathrm{SM}} \;=\; \left\{ x \in \mathrm{SU}(5) : x = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \right\} \tag{1}$$

$$\cong\; (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))/\mathbb{Z}_6.$$

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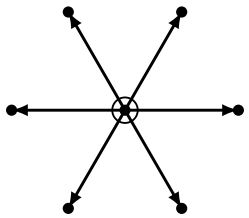
Note that this equation presents G_{SM} as a subgroup of $\mathrm{SU}(5)$. It is well known [2] that if we restrict the natural representation of $\mathrm{SU}(5)$ on the exterior algebra $\Lambda\mathbb{C}^5$ to this subgroup, we get exactly the usual 32-dimensional representation of G_{SM} on one generation of Standard Model fermions and their antiparticles. This in turn gives a representation of $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5$ on $\Lambda\mathbb{C}^5$. We call this the **Standard Model representation** of $\mathfrak{g}_{\mathrm{SM}}$.

While elegant, this construction does not address the *three* generations of the Standard Model. This is where $\mathfrak{e}_7$ comes in. It is worth sketching how this works before giving full details in the body of the paper.

We first fix a suitable inclusion of $\mathfrak{g}_{\mathrm{SM}}$ in $\mathfrak{e}_7$. There is then a unique $\mathfrak{sl}_3$ subalgebra of $\mathfrak{e}_7$ that commutes with everything in $\mathfrak{g}_{\mathrm{SM}}$. This will act as symmetries relating the three generations, so we call it $\mathfrak{sl}_3^{\mathrm{gen}}$.

We next choose a Cartan subalgebra of $\mathfrak{e}_7$ compatible with that of $\mathfrak{g}_{\mathrm{SM}}$ and $\mathfrak{sl}_3^{\mathrm{gen}}$. This gives a way to split the eight dimensions of $\mathfrak{sl}_3^{\mathrm{gen}}$ into two Cartan directions and six root spaces. In the Cartan of $\mathfrak{sl}_3^{\mathrm{gen}}$, the six roots lie on three lines:



Each line corresponds to a copy of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3^{\mathrm{gen}}$. The centralizer in $\mathfrak{e}_7$ of each such $\mathfrak{sl}_2$ is isomorphic to $\mathfrak{so}_{12}$. We get three different $\mathfrak{sl}_2 \oplus \mathfrak{so}_{12}$ subalgebras of $\mathfrak{e}_7$ this way. Their intersection is a copy of $\mathfrak{sl}_6 \oplus \mathbb{C}^2$ that contains $\mathfrak{g}_{\mathrm{SM}}$.

Now to the main point: how to get three generations. Each of the three $\mathfrak{sl}_2 \oplus \mathfrak{so}_{12}$ subalgebras is the direct sum of their common intersection $\mathfrak{sl}_6 \oplus \mathbb{C}^2$ and a 32-dimensional subspace. The subalgebra $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$ acts on each of these 32-dimensional subspaces via the $\mathfrak{e}_7$ Lie bracket. In each case this action is equivalent to the Standard Model representation of $\mathfrak{g}_{\mathrm{SM}}$ on $\Lambda\mathbb{C}^5$. If we call these subspaces V_1, V_2 and V_3, we have

$$\mathfrak{e}_7 \;\cong\; (\mathfrak{sl}_6 \oplus \mathbb{C}^2) \,\oplus\, V_1 \,\oplus\, V_2 \,\oplus\, V_3.$$

The Lie algebra $\mathfrak{g}_{\mathrm{SM}}$ is contained in $\mathfrak{sl}_6 \oplus \mathbb{C}^2$. We thus get an inclusion

$$\mathfrak{g}_{\mathrm{SM}} \oplus\, V_1 \,\oplus\, V_2 \,\oplus\, V_3 \;\subset\; \mathfrak{e}_7$$

where the $\mathfrak{e}_7$ Lie bracket gives three copies of the Standard Model representation of $\mathfrak{g}_{\mathrm{SM}}$.

All this beauty comes at a price. The three generations only account for $3{\cdot}32 = 96$ dimensions of $\mathfrak{e}_7$. The rest are accounted for by $\mathfrak{sl}_6 \oplus \mathbb{C}^2$, which has dimension $35 + 2 = 37 = 133 - 96$. We can split $\mathfrak{sl}_6 \oplus \mathbb{C}^2$ into a 7-dimensional Cartan subalgebra and 30 root spaces. This Lie algebra contains $\mathfrak{g}_{\mathrm{SM}}$, which has a 4-dimensional Cartan and 8 root spaces. The remainder consists of 3 Cartan dimensions and Nasmith's "22 additional root spaces" [10, §6]. The latter span a 22-dimensional representation of $\mathfrak{g}_{\mathrm{SM}}$ as in Table 1.

There is a unique copy of $\mathfrak{sl}_5$ in $\mathfrak{e}_7$ between $\mathfrak{g}_{\mathrm{SM}}$ and the aforementioned $\mathfrak{sl}_6$. Indeed there is a chain

$$\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5 \subset \mathfrak{sl}_6 \subset \mathfrak{sl}_6 \oplus \mathbb{C}^2 \subset \mathfrak{e}_7$$

with compatible root systems for all these Lie algebras. The first row in Table 1 corresponds to the 3 Cartan dimensions in $\mathfrak{e}_7$ not contained in $\mathfrak{g}_{\mathrm{SM}}$. The second corresponds to the 12 root spaces of $\mathfrak{e}_7$ lying in $\mathfrak{sl}_5$ but not in $\mathfrak{g}_{\mathrm{SM}}$. In an $\mathrm{SU}(5)$ grand unified theory, these correspond

$\mathfrak{g}_{\mathrm{SM}}$ content	SU(5) interpretation	dim
$(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1})$	SU(5) singlets	3
$(\mathbf{3},\mathbf{2}) \oplus (\bar{\mathbf{3}},\mathbf{2})$	X, Y leptoquark gauge bosons of SU(5)	12
$(\mathbf{3},\mathbf{1}) \oplus (\bar{\mathbf{3}},\mathbf{1})$	color-triplet part of $\mathbf{5} \oplus \bar{\mathbf{5}}$	6
$(\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2})$	weak-doublet part of $\mathbf{5} \oplus \bar{\mathbf{5}}$	4
		25

TABLE 1. Summands of $\mathfrak{e}_7$ not in $\mathfrak{g}_{\mathrm{SM}}$ or the three generations.

to the extra gauge bosons not present in the Standard Model, the so-called X and Y bosons, which serve to convert quarks into leptons and vice versa. The last two rows together form the $\mathbf{5} \oplus \bar{\mathbf{5}}$ of $\mathfrak{sl}_5$, corresponding to the 10 root spaces lying in $\mathfrak{sl}_6$ but not in $\mathfrak{sl}_5$. In an SU(5) grand unified theory, the $\mathbf{5}$ is the multiplet that contains the Standard Model Higgs boson, and this multiplet has antiparticles forming the $\bar{\mathbf{5}}$.

Relation to earlier work. Our work arose as an attempt to understand the work of Nasmith [10, 11]. But it also has other antecedents. Kugo and Yanagida [8] studied generation unification using supergravity on the manifold $\mathrm{E}_7/(\mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{U}(1))$ and identified three families of quarks and leptons with pseudo Nambu–Goldstone fermions, arguing that the triplication of generations points to an underlying E_7 symmetry. Sato and Yanagida [13] refined this to an $\mathrm{E}_7/(\mathrm{SU}(5) \times \mathrm{U}(1)^3)$ model, noting in particular that the model supplies three right-handed neutrinos.

Our work also has strong connections to Lisi's study of $\mathfrak{e}_7$ using Clifford algebras, the nonassociative algebra $\mathbb{H} \otimes \mathbb{O}$, and triality [9]. For example, our three decompositions of $\mathfrak{e}_7$ as a direct sum of a subalgebra $\mathfrak{sl}_2 \oplus \mathfrak{so}_{12}$ and a 64-dimensional representation of this subalgebra echo his work, which uses one such decomposition of a real form of $\mathfrak{e}_7$. However, he attempts to include spin degrees of freedom, which doubles the size of the Standard Model representation from $\Lambda\mathbb{C}^5$ to

$$(\Lambda^{\mathrm{even}}\mathbb{C}^5 \otimes \mathbf{2}) \ \oplus \ (\Lambda^{\mathrm{odd}}\mathbb{C}^5 \otimes \overline{\mathbf{2}}),$$

where $\mathbf{2}$ is the defining representation of $\mathfrak{sl}_2$ and $\overline{\mathbf{2}}$ is its conjugate. For this reason he cannot fit three linearly independent generations of fermions and their antiparticles into $\mathfrak{e}_7$. Like Nasmith, we simply *ignore* how the particles transform under Lorentz transformations: tackling this question would require a larger algebraic structure and new ideas. Distler and Garibaldi [5] have put strong limitations on including the complexified Lorentz Lie algebra $\mathfrak{sl}_2$ and chiral spin-1/2 fermions in any form of $\mathfrak{e}_8$.

Following Nasmith, our work relies heavily on the chain of Lie algebras shown in Table 2, which arises from $\mathfrak{e}_7$ by a systematic process of 'removing roots':

$$\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \subset \mathfrak{sl}_5 \subset \mathfrak{so}_{10} \subset \mathfrak{e}_6 \subset \mathfrak{e}_7.$$

This chain also plays a key role in our work with Bokor and Boyle [3]. There we exploited the fact that each step down the chain gives a hermitian Jordan triple system. (This fact does not hold for $\mathfrak{e}_7 \subset \mathfrak{e}_8$.) We showed that the chain going down from $\mathfrak{e}_6$ naturally accounts for the representation of $\mathfrak{g}_{\mathrm{SM}}$ on *one* generation of Standard Model fermions and their antiparticles. We raised the possibility that the chain going down from $\mathfrak{e}_7$ could account for *three*, but did not settle the question. The work here may shed some light on that.

Plan. Section 2 defines a 'good' embedding of $\mathfrak{g}_{\mathrm{SM}}$ in $\mathfrak{e}_7$ and shows how to construct one. From then on we choose a particular good $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$ and work with that. Section 3 shows that the centralizer of this $\mathfrak{g}_{\mathrm{SM}}$ in $\mathfrak{e}_7$ contains only one subalgebra isomorphic to $\mathfrak{sl}_3$, which we call $\mathfrak{sl}_3^{\mathrm{gen}}$. Section 4 studies three copies of $\mathfrak{sl}_2$ in $\mathfrak{sl}_3^{\mathrm{gen}}$, which we call $\mathfrak{sl}_2(\beta_k)$ for $k = 1, 2, 3$. Section 5 gives an analysis of the roots of $\mathfrak{e}_7$ crucial for all the proofs that follow. Section 6 shows that the centralizer of each subalgebra $\mathfrak{sl}_2(\beta_k)$ is a subalgebra isomorphic to $\mathfrak{so}_{12}$, which we call $\mathfrak{so}_{12}(\beta_k)$. It also studies three subalgebras $\mathfrak{m}_k = \mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k) \subset \mathfrak{e}_7$. Section 7 shows that the centralizer of $\mathfrak{sl}_3^{\mathrm{gen}}$ in $\mathfrak{e}_7$ is a copy of $\mathfrak{sl}_6$ lying in the intersection of all three of these subalgebras $\mathfrak{m}_k$. We call this centralizer $\mathfrak{sl}_6^{\mathrm{SM}}$. Section 8 show that there is a unique copy of $\mathfrak{sl}_5$ containing $\mathfrak{g}_{\mathrm{SM}}$ and contained in $\mathfrak{sl}_6^{\mathrm{SM}}$. We call this $\mathfrak{sl}_5^{\mathrm{SM}}$.

Based on all this work, Section 9 constructs three 30-dimensional subspaces of $\mathfrak{e}_7$ that transform under $\mathfrak{g}_{\mathrm{SM}}$ as one generation of fermions and antifermions *not including* right-handed neutrinos and their antiparticles. Section 10 shows how to construct larger 32-dimensional subspaces V_1, V_2, V_3 that transform under $\mathfrak{g}_{\mathrm{SM}}$ as one generation of fermions including right-handed neutrinos and their antiparticles.

2. The Standard Model Lie algebra in $\mathfrak{e}_7$

All our constructions rely on choosing a Lie subalgebra of $\mathfrak{e}_7$ that is isomorphic to $\mathfrak{g}_{\mathrm{SM}}$. We need to choose one with good properties. If one such subalgebra counts as 'good' and it is mapped to another by any automorphism of $\mathfrak{e}_7$, the other also counts as 'good'. However, there are probably many subalgebras of $\mathfrak{e}_7$ that are isomorphic to $\mathfrak{g}_{\mathrm{SM}}$ but not mapped to each other by automorphisms of $\mathfrak{e}_7$. For example, it is known that there is a countable infinity of subalgebras of $\mathfrak{so}_{10}$ isomorphic to $\mathfrak{g}_{\mathrm{SM}}$, not related to each other by automorphisms of $\mathfrak{so}_{10}$ [1]. Thus, we should be careful about our choice.

Here we describe two methods of choosing a good $\mathfrak{g}_{\mathrm{SM}}$ subalgebra of $\mathfrak{e}_7$: one that is quick to describe, and another that is more useful. Both require the concept of a 'regular' Lie subalgebra [6, Chap. II]. A subalgebra $\mathfrak{g}'$ of a semisimple Lie algebra $\mathfrak{g}$ with chosen Cartan $\mathfrak{h}$ is **regular** if $[\mathfrak{h}, \mathfrak{g}'] \subseteq \mathfrak{g}'$. Regular subalgebras are a powerful tool because they are easy to work with using root systems.

For the quick method, start with the subgroup $G_{\mathrm{SM}} \subset \mathrm{SU}(5)$ given in Equation (1). Taking Lie algebras and complexifying, we obtain the Lie subalgebra $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5$, and this is regular with respect to the standard Cartan of $\mathfrak{sl}_5$. Then choose any regular subalgebra of $\mathfrak{e}_7$ isomorphic to $\mathfrak{sl}_5$. All such regular subalgebras are related by automorphisms of $\mathfrak{e}_7$: this follows from the fact that all A_4 root subsystems of E_7 lie in a single orbit of the E_7 Weyl group [12, Table 10.2]. Composing these two embeddings

$$\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5 \subset \mathfrak{e}_7,$$

$\mathfrak{g}_{\mathrm{SM}}$ becomes a regular subalgebra of $\mathfrak{e}_7$. Any subalgebra of $\mathfrak{e}_7$ obtained by this procedure counts as 'good' for our purposes.

In the second method, we start with $\mathfrak{e}_7$ and construct a sequence of smaller and smaller regular subalgebras by successively removing dots from its Dynkin diagram until we are led to $\mathfrak{g}_{\mathrm{SM}}$. This is an instance of a general procedure which we call 'removing a root'.

Suppose $\mathfrak{g}$ is a semisimple Lie algebra with Cartan $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be its root system. For each $r \in \Phi$, define the root space $\mathfrak{g}_r$ by

$$\mathfrak{g}_r = \{x \in \mathfrak{g} \ : \ [a, x] = r(a)x \text{ for all } a \in \mathfrak{h}\}.$$

We then have

$$\mathfrak{g} = \mathfrak{h} \ \oplus \bigoplus_{r \in \Phi} \mathfrak{g}_r.$$

Choose a set of simple roots $\alpha_1, \ldots, \alpha_n \in \Phi$. We now trim down $\mathfrak{g}$ to a smaller semisimple Lie algebra by removing the last of these simple roots, using the following procedure.

Let $\Phi' \subset \Phi$ be the intersection of Φ and the set of integer linear combinations of $\alpha_1, \ldots, \alpha_{n-1}$. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the subspace on which the root $\alpha_n \colon \mathfrak{h} \to \mathbb{C}$ vanishes. Then

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{r \in \Phi'} \mathfrak{g}_r$$

is a semisimple Lie algebra in its own right. It is a regular subalgebra of $\mathfrak{g}$, with rank one less than that of $\mathfrak{g}$. We take its Cartan to be $\mathfrak{h}'$, and its root system is then Φ'. This allows us to repeat the procedure as long as Φ' is nonempty.

It is also important to note that $\mathfrak{g}'$ is contained in the larger Lie subalgebra

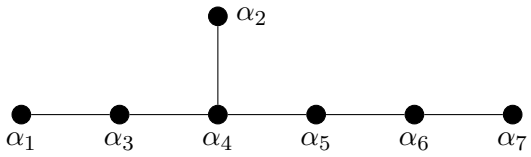
$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{r \in \Phi'} \mathfrak{g}_r.$$

This is also a regular subalgebra of $\mathfrak{g}$, but it is not semisimple, merely **reductive**: that is isomorphic to the direct sum of a semisimple Lie algebra and an abelian one. It is isomorphic to $\mathfrak{g}' \oplus \mathbb{C}$, since $\mathfrak{h}$ is one dimension bigger than $\mathfrak{h}'$. This subalgebra $\mathfrak{l}$ is called a **maximal Levi subalgebra** of $\mathfrak{g}$. We shall see that $\mathfrak{g}_{\mathrm{SM}} \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \oplus \mathbb{C}$ arises as a maximal Levi subalgebra of $\mathfrak{sl}_5$.

We start the procedure with $\mathfrak{e}_7$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{e}_7$ and let $\Phi \subset \mathfrak{h}^*$ be the resulting root system, so

$$\mathfrak{e}_7 \;=\; \mathfrak{h} \oplus \bigoplus_{r \in \Phi} (\mathfrak{e}_7)_r,$$

where each root space $(\mathfrak{e}_7)_r$ has dimension one. Choose a set of simple roots $\alpha_1, \ldots, \alpha_7 \in \Phi$ such that any pair lies at an angle of 120° when connected by an edge in this diagram, and 90° otherwise:



We then carry out the procedure just described, repeatedly, removing roots one at a time starting from α_7 and working our way down to α_4. We obtain a series of regular semisimple Lie subalgebras of $\mathfrak{e}_7$. These are sometimes called the $\boldsymbol{\mathfrak{e}_n}$ **series**, or more precisely the portion of the $\mathfrak{e}_n$ series going from $n = 7$ down to $n = 3$. We show them in Table 2, along with the dimension of each Lie algebra, its rank, and its number of roots.

At the last stage, the maximal Levi subalgebra is $\mathfrak{e}_3 \oplus \mathbb{C} \cong \mathfrak{g}_{\mathrm{SM}}$. By construction this is a regular subalgebra of $\mathfrak{e}_7$. Any subalgebra of $\mathfrak{e}_7$ that is carried to this one under some automorphism is defined to be **good**, and all the constructions of this paper apply to it.

In what follows we often work with a fixed Cartan $\mathfrak{h} \subset \mathfrak{e}_7$, which gives a Cartan for all the smaller Lie algebras in Table 2, but we state as many results as we can in a Cartan-independent way, depending only on a chosen good $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$. We identify $\mathfrak{h}$ with its dual using the Killing form, normalized so that $\|r\|^2 = 2$ for every root: this is possible since the E_7 root system is simply laced. We let V be the real span of Φ, which is a 7-dimensional real inner product space contained in $\mathfrak{h}$. We have

$$\langle r, s \rangle \in \{-2, -1, 0, 1, 2\} \qquad \text{for all } r, s \in \Phi \tag{2}$$

with the value 2 only when $s = r$, and -2 only when $s = -r$.

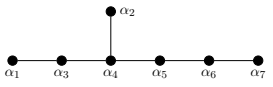
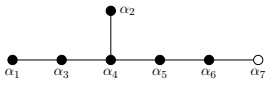
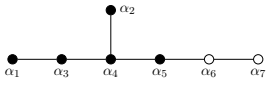
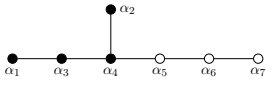
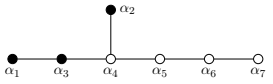
Dynkin diagram	semisimple Lie algebra $\mathfrak{g}$	$\dim \mathfrak{g} = \dim \mathfrak{h} + \lvert\Phi\rvert$
	$\mathfrak{e}_7$	$133 = 7 + 126$
	$\mathfrak{e}_6$	$78 = 6 + 72$
	$\mathfrak{e}_5 = \mathfrak{so}_{10}$	$45 = 5 + 40$
	$\mathfrak{e}_4 = \mathfrak{sl}_5$	$24 = 4 + 20$
	$\mathfrak{e}_3 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_2$	$11 = 3 + 8$

TABLE 2. The $\mathfrak{e}_n$ series from $n = 7$ down to $n = 3$.

3. The generation $\mathfrak{sl}_3$

Proposition 1. *The centralizer of $\mathfrak{g}_{\mathrm{SM}}$ in $\mathfrak{e}_7$ is isomorphic to $\mathfrak{sl}_3 \oplus \mathbb{C}^2$, and it contains exactly one subalgebra isomorphic to $\mathfrak{sl}_3$.*

Proof. This description of the centralizer is in [10, §6]. We prove it here using the general theory of regular subalgebras [6], since the ideas are also used in later centralizer computations.

Suppose $\mathfrak{g}$ is a simply laced semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ. Let $\mathfrak{g}' \subseteq \mathfrak{g}$ be a regular reductive subalgebra, so

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{r \in \Phi'} \mathfrak{g}_r$$

for some subspace $\mathfrak{h}' \subseteq \mathfrak{h}$ and some subsystem $\Phi' \subseteq \Phi$ with $\mathrm{span}(\Phi') \subseteq \mathfrak{h}'$. Note that $\mathfrak{h}'$ may be strictly larger than $\mathrm{span}(\Phi')$, so that $\mathfrak{g}'$ may have a nontrivial center, as is the case for $\mathfrak{g}_{\mathrm{SM}}$.

An element of $\mathfrak{g}$ centralizes $\mathfrak{g}'$ if and only if it commutes with $\mathfrak{h}'$ and with every root space $\mathfrak{g}_r$ for $r \in \Phi'$. One can show the centralizer $C_{\mathfrak{g}}(\mathfrak{g}')$ is spanned by

- the subspace of $\mathfrak{h}$ orthogonal to every element of Φ', and
- the root spaces $\mathfrak{g}_r$ for those roots $r \in \Phi$ orthogonal to every element of Φ'.

(If $\mathfrak{g}$ were not simply laced, we would need to replace the second item by a stronger condition.) We can summarize this result as

$$C_{\mathfrak{g}}(\mathfrak{g}') = \Phi'^{\perp} \oplus \bigoplus_{r \in \Phi \cap \Phi'^{\perp}} \mathfrak{g}_r, \tag{3}$$

where $\Phi'^{\perp}$ denotes the subspace of $\mathfrak{h}$ orthogonal to every element of Φ'.

In the case at hand, $\mathfrak{g}' = \mathfrak{g}_{\mathrm{SM}}$ is a regular subalgebra of $\mathfrak{g} = \mathfrak{e}_7$, with root system $\Phi' \subset \Phi$, and

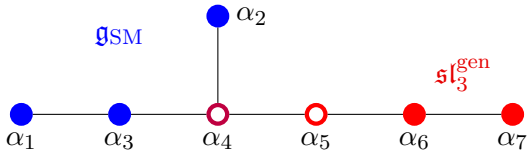
$$C_{\mathfrak{e}_7}(\mathfrak{g}_{\mathrm{SM}}) = \Phi'^{\perp} \oplus \bigoplus_{r \in \Phi \cap \Phi'^{\perp}} (\mathfrak{e}_7)_r.$$

Here $\Phi \cap \Phi'^{\perp}$ is a root system of type A_2. Thus, the second summand above and the 2-dimensional subspace of the first spanned by these A_2 roots together form a copy of $\mathfrak{sl}_3$. Since $\mathfrak{h}$ has dimension 7 while the span of Φ' has dimension 3, $\Phi'^{\perp}$ has dimension 4; removing the two

dimensions spanned by the A_2 roots leaves a 2-dimensional abelian Lie algebra that commutes with this $\mathfrak{sl}_3$. Thus $C_{\mathfrak{e}_7}(\mathfrak{g}_{\rm SM}) \cong \mathfrak{sl}_3 \oplus \mathbb{C}^2$ as desired.

To show that the $\mathfrak{sl}_3$ subalgebra is unique, let $\mathfrak{a} \subseteq \mathfrak{sl}_3 \oplus \mathbb{C}^2$ be any subalgebra isomorphic to $\mathfrak{sl}_3$, and let p denote projection onto the abelian summand $\mathbb{C}^2$. Then $p(\mathfrak{a})$ is an abelian quotient of $\mathfrak{a}$; since $\mathfrak{sl}_3$ is simple, $p(\mathfrak{a}) = 0$. Hence $\mathfrak{a} \subseteq \mathfrak{sl}_3$, and equality follows by comparing dimensions. □

We call the unique copy of $\mathfrak{sl}_3$ in the centralizer of $\mathfrak{g}_{\rm SM}$ the **generation $\mathfrak{sl}_3$**, and denote it as $\mathfrak{sl}_3^{\rm gen}$. The proof of the proposition leads to this picture:



Each root α_i corresponds to a copy of $\mathfrak{sl}_2$ in $\mathfrak{e}_7$ with basis elements e_i, f_i, h_i obeying the usual relations

$$[e_i, f_i] = h_i, \quad [h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i.$$

The blue dots correspond to the semisimple part of $\mathfrak{g}_{\rm SM}$: that is, e_i, f_i and h_i for $i = 1, 2, 3$ generate the subalgebra $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2 \subset \mathfrak{g}_{\rm SM}$. The red dots correspond to the semisimple part of the centralizer of $\mathfrak{g}_{\rm SM}$, which is the generation $\mathfrak{sl}_3$: e_i, f_i, and h_i for $i = 6, 7$ generate $\mathfrak{sl}_3^{\rm gen}$. The hollow purple and red dots correspond to the center of $C_{\mathfrak{e}_7}(\mathfrak{g}_{\rm SM})$: h_4 and h_5 span this center, which is a copy of $\mathbb{C}^2$. The hollow purple dot also corresponds to the center of $\mathfrak{g}_{\rm SM}$, which is spanned by h_4. The overlap is no coincidence: by definition, the center of $\mathfrak{g}_{\rm SM}$ is the intersection of $\mathfrak{g}_{\rm SM}$ and its centralizer. In physics, the center of $\mathfrak{g}_{\rm SM}$ gives rise to hypercharge symmetry.

4. Three copies of $\mathfrak{sl}_2$ in the generation $\mathfrak{sl}_3$

Being a regular subalgebra of $\mathfrak{e}_7$, $\mathfrak{sl}_3^{\rm gen}$ has a root system $A \subset \Phi$ of type A_2, spanning a plane P in the real vector space V spanned by the roots of $\mathfrak{e}_7$. We call P the **generation plane**. It is convenient to name the weights of the defining representation of $\mathfrak{sl}_3^{\rm gen}$: these are $w_1, w_2, w_3 \in P$ with

$$w_1 + w_2 + w_3 = 0, \qquad |w_i|^2 = \tfrac{2}{3}, \qquad \langle w_i, w_j \rangle = -\tfrac{1}{3} \ \ (i \neq j), \tag{4}$$

and then

$$A \ = \ \{\, w_i - w_j \ : \ i \neq j \,\}, \qquad |A| = 6.$$

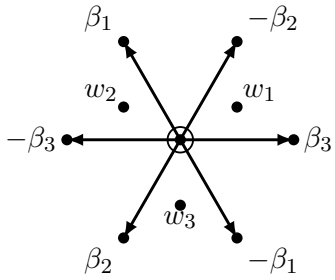
The six elements of A lie on three lines through the origin. For $k \in \{1, 2, 3\}$ set

$$\beta_k \ := \ w_i - w_j, \qquad i, j, k \text{ a cyclic permutation of } 1, 2, 3, \tag{5}$$

so that $A = \{\pm\beta_1, \pm\beta_2, \pm\beta_3\}$. From (4),

$$\langle w_k, \beta_k \rangle = 0, \qquad \langle w_i, \beta_k \rangle = \pm 1 \ \ (i \neq k). \tag{6}$$

Thus each line of A singles out one of the three weights w_k: the one orthogonal to it.



Each line gives a copy of $\mathfrak{sl}_2$ in the generation $\mathfrak{sl}_3$:

$$\mathfrak{sl}_2(\beta_k) \;=\; \mathbb{C}e_{\beta_k} \oplus \mathbb{C}h_{\beta_k} \oplus \mathbb{C}f_{\beta_k} \;\subset\; \mathfrak{sl}_3^{\mathrm{gen}},$$

We call these the three **generation $\mathfrak{sl}_2$'s**. As we shall see, the generation $\mathfrak{sl}_3$ generates transformations that mix up fermions of all three generations. The kth generation $\mathfrak{sl}_2$ generates transformations that leave fermions in the kth generation fixed, while mixing up fermions in the other two generations.

5. How the roots of $\mathfrak{e}_7$ see the plane P

Everything to come rests on the following trichotomy, which says that a root of $\mathfrak{e}_7$ can project down to the generation plane in only three qualitatively different ways. This is a reformulation of Nasmith's Lemma 2.1 [10], which in turn goes back to ideas in [4].

Recall that Φ is the set of roots of $\mathfrak{e}_7$, while $A \subset \Phi$ is the set of roots of $\mathfrak{g}_{\mathrm{SM}}$. Similarly V is the real span of Φ, while $P \subset V$, the generation plane, is the real span of A. Let $\pi\colon V \to P$ denote orthogonal projection.

Lemma 2. *For every $r \in \Phi$,*

$$\pi(r) \;\in\; \{0\} \;\sqcup\; \{\pm w_1, \pm w_2, \pm w_3\} \;\sqcup\; A,$$

and $\pi(r) \in A$ if and only if $r \in A$.

Proof. For $\delta \in A$ we have $\langle \pi(r), \delta\rangle = \langle r, \delta\rangle \in \mathbb{Z}$ by (2), since $\delta \in P$. As A is simply laced with $\delta^\vee = \delta$, this says precisely that $\pi(r)$ lies in the weight lattice Λ of A.

Now Λ is the hexagonal lattice whose nonzero vectors have squared lengths $\frac{2}{3}$, 2, $\frac{8}{3}, \ldots$; the six vectors of squared length $\frac{2}{3}$ are $\pm w_1, \pm w_2, \pm w_3$, and the six of squared length 2 are the roots A. Since π is an orthogonal projection, $|\pi(r)|^2 \le |r|^2 = 2$, so only 0, $\pm w_i$ and A can occur.

Finally, $|\pi(r)|^2 = 2 = |r|^2$ forces $r = \pi(r) \in P$. But $\Phi \cap P$ is a rank-two subsystem of Φ containing A, and the only rank-two simply laced system containing A_2 is A_2 itself; hence $r \in A$. □

Now partition the roots outside A according to their projection to the generation plane:

$$\Phi_0 := \{\, r \in \Phi : \pi(r) = 0 \,\}, \qquad \Phi_k := \{\, r \in \Phi : \pi(r) = \pm w_k \,\} \quad (k = 1, 2, 3). \tag{7}$$

By Lemma 2 this is a genuine partition, so

$$\Phi \;=\; A \;\sqcup\; \Phi_0 \;\sqcup\; \Phi_1 \;\sqcup\; \Phi_2 \;\sqcup\; \Phi_3. \tag{8}$$

Lemma 3. $|\Phi_0| = 30$, *and Φ_0 is a root subsystem of type* A_5. *Moreover* $|\Phi_1| = |\Phi_2| = |\Phi_3| = 30$.

Proof. The set Φ_0 consists of the $\mathfrak{e}_7$ roots orthogonal to the plane P. This set spans $P^\perp \cap V$, which is 5-dimensional. To determine its cardinality we can start by counting the roots of E_7 orthogonal to any given root—it doesn't matter which one, since the Weyl group $W(\mathrm{E}_7)$ acts transitively on roots. The answer is 60. Since for each $k = 1, 2, 3$ we have

$$\{\, r \in \Phi : r \perp \beta_k \,\} \;=\; \Phi_0 \sqcup \Phi_k,$$

it follows that $|\Phi_0| + |\Phi_k| = 60$. Summing over k gives

$$3\,|\Phi_0| \;+\; \big(|\Phi_1| + |\Phi_2| + |\Phi_3|\big) \;=\; 180.$$

On the other hand the four set Φ_i partition $\Phi - A$ by Equation (8), so

$$|\Phi_0| + |\Phi_1| + |\Phi_2| + |\Phi_3| \;=\; |\Phi| - |A| \;=\; 126 - 6 \;=\; 120.$$

Subtracting the second identity from the first yields $2\,|\Phi_0| = 60$, hence $|\Phi_0| = 30$. Since $|\Phi_0| + |\Phi_k| = 60$, we get

$$|\Phi_1| = |\Phi_2| = |\Phi_3| \;=\; 30.$$

Finally Φ_0 is a root system in its own right, having rank 5 and 30 roots. Among rank-five simply laced systems only A_5 has 30 roots (D_5 has 40, and the largest reducible option $\mathrm{D}_4 \times \mathrm{A}_1$ has 26), so Φ_0 is of type A_5. □

We study the Lie subalgebra $\mathfrak{sl}_6 \subset \mathfrak{e}_6$ with root system Φ_0 in Section 7.

6. Three copies of $\mathfrak{so}_{12}$ in $\mathfrak{e}_7$

We now show that the three generation $\mathfrak{sl}_2$'s are centralized by three different copies of $\mathfrak{so}_{12}$.

Lemma 4. *For each k, $\Phi_0 \sqcup \Phi_k$ is a root subsystem of type D_6.*

Proof. Since $\beta_k \in P$, we have $r \perp \beta_k$ if and only if $\pi(r) \perp \beta_k$, where $\pi\colon V \to P$ is the orthogonal projection onto the generation plane. Run through the possibilities of Lemma 2: 0 is orthogonal to β_k; $\pm w_k$ is orthogonal to β_k by Equation (6); $\pm w_i$ for $i \neq k$ is not, again by Equation (6); and no root of A is orthogonal to β_k, because in a system of type A_2 any two roots have inner product ± 2 or ± 1. Hence the orthogonal roots are exactly $\Phi_0 \sqcup \Phi_k$, of cardinality $30 + 30 = 60$ by Lemma 3. The set is closed and has rank 6; among rank-6 simply laced systems, 60 roots identifies D_6. (In particular, A_6 has 42 and E_6 has 72.) □

Write $\mathfrak{so}_{12}(\beta_k)$ for the corresponding regular subalgebra, with Cartan subalgebra $\mathfrak{h} \cap \beta_k^\perp$.

Proposition 5. *For each $k \in \{1, 2, 3\}$ the subalgebras $\mathfrak{sl}_2(\beta_k)$ and $\mathfrak{so}_{12}(\beta_k)$ are centralizers of each other in $\mathfrak{e}_7$. Their direct sum is a maximal-rank subalgebra $\mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k)$ of $\mathfrak{e}_7$, and*

$$\mathfrak{e}_7 \;=\; \mathfrak{sl}_2(\beta_k) \;\oplus\; \mathfrak{so}_{12}(\beta_k) \;\oplus\; (\mathbf{2} \otimes \mathbf{32}) \tag{9}$$

where $\mathbf{2} \otimes \mathbf{32}$ is the tensor product of the defining representation of $\mathfrak{sl}_2(\beta_k)$ with a chiral spinor representation of $\mathfrak{so}_{12}(\beta_k)$.

Proof. By Lemma 4 the roots orthogonal to β_k are $\Phi_0 \sqcup \Phi_k$, of type D_6; let $\mathfrak{so}_{12}(\beta_k)$ be the corresponding regular subalgebra, with Cartan subalgebra $\mathfrak{h} \cap \beta_k^\perp$. For $r \perp \beta_k$ one has $|r \pm \beta_k|^2 = |r|^2 + |\beta_k|^2 = 4$, so $r \pm \beta_k \notin \Phi$ and thus $[\mathfrak{g}_{\pm\beta_k}, \mathfrak{g}_r] = 0$; the Cartan parts commute as well, so $\mathfrak{sl}_2(\beta_k)$ and $\mathfrak{so}_{12}(\beta_k)$ commute with one another. Since $\operatorname{rank}\mathfrak{sl}_2 + \operatorname{rank}\mathfrak{so}_{12} = 1 + 6 = 7 = \operatorname{rank}\mathfrak{e}_7$, the sum $\mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k)$ has maximal rank and each summand is the other's centralizer.

Write M for the direct sum of the root spaces $\mathfrak{g}_r$ for which $r \not\perp \beta_k$ and $r \neq \pm\beta_k$, so that

$$\mathfrak{e}_7 = \mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k) \oplus M$$

as a representation of $\mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k)$. The dimension of M is $133 - 3 - 66 = 64$.

First we identify M as a representation of $\mathfrak{sl}_2(\beta_k)$. Because E_7 is simply laced, any two roots r, α satisfy $\langle r, \alpha \rangle \in \{0, \pm 1, \pm 2\}$, with ± 2 occurring only for $r = \pm\alpha$. Since the 64 roots r contributing to M have $r \neq \pm\beta_k$ and $r \not\perp \beta_k$, these must have $\langle r, \beta_k \rangle = \pm 1$. Thus $\mathrm{ad}(h_{\beta_k})$ has only the eigenvalues ± 1 on M, with eigenspaces M_+ and M_-, and the raising and lowering operators $e_{\pm\beta_k}$ carry $M_\mp$ isomorphically onto $M_\pm$. It follows that every irreducible $\mathfrak{sl}_2(\beta_k)$-summand of M is a copy of the 2-dimensional defining representation, which we call $\mathbf{2}$, and

$$M \;\cong\; \mathbf{2} \otimes M_+,$$

where M_+ is spanned by the $\mathfrak{e}_7$ roots r with $\langle r, \beta_k \rangle = 1$.

It remains to identify M_+ as a representation of $\mathfrak{so}_{12}(\beta_k)$. It has dimension $64/2 = 32$, and its $\mathfrak{so}_{12}(\beta_k)$ weights are the orthogonal projections onto $\mathfrak{h} \cap \beta_k^\perp$ of the $\mathfrak{e}_7$ roots r with $\langle r, \beta_k \rangle = 1$. These 32 weights are distinct, and each has the same length. The only representations of $\mathfrak{so}_{12}$ with these properties are the two chiral spinor (or 'half-spin') representations, commonly called $\mathbf{32}$ and $\overline{\mathbf{32}}$. These two are interchanged by the outer automorphism of $\mathfrak{so}_{12}$ that swaps the two spinor nodes of the D_6 diagram, so which one is M_+ depends on a convention about we identify $\mathfrak{so}_{12}(\beta_k)$ with $\mathfrak{so}_{12}$, and we choose a convention such that $M_+ \cong \mathbf{32}$. □

From the proof we see that

$$\mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k) = \mathfrak{h} \oplus \bigoplus_{r \in \{\pm\beta_k\} \sqcup \Phi_0 \sqcup \Phi_k} (\mathfrak{e}_7)_r \tag{10}$$

Thus, the three decompositions of $\mathfrak{e}_7$ in Proposition 5 are truly different. Proposition 7 says how they overlap.

7. The standard $\mathfrak{sl}_6$

There is a unique copy of $\mathfrak{sl}_6$ lying in all three of the $\mathfrak{sl}_2 \oplus \mathfrak{so}_{12}$ subalgebras of $\mathfrak{e}_7$ constructed in Proposition 5. This is the largest subalgebra of $\mathfrak{e}_7$ that commutes with $\mathfrak{g}_{\mathrm{SM}}$.

Proposition 6. *The centralizer of $\mathfrak{sl}_3^{\mathrm{gen}}$ in $\mathfrak{e}_7$ is isomorphic to $\mathfrak{sl}_6$. Its root system is Φ_0. In fact $\mathfrak{sl}_3^{\mathrm{gen}}$ and this copy of $\mathfrak{sl}_6$ are centralizers of each other, and their direct sum $\mathfrak{sl}_3^{\mathrm{gen}} \oplus \mathfrak{sl}_6$ is a maximal subalgebra of $\mathfrak{e}_7$.*

Proof. Since $\mathfrak{sl}_3^{\mathrm{gen}}$ is a regular simple subalgebra of $\mathfrak{e}_7$, we can use same method as in Proposition 1 to determine its centralizer:

$$C_{\mathfrak{e}_7}(\mathfrak{sl}_3^{\mathrm{gen}}) = P^\perp \oplus \bigoplus_{r \in \Phi_0} (\mathfrak{e}_7)_r$$

where $P^\perp$ is the orthogonal complement of the generation plane in $\mathfrak{h}$, and Φ_0 consists of all roots in $P^\perp$. The root system Φ_0 is of type A_5 by Lemma 3. Since P has dimension 2, $P^\perp$ has dimension $7 - 2 = 5$, matching the rank of A_5, so there is no additional abelian summand and the centralizer is isomorphic to $\mathfrak{sl}_6$.

A similar argument shows that the centralizer of this $\mathfrak{sl}_6$ is $\mathfrak{sl}_3^{\mathrm{gen}}$. Dynkin showed that for this $\mathfrak{sl}_6 \subset \mathfrak{e}_7$, the direct sum $\mathfrak{sl}_3^{\mathrm{gen}} \oplus \mathfrak{sl}_6$ is a maximal subalgebra of $\mathfrak{e}_7$: see [6, Table 12] and refer to Table 1 for his system of numbering roots of $\mathfrak{e}_7$. In fact he described this maximal subalgebra using the procedure of removing a root, described in Section 2, applied to the root we call α_5. □

We call this copy of $\mathfrak{sl}_6$ in $\mathfrak{e}_7$ the **standard $\mathfrak{sl}_6$**, and denote it as $\mathfrak{sl}_6^{\mathrm{SM}}$.

Proposition 7. *Let* $\mathfrak{m}_k = \mathfrak{sl}_2(\beta_k) \oplus \mathfrak{so}_{12}(\beta_k)$. *Then*

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 = \mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P)$$

for the 2-dimensional abelian Lie subalgebra $\mathbb{C} \otimes P \subset \mathfrak{e}_6$, *which is the complexification of the generation plane* $P \subset \mathfrak{h}$. *Moreover, the pairwise intersections of the* $\mathfrak{m}_k$ *already equal the triple intersection.*

Proof. Recall from Equation (10) that

$$\mathfrak{m}_k = \mathfrak{h} \oplus \bigoplus_{r \in \{\pm\beta_k\} \sqcup \Phi_0 \sqcup \Phi_k} (\mathfrak{e}_7)_r. \tag{11}$$

If $i \neq k$, any root $r \in \Phi_k$ lies outside $\{\pm\beta_i\} \sqcup \Phi_0 \sqcup \Phi_i$, and so do $\pm\beta_k$. Thus,

$$\mathfrak{m}_i \cap \mathfrak{m}_k = \mathfrak{h} \oplus \bigoplus_{r \in \Phi_0} (\mathfrak{e}_7)_r.$$

This intersection is $\mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P)$, since $\mathfrak{h}$ is the direct sum of the subspace $\mathbb{C} \otimes P \subset \mathfrak{h}$ and its orthogonal complement in $\mathfrak{h}$, which is the Cartan subalgebra of $\mathfrak{sl}_6^{\mathrm{SM}}$, while the roots of $\mathfrak{sl}_6^{\mathrm{SM}}$ are exactly those in Φ_0, by Proposition 6.

Since this is true for all $i \neq k$, the triple intersection is

$$\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 = \mathfrak{h} \oplus \bigoplus_{r \in \Phi_0} (\mathfrak{e}_7)_r = \mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P). \tag{12}$$

proving the proposition. □

8. The standard $\mathfrak{sl}_5$

Sitting between $\mathfrak{sl}_6^{\mathrm{SM}}$ and $\mathfrak{g}_{\mathrm{SM}}$ there is a unique copy of $\mathfrak{sl}_5$. In some ways this is more familiar to particle physicists than the previous constructions. The inclusion $G_{\mathrm{SM}} \subset \mathrm{SU}(5)$ from Equation (1), which is the basis of the Georgi–Glashow grand unified theory [7], gives an inclusion of complex Lie algebras $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5$ that matches the one here. However, we obtain it in a different way.

Proposition 8. *There is a unique subalgebra of* $\mathfrak{sl}_6^{\mathrm{SM}}$ *isomorphic to* $\mathfrak{sl}_5$ *that contains* $\mathfrak{g}_{\mathrm{SM}}$. *Moreover this* $\mathfrak{sl}_5$ *subalgebra is regular in* $\mathfrak{e}_7$.

Proof. We first construct a regular $\mathfrak{sl}_5$ between $\mathfrak{g}_{\mathrm{SM}}$ and $\mathfrak{e}_7$, then show it lies in $\mathfrak{sl}_6^{\mathrm{SM}}$, and finally show it is the only $\mathfrak{sl}_5$ between $\mathfrak{g}_{\mathrm{SM}}$ and $\mathfrak{sl}_6^{\mathrm{SM}}$—regular or not.

A regular $\mathfrak{sl}_5$ *containing* $\mathfrak{g}_{\mathrm{SM}}$. Since $\mathfrak{g}_{\mathrm{SM}}$ is a regular subalgebra of $\mathfrak{e}_7$, its Cartan subalgebra is $\mathfrak{h}_{\mathrm{SM}} = \mathfrak{h} \cap \mathfrak{g}_{\mathrm{SM}}$; let $U = \mathfrak{h}_{\mathrm{SM}} \cap V$ be its real form, so that $\dim_{\mathbb{R}} U = 4$ as $\mathfrak{g}_{\mathrm{SM}}$ has rank 4.

Let $\mathfrak{a}$ be any regular subalgebra of type A_4 containing $\mathfrak{g}_{\mathrm{SM}}$. Being regular, $\mathfrak{a}$ has Cartan subalgebra $\mathfrak{a} \cap \mathfrak{h}$. This intersection contains $\mathfrak{h}_{\mathrm{SM}}$, and both have dimension 4, so $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{h}_{\mathrm{SM}}$. Hence the roots of $\mathfrak{a}$ form a subsystem of type A_4 that lies in $\Phi \cap U$ and spans U. A direct count in the coordinates of Section 2 shows $\Phi \cap U$ consists of exactly 20 roots and is itself a subsystem of type A_4. Since a rank-4 subsystem of an A_4 system is the whole system, the roots of $\mathfrak{a}$ are exactly $\Phi \cap U$. Thus there is at most one such $\mathfrak{a}$, namely

$$\mathfrak{sl}_5^{\mathrm{SM}} := \mathfrak{h}_{\mathrm{SM}} \oplus \bigoplus_{r \in \Phi \cap U} (\mathfrak{e}_7)_r,$$

and one can check directly that this is a subalgebra of type A_4; it is the unique regular $\mathfrak{sl}_5$ in $\mathfrak{e}_7$ containing $\mathfrak{g}_{\mathrm{SM}}$.

It lies in $\mathfrak{sl}_6^{\mathrm{SM}}$. We must show $\Phi \cap U \subseteq \Phi_0$. The generation plane P is spanned by the roots A, and $\mathfrak{h}_{\mathrm{SM}}$ is orthogonal to P: the center of $\mathfrak{g}_{\mathrm{SM}}$ and the roots of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_3 \subset \mathfrak{g}_{\mathrm{SM}}$ all lie in $C_{\mathfrak{e}_7}(\mathfrak{sl}_3^{\mathrm{gen}})$, whose Cartan is $\mathfrak{h} \cap P^\perp$ by Proposition 6. Hence $U \subseteq P^\perp$, so every root in $\Phi \cap U$ is orthogonal to P, i.e. lies in Φ_0. Therefore $\mathfrak{sl}_5^{\mathrm{SM}} \subseteq \mathfrak{sl}_6^{\mathrm{SM}}$.

Uniqueness in $\mathfrak{sl}_6^{\mathrm{SM}}$. Now let $\mathfrak{s} \subseteq \mathfrak{sl}_6^{\mathrm{SM}}$ be *any* subalgebra isomorphic to $\mathfrak{sl}_5$ with $\mathfrak{g}_{\mathrm{SM}} \subseteq \mathfrak{s}$; we show $\mathfrak{s} = \mathfrak{sl}_5^{\mathrm{SM}}$, so that the regularity hypothesis was superfluous. Identify $\mathfrak{sl}_6^{\mathrm{SM}}$ with $\mathfrak{sl}(\mathbb{C}^6)$ acting on its defining representation $\mathbb{C}^6$. Since $\mathfrak{sl}_5$ has no faithful representation of dimension 6 and its smallest nontrivial irreducible representations have dimension 5, the action of $\mathfrak{s}$ on $\mathbb{C}^6$ decomposes as

$$\mathbb{C}^6 \;=\; W \oplus L, \qquad \dim W = 5, \quad \dim L = 1,$$

with W a faithful irreducible representation of $\mathfrak{s}$ and L trivial. Then $\mathfrak{s} \subseteq \mathfrak{sl}(W) \subseteq \mathfrak{sl}(\mathbb{C}^6)$, and comparing dimensions ($\dim \mathfrak{s} = 24 = \dim \mathfrak{sl}(W)$) gives $\mathfrak{s} = \mathfrak{sl}(W)$. Thus, $\mathfrak{s}$ is determined by the hyperplane $W \subset \mathbb{C}^6$, and $\mathfrak{g}_{\mathrm{SM}} \subseteq \mathfrak{s}$ acts as zero on the line L.

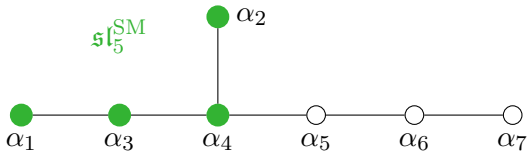
We claim the choice of the line L is forced, and with it W and $\mathfrak{s}$. Under $\mathfrak{g}_{\mathrm{SM}}$, the defining representation of $\mathfrak{sl}_6$ decomposes as

$$\mathbb{C}^6\big|_{\mathfrak{g}_{\mathrm{SM}}} \;=\; X \oplus Y \oplus T,$$

where X is a 3-dimensional irreducible representation of $\mathfrak{sl}_3 \subseteq \mathfrak{g}_{\mathrm{SM}}$ on which $\mathfrak{sl}_2$ acts trivially, Y is a 2-dimensional irreducible representation of $\mathfrak{sl}_2 \subseteq \mathfrak{g}_{\mathrm{SM}}$ on which $\mathfrak{sl}_3$ acts trivially, and T is a line on which $\mathfrak{sl}_3$, $\mathfrak{sl}_2$, and the center of $\mathfrak{g}_{\mathrm{SM}}$ all act trivially. Since X and Y are nontrivial irreducible representations of the simple summands of $\mathfrak{g}_{\mathrm{SM}}$, neither contains a nonzero $\mathfrak{g}_{\mathrm{SM}}$-invariant vector; hence the space of $\mathfrak{g}_{\mathrm{SM}}$-invariants in $\mathbb{C}^6$ is exactly T. In particular the trivial representation of $\mathfrak{g}_{\mathrm{SM}}$ occurs in $\mathbb{C}^6$ with multiplicity one, so the $\mathfrak{g}_{\mathrm{SM}}$-fixed line L must equal T. Then $W = \mathbb{C}^6/L$ is determined, and so is $\mathfrak{s} = \mathfrak{sl}(W)$.

There is therefore at most one $\mathfrak{sl}_5$ between $\mathfrak{g}_{\mathrm{SM}}$ and $\mathfrak{sl}_6^{\mathrm{SM}}$. As $\mathfrak{sl}_5^{\mathrm{SM}}$ is one, it is the only one, completing the proof. □

We call the unique $\mathfrak{sl}_5$ subalgebra containing $\mathfrak{g}_{\mathrm{SM}}$ and contained in $\mathfrak{sl}_6^{\mathrm{SM}}$ the **standard** $\mathfrak{sl}_5$, and denote it as $\mathfrak{sl}_5^{\mathrm{SM}}$. This subalgebra is generated by the elements e_i, f_i and h_i for $i = 1, 2, 3, 4$, so we can visualize it as follows:



9. Three generations omitting right-handed neutrinos

So far, starting from only a good choice of subalgebra $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$ as in Section 2, we have constructed a chain of subalgebras:

$$\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5^{\mathrm{SM}} \subset \mathfrak{sl}_6^{\mathrm{SM}} \subset \mathfrak{e}_7.$$

Each one is uniquely determined:

- The centralizer of $\mathfrak{g}_{\mathrm{SM}}$ contains a unique copy of $\mathfrak{sl}_3$, namely $\mathfrak{sl}_3^{\mathrm{gen}}$. (Proposition 1.)
- The centralizer of $\mathfrak{sl}_3^{\mathrm{gen}}$ is copy of $\mathfrak{sl}_6$, namely $\mathfrak{sl}_6^{\mathrm{SM}}$. (Proposition 6.)
- There is a unique copy of $\mathfrak{sl}_5$ between $\mathfrak{g}_{\mathrm{SM}}$ and $\mathfrak{sl}_6^{\mathrm{SM}}$, namely $\mathfrak{sl}_5^{\mathrm{SM}}$. (Proposition 8.)

We are now ready to exploit this chain.

For each $k = 1, 2, 3$ we identify the 30-dimensional space spanned by the root spaces $(\mathfrak{e}_7)_r$ with $r \in \Phi_k$ as a representation first of $\mathfrak{sl}_6^{\mathrm{SM}}$, then of $\mathfrak{sl}_5^{\mathrm{SM}}$, and finally of $\mathfrak{g}_{\mathrm{SM}}$. In the end, we obtain the usual representation of $\mathfrak{g}_{\mathrm{SM}}$ on one generation of Standard Model fermions together with their antiparticles, *except for* the right-handed neutrino and its antiparticle.

Lemma 9. *For each $k = 1, 2, 3$ we have*

$$\bigoplus_{r \in \Phi_k} (\mathfrak{e}_7)_r \;\cong\; \Lambda^2\mathbb{C}^6 \,\oplus\, \Lambda^4\mathbb{C}^6$$

as representations of $\mathfrak{sl}_6^{\mathrm{SM}}$.

Proof. For short let us write

$$N_k = \bigoplus_{r \in \Phi_k} (\mathfrak{e}_7)_r.$$

Note that N_k is an $\mathfrak{sl}_6^{\mathrm{SM}}$-submodule of $\mathfrak{e}_7$: for $s \in \Phi_0$ and $r \in \Phi_k$ we have $\pi(r+s) = \pi(r) = \pm w_k$, so whenever $r + s$ is a root it again lies in Φ_k, showing that $[(\mathfrak{e}_7)_s, (\mathfrak{e}_7)_r] \subseteq N_k$.

Split Φ_k into two subsets, defining

$$\Phi_k^{\pm} \;=\; \{\, r \in \Phi_k : \pi(r) = \pm w_k \,\}, \qquad \Phi_k = \Phi_k^+ \sqcup \Phi_k^-,$$

and set

$$N_k^{\pm} = \bigoplus_{r \in \Phi_k^{\pm}} (\mathfrak{e}_7)_r.$$

Since the projection $\pi\colon V \to P$ is unaffected by adding a root of Φ_0, each $N_k^{\pm}$ is itself an $\mathfrak{sl}_6^{\mathrm{SM}}$-submodule, and $N_k = N_k^+ \oplus N_k^-$. The reflection $r \mapsto -r$ carries Φ_k^+ bijectively onto Φ_k^-, so $|\Phi_k^+| = |\Phi_k^-| = \frac{1}{2}|\Phi_k| = 15$ by Lemma 3, and N_k^- is the dual of N_k^+.

N_k^+ is a 15-dimensional representation of $\mathfrak{sl}_6^{\mathrm{SM}}$ with 15 distinct weights all of the same length: namely, the roots of Φ_k^+ projected down to the Cartan of $\mathfrak{sl}_6^{\mathrm{SM}}$. The only representations of $\mathfrak{sl}_6$ with these properties are its fundamental representations on $\Lambda^2\mathbb{C}^6$ and its dual $\Lambda^4\mathbb{C}^6$. We must thus have $N_k \cong \Lambda^2\mathbb{C}^6 \,\oplus\, \Lambda^4\mathbb{C}^6$. □

Next we restrict along the inclusion $\mathfrak{sl}_5^{\mathrm{SM}} \subset \mathfrak{sl}_6^{\mathrm{SM}}$.

Theorem 10. *For each $k = 1, 2, 3$ we have*

$$\bigoplus_{r \in \Phi_k} (\mathfrak{e}_7)_r \cong \Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5$$

as representations of $\mathfrak{sl}_5^{\mathrm{SM}}$.

Proof. By Lemma 9 the span is $\Lambda^2\mathbb{C}^6 \oplus \Lambda^4\mathbb{C}^6$ as an $\mathfrak{sl}_6^{\mathrm{SM}}$-module. Restricting to $\mathfrak{sl}_5^{\mathrm{SM}} \subset \mathfrak{sl}_6^{\mathrm{SM}}$ and splitting $\mathbb{C}^6$ as $\mathbb{C}^5 \oplus \mathbb{C}$, we have

$$\begin{aligned} \Lambda^2(\mathbb{C}^5 \oplus \mathbb{C}) &\cong (\Lambda^0\mathbb{C}^5 \otimes \Lambda^2\mathbb{C}) \oplus (\Lambda^1\mathbb{C}^5 \otimes \Lambda^1\mathbb{C}) \oplus (\Lambda^2\mathbb{C}^5 \otimes \Lambda^0\mathbb{C}) \\ &\cong \Lambda^1\mathbb{C}^5 \oplus \Lambda^2\mathbb{C}^5 \end{aligned}$$

as representations of $\mathfrak{sl}_5^{\mathrm{SM}}$, and dually

$$\Lambda^4\mathbb{C}^6 \;\cong\; \Lambda^4\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5$$

so the proposition follows. □

Restricting this representation to $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{sl}_5^{\mathrm{SM}}$ we obtain the representation of $\mathfrak{g}_{\mathrm{SM}}$ on one generation of Standard Model fermions together with their antiparticles *omitting* the right-handed neutrino and its antiparticle, which transform trivially under $\mathfrak{g}_{\mathrm{SM}}$ and would naturally lie in $\Lambda^0\mathbb{C}^5$ and $\Lambda^5\mathbb{C}^5$. For details of how $\Lambda\mathbb{C}^5$ captures one generation of fermions and their antiparticles see [2, Table 4], which we reproduce here in Table 3. In this table we give $\mathbb{C}^5$ a basis called r, g, b, u, d. We write wedge products simply as products, write c for r, g or b, and write $\overline{c}$ for gb, br or rg.

$\Lambda^0\mathbb{C}^5$	$\Lambda^1\mathbb{C}^5$	$\Lambda^2\mathbb{C}^5$	$\Lambda^3\mathbb{C}^5$	$\Lambda^4\mathbb{C}^5$	$\Lambda^5\mathbb{C}^5$
$\overline{\nu}_L = 1$	$e_R^+ = u$	$e_L^+ = ud$	$e_R^- = rgb$	$e_L^- = drgb$	$\nu_R = udrgb$
	$\overline{\nu}_R = d$	$u_L^c = uc$	$\overline{u}_R^{\overline{c}} = d\overline{c}$	$\nu_L = urgb$	
	$d_R^c = c$	$d_L^c = dc$	$\overline{d}_R^{\overline{c}} = u\overline{c}$	$\overline{d}_L^{\overline{c}} = ud\overline{c}$	
		$\overline{u}_L^{\overline{c}} = \overline{c}$	$u_R^c = udc$		

TABLE 3. First-generation fermions as exterior algebra elements.

The statement of Theorem 10 depends crucially on a chosen Cartan $\mathfrak{h} \subset \mathfrak{e}_7$ and also chosen weights w_k for $\mathfrak{sl}_3^{\mathrm{gen}}$. Next we state a result that depends only on our choice of $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$. With only this structure, there is no way to separate out and name the three generations, since the generation symmetry is unbroken: $\mathfrak{sl}_3^{\mathrm{gen}}$ commutes with $\mathfrak{g}_{\mathrm{SM}}$. We shall see that e_7 splits into four parts: $\mathfrak{sl}_3^{\mathrm{gen}}$, its centralizer $\mathfrak{sl}_6^{\mathrm{SM}}$, and two 45-dimensional subspaces. One of these consists of all three generations of left-handed fermions, not including left-handed antineutrinos. The other consists of their antiparticles.

To state this result, let $\mathbf{3}$ and $\mathbf{3}^*$ be the two irreducible 3-dimensional representations of $\mathfrak{sl}_3^{\mathrm{gen}}$, and let $\mathbf{15}$ and $\mathbf{15}^*$ be the two irreducible 15-dimensional representations of $\mathfrak{sl}_6^{\mathrm{SM}}$. In each case, saying which representation is which depends on arbitrary conventions, since the Dynkin diagram of each Lie subalgebra of $\mathfrak{e}_7$ has a $\mathbb{Z}_2$ symmetry that exchanges dual representations.

Theorem 11. *As a representation of its maximal subalgebra $\mathfrak{sl}_3^{\mathrm{gen}} \oplus \mathfrak{sl}_6$, the Lie algebra $\mathfrak{e}_7$ decomposes into irreducible representations as follows:*

$$\mathfrak{e}_7 = \mathfrak{sl}_3^{\mathrm{gen}} \oplus \mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbf{3} \otimes \mathbf{15}) \oplus (\mathbf{3}^* \otimes \mathbf{15}^*).$$

Proof. As a purely representation-theoretic result this is well known [14, Table 52], but we outline a self-contained proof. Clearly $\mathfrak{sl}_3^{\mathrm{gen}}$ and $\mathfrak{sl}_6^{\mathrm{SM}}$ are irreducible summands of $\mathfrak{e}_7$ as a representation of $\mathfrak{sl}_3^{\mathrm{gen}} \oplus \mathfrak{sl}_6$. By the proof of Lemma 9, the rest breaks up into two 45-dimensional summands,

$$\textstyle\bigoplus_{k=1}^3 N_k^+ \;\oplus\; \bigoplus_{k=1}^3 N_k^-.$$

The proof also shows that $\mathfrak{sl}_6^{\mathrm{SM}}$ has equivalent 15-dimensional irreducible representations on each space N_k^+, and dual representations on each space N_k^-. On the other hand, $\mathfrak{sl}_3^{\mathrm{gen}}$ acts irreducibly via its defining representation on the copy of $\mathbb{C}^3$ whose basis is the three roots w_k, and via the dual of the defining representation on the copy of $\mathbb{C}^3$ whose basis is the three roots $-w_k$. ∎

We can choose to make the identifications

$$\mathbf{15} \cong \Lambda^2\mathbb{C}^5 \oplus \Lambda^4\mathbb{C}^5, \qquad \mathbf{15}^* \cong \Lambda^1\mathbb{C}^5 \oplus \Lambda^3\mathbb{C}^5.$$

Then $\mathbf{3} \otimes \mathbf{15}$ corresponds to all the left-handed fermions in three generations except the left-handed antineutrinos, as in Table 3, while $\mathbf{3}^* \otimes \mathbf{15}^*$ corresponds to all the right-handed fermions in three generations except the right-handed neutrinos.

10. Three generations in $\mathfrak{e}_7$

We can also include right-handed neutrinos and their antiparticles in this setup. They behave unlike the other particles. However, we can include them in such a way that we get three copies of the Standard Model representation in $\mathfrak{e}_7$.

Theorem 10 implies that unlike the other fermions of the kth generation, the kth right-handed neutrino cannot fit into the root spaces associated to Φ_k. However, Nasmith [10] noted that it can be fit into the root space associated to β_k, one of the six roots of the generation $\mathfrak{sl}_3$. Its antiparticle then naturally goes into the root space associated to $-\beta_k$. Three copies of the Standard Model representation of $\mathfrak{g}_{\mathrm{SM}}$ on $\Lambda\mathbb{C}^5$ then fill up all of $\mathfrak{e}_7$ except for a copy of $\mathfrak{sl}_6 \oplus \mathbb{C}^2$. The copy of $\mathfrak{sl}_6$ here is $\mathfrak{sl}_6^{\mathrm{gen}}$, while the copy of $\mathbb{C}^2$ is $\mathbb{C} \otimes P$, the complexification of the generation plane.

To implement this, for $k = 1, 2, 3$ let

$$V_k = \bigoplus_{r \in \{\pm\beta_k\} \sqcup \Phi_k} (\mathfrak{e}_7)_r. \tag{13}$$

This space is 32-dimensional, and by Equations (11) and (12) we have

$$\mathfrak{m}_k = (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3) \oplus V_k. \tag{14}$$

In fact V_k captures one generation of Standard Model fermions and their antiparticles.

Theorem 12. *As a representation of $\mathfrak{sl}_6^{\mathrm{SM}}$,*

$$V_k \;\cong\; \Lambda^0\mathbb{C}^6 \oplus \Lambda^2\mathbb{C}^6 \oplus \Lambda^4\mathbb{C}^6 \oplus \Lambda^6\mathbb{C}^6 \;=\; \Lambda^{\mathrm{even}}\mathbb{C}^6.$$

Consequently, as a representation of $\mathfrak{sl}_5^{\mathrm{SM}}$ we have

$$V_k \;\cong\; \Lambda\mathbb{C}^5,$$

and as a representation of $\mathfrak{g}_{\mathrm{SM}}$, V_k is the Standard Model representation.

Proof. The first statement follows from Lemma 9 together with

$$(\mathfrak{e}_7)_\beta \cong \Lambda^0\mathbb{C}^6, \quad (\mathfrak{e}_7)_{-\beta} \cong \Lambda^6\mathbb{C}^6.$$

For the second, split $\mathbb{C}^6 = \mathbb{C}^5 \oplus \mathbb{C}$ as $\mathfrak{sl}_5$-modules. Since $\Lambda(V_1 \oplus V_2) \cong \Lambda V_1 \otimes \Lambda V_2$, taking the even part and using $\Lambda\mathbb{C} = \Lambda^0\mathbb{C} \oplus \Lambda^1\mathbb{C}$ gives

$$\Lambda^{\mathrm{even}}(\mathbb{C}^5 \oplus \mathbb{C}) \;\cong\; \left(\Lambda^{\mathrm{even}}\mathbb{C}^5 \otimes \Lambda^0\mathbb{C}\right) \oplus \left(\Lambda^{\mathrm{odd}}\mathbb{C}^5 \otimes \Lambda^1\mathbb{C}\right) \;\cong\; \Lambda^{\mathrm{even}}\mathbb{C}^5 \oplus \Lambda^{\mathrm{odd}}\mathbb{C}^5 \;=\; \Lambda\mathbb{C}^5. \quad \square$$

Theorem 13. *We have a direct sum decomposition*

$$\mathfrak{e}_7 \;=\; \mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P) \oplus V_1 \oplus V_2 \oplus V_3.$$

where each summand is an $\mathfrak{sl}_6^{\mathrm{SM}}$-subrepresentation of $\mathfrak{e}_7$, with $\mathfrak{sl}_6^{\mathrm{SM}}$ acting by the $\mathfrak{e}_7$ Lie bracket.

Proof. By Proposition 7 and Equation (12) we have

$$\mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P) = \mathfrak{h} \oplus \bigoplus_{r \in \Phi_0} (\mathfrak{e}_7)_r. \tag{15}$$

By this formula and Equation (13) it follows that the spaces V_k are mutually orthogonal and all orthogonal to $\mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P)$, since by Equation (7) the sets Φ_k $(k = 0, 1, 2, 3)$ and the sets $\{\pm\beta_k\}$ $(k = 1, 2, 3)$ are all disjoint. Since the union of all these sets is the set Φ of $\mathfrak{e}_7$ roots, we have

$$\mathfrak{sl}_6^{\mathrm{SM}} \oplus (\mathbb{C} \otimes P) \oplus V_1 \oplus V_2 \oplus V_3 = \mathfrak{h} \oplus \bigoplus_{r \in \Phi} (\mathfrak{e}_7)_r = \mathfrak{e}_7.$$

To show that the subspaces V_k are preserved by $\mathfrak{sl}_6^{\mathrm{SM}}$, first let $s \in \Phi_0$ and $r \in \Phi_k$: then $\pi(r + s) = \pi(r) = \pm w_k$, so if $r + s$ is a root it again lies in Φ_k. This shows that

$$[(\mathfrak{e}_7)_s, \bigoplus_{r \in \Phi_k} (\mathfrak{e}_7)_r] \subseteq \bigoplus_{r \in \Phi_k} (\mathfrak{e}_7)_r,$$

so this direct sum is preserved by $\mathfrak{sl}_6^{\mathrm{SM}}$. Next let $s \in \Phi_0$ and consider β_k: if $\beta_k + s$ were a root, its projection onto the generation plane would be β_k, of squared length 2, forcing $\beta_k + s \in P$ by the argument in Lemma 2 and hence $s = 0$. Thus

$$[(\mathfrak{e}_7)_s, (\mathfrak{e}_7)_{\pm\beta_k}] = 0.$$

This shows that V_k is preserved by $\mathfrak{sl}_6^{\mathrm{SM}}$.

Clearly $\mathfrak{sl}_6^{\mathrm{SM}}$ is preserved by bracketing with $\mathfrak{sl}_6^{\mathrm{SM}}$. Finally, $\mathbb{C} \otimes P$ is annihilated by bracketing with $\mathfrak{sl}_6^{\mathrm{SM}}$ because it is spanned by the roots β_k, which lie in $\mathfrak{sl}_3^{\mathrm{gen}}$, which commutes with $\mathfrak{sl}_6^{\mathrm{SM}}$ by Proposition 6. □

The statement of this final theorem depends not only on the choice of a good $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$ but also a Cartan subalgebra $\mathfrak{h}$ and three roots β_k of $\mathfrak{sl}_3^{\mathrm{gen}}$. Theorem 11, by comparison, depends only on $\mathfrak{g}_{\mathrm{SM}} \subset \mathfrak{e}_7$. In the framework discussed here, the right-handed neutrinos and their antiparticles are fundamentally different from other fermions. Not only do they transform trivially under $\mathfrak{g}_{\mathrm{SM}}$, their 6-dimensional space cannot be singled out from the 8-dimensional space $\mathfrak{sl}_3^{\mathrm{gen}}$ without choosing a Cartan of that Lie algebra. Proposition 1 gives a 10-dimensional space of particles that transform trivially under $\mathfrak{g}_{\mathrm{SM}}$, and a natural way to pick out the 8-dimensional subspace $\mathfrak{sl}_3^{\mathrm{gen}}$, but decomposing it further requires extra choices.

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