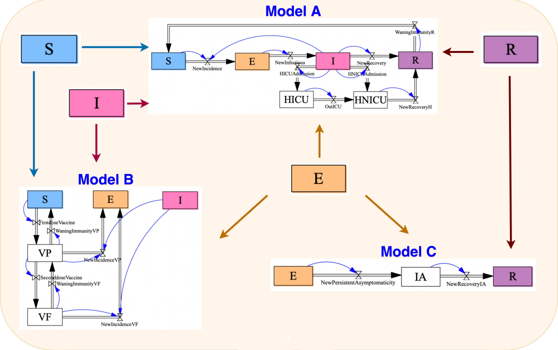
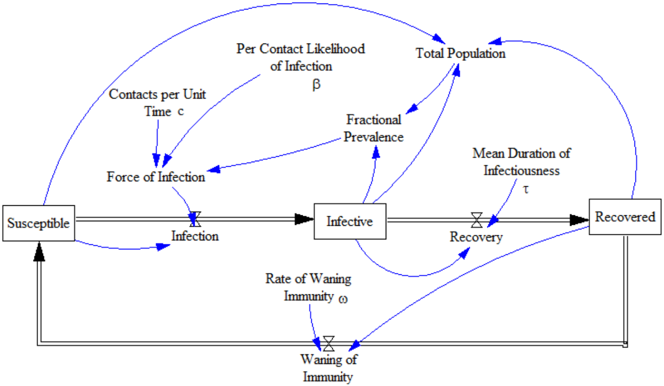
CATEGORY THEORY IN EPIDEMIOLOGY



John Baez, Xiaoyan Li, Nathaniel Osgood, Sophie Libkind and Evan Patterson

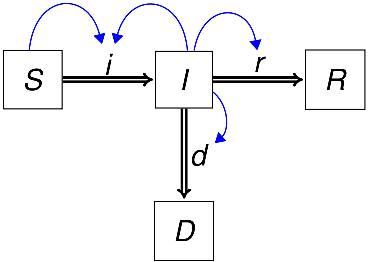
African Mathematics Seminar, 2022 November 2

In "System Dynamics", dynamical systems are modeled using "stock-flow diagrams":



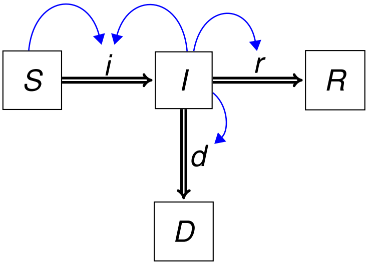
They are widely used in economics, epidemiology, etc.

Here is a simple example of a stock-flow diagram:



The boxes are **stocks**, the double arrows are **flows**, and the blue arrows are **links** from stocks to flows.

If we equip each flow with a **flow function**, a stock-flow diagram gives differential equations describing how the stocks change with time.



$$\frac{dS}{dt} = -\phi_i(S, I)$$

$$\frac{dI}{dt} = \phi_i(S, I) - \phi_r(I) - \phi_d(I)$$

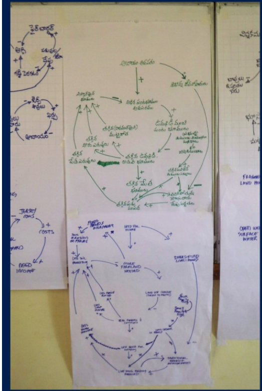
$$\frac{dR}{dt} = \phi_r(I)$$

$$\frac{dD}{dt} = \phi_d(I)$$

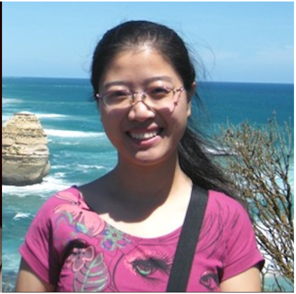
Why don’t we just write down the differential equations?

One reason is that ordinary people — nonmathematicians! — find it easier to understand diagrams than differential equations.

This lets community members work with experts to help build models.


COSTS
MORE FARMLAND NEEDED
DIVERSIFIED LIVELIHOOD
SURFACE WATER

There is a community of epidemiologists who use stock-flow diagrams to model the spread of disease. This includes my coauthors Nathaniel Osgood and Xiaoyan Li, who do COVID modeling for the government of Canada.

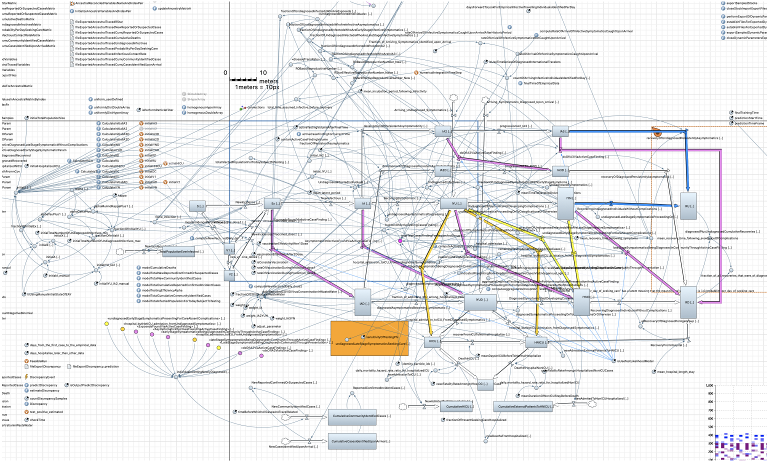
 

Most stock-flow modeling is done using software called AnyLogic. It's powerful, but it has several big problems:

- It has no support for *"composing"* models: that is, taking several smaller models and putting them together to form a larger model.
- It has no support for *"stratifying"* models: that is, taking a model and splitting one stock into several stocks (e.g. age groups).
- It has no support for *collaboratively* building models.
- It is not *free* and not *open-source!*

Our new work aims to fix all these problems.

The ability to *compose* models is crucial because realistic models are complicated and built out of many smaller parts. Here is Osgood and Li's COVID model used by the government of Canada:



The key to compositional modeling is:

1. developing mathematics so that
   - putting together smaller models to form bigger ones is an example of *composing morphisms in a category*
   - converting models to systems of differential equations is a *functor between categories*
2. writing software to implement these ideas, in a system that is friendly to category theory.

We have now created software that supports compositional modeling with stock-flow diagrams, working with Evan Patterson and Sophie Libkind at the Topos Institute, who are experts on categories and computing:

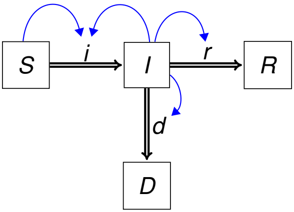
We used AlgebraicJulia: a framework for high-performance scientific computing using category theory. This was developed by a team including James Fairbanks, Evan Patterson, Sophie Libkind, and others.

Let me sketch the math underlying this software. I'll do a simplified version of stock-flow diagrams!

In its simplest form, a **stock-flow diagram** consists of finite sets and functions:

$$
\begin{tikzcd}
\text{Flows} \arrow[rr, shift left, "u"] \arrow[rr, shift right, "d"'] & & \text{Stocks} \\
& \text{Links} \arrow[ul, "t"] \arrow[ur, "s"'] &
\end{tikzcd}
$$

together with, for each $f \in \text{Flows}$, a **flow function** $\phi_f \colon \mathbb{R}^{L(f)} \to \mathbb{R}$ where $L(f)$ is the set of all $\ell \in \text{Links}$ with $t(\ell) = f$.

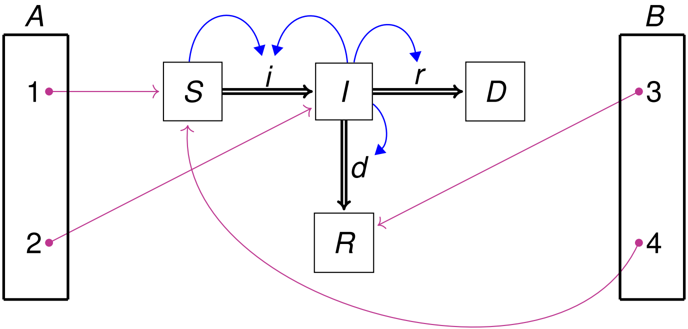


$$\phi_i \colon \mathbb{R}^2 \to \mathbb{R} \text{ gives } \phi_i(S, I)$$

$$\phi_r \colon \mathbb{R} \to \mathbb{R} \text{ gives } \phi_r(I)$$

$$\phi_d \colon \mathbb{R} \to \mathbb{R} \text{ gives } \phi_d(I)$$

An **open stock-flow diagram** is a stock-flow diagram equipped with maps $i\colon A \to \mathrm{Stocks}$, $o\colon B \to \mathrm{Stocks}$ for some finite sets A, B.

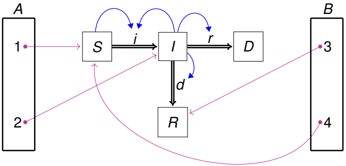


We call this an open stock-flow diagram **from A to B** and write it as $A \xrightarrow{F} B$.

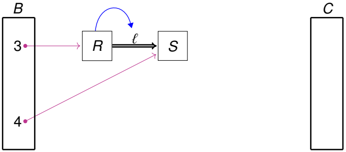
We can **compose** open stock-flow diagrams $A \xrightarrow{F} B$ and $B \xrightarrow{G} C$ by "gluing them together along B".

We get an open stock-flow diagram called $A \xrightarrow{GF} C$.

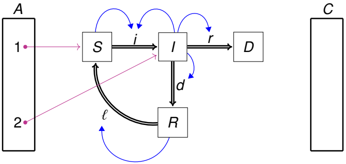
$$A \xrightarrow{F} B$$



$$B \xrightarrow{G} C$$



$$A \xrightarrow{GF} C$$



Thus, we get a category **Open**(**StockFlow**) with:

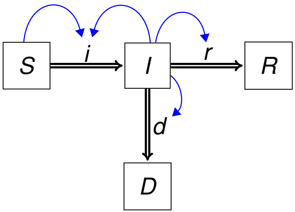
- finite sets as objects,
- open stock-flow diagrams as morphisms.

Next we can construct a category **Open**(**Dynam**) of "open dynamical systems", and a functor

$$\Phi \colon \mathbf{Open}(\mathbf{StockFlow}) \to \mathbf{Open}(\mathbf{Dynam})$$

This turns any open stock-flow diagram into an open dynamical system.

We have already seen how this works without the "openness":



$$\frac{dS}{dt} = -\phi_i(S, I)$$

$$\frac{dI}{dt} = \phi_i(S, I) - \phi_r(I) - \phi_d(I)$$

$$\frac{dR}{dt} = \phi_r(I)$$

$$\frac{dD}{dt} = \phi_d(I)$$

A **dynamical system on** some finite set of variables X is a vector field v on $\mathbb{R}^X$. This lets write down a system of first-order ordinary differential equations.

For example, if $X = \{S, I, D, R\}$ and v is the 4-component vector field (v_S, v_I, v_D, v_R) on $\mathbb{R}^X \cong \mathbb{R}^4$, we get

$$\begin{aligned}
\frac{d}{dt} S(t) &= v_S(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt} I(t) &= v_I(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt} D(t) &= v_D(S(t), I(t), D(t), R(t)) \\
\frac{d}{dt} R(t) &= v_R(S(t), I(t), D(t), R(t))
\end{aligned}$$

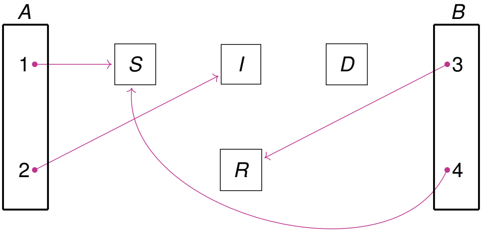
A **dynamical system on** some finite set of variables X is a vector field v on $\mathbb{R}^X$. This lets write down a system of first-order ordinary differential equations.

For example, if $X = \{S, I, D, R\}$ and v is the 4-component vector field (v_S, v_I, v_D, v_R) on $\mathbb{R}^X \cong \mathbb{R}^4$, we get

$$\begin{aligned} \frac{dS}{dt} &= v_S \\ \frac{dI}{dt} &= v_I \\ \frac{dD}{dt} &= v_D \\ \frac{dR}{dt} &= v_R \end{aligned}$$

An **open dynamical system** $A \xrightarrow{v} B$ is a dynamical system v on some finite set X equipped with maps $i\colon A \to X$, $o\colon B \to X$ for some finite sets A, B.

For example:



$$\frac{dS}{dt} = v_S$$

$$\frac{dI}{dt} = v_I$$

$$\frac{dR}{dt} = v_R$$

$$\frac{dD}{dt} = v_D$$

Here $X = \{S, I, D, R\}$.

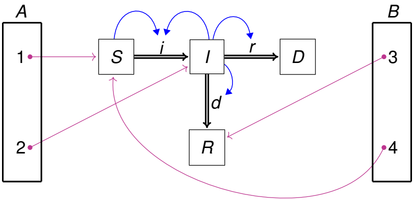
Just as we constructed the category **Open**(**StockFlow**), we can construct a category **Open**(**Dynam**) with:

- finite sets as objects,
- open dynamical systems as morphisms.

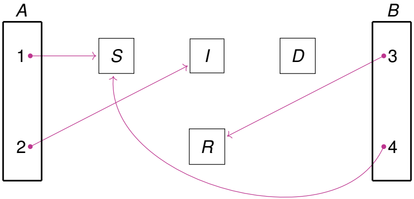
The process we've already seen for converting stock flow diagrams into dynamical systems then gives a functor

$$\Phi\colon \mathbf{Open}(\mathbf{StockFlow}) \to \mathbf{Open}(\mathbf{Dynam})$$

For example, Φ maps this open stock flow diagram:



to this *open* dynamical system:



$$\frac{dS}{dt} = -\phi_i(S, I)$$

$$\frac{dI}{dt} = \phi_i(S, I) - \phi_r(I) - \phi_d(I)$$

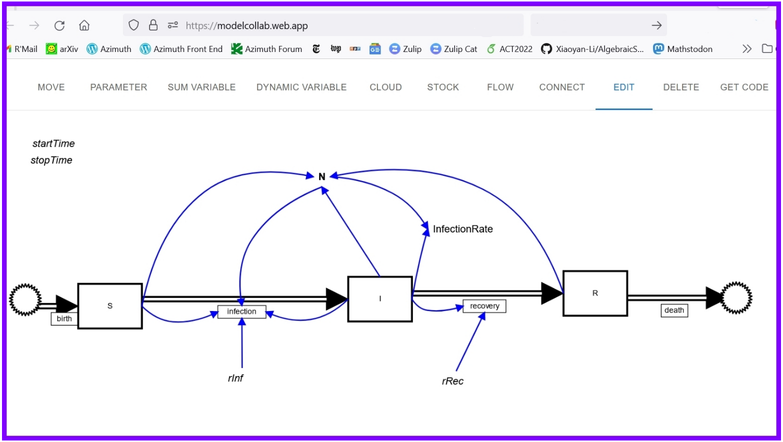
$$\frac{dR}{dt} = \phi_r(I)$$

$$\frac{dD}{dt} = \phi_d(I)$$

By implementing these ideas in AlgebraicJulia, my coauthors created a software package called **StockFlow**, which is now available on GitHub. This lets you:

- compose open stock-flow diagrams
- turn open stock-flow diagrams into open dynamical systems
- *solve* the differential equations given by these dynamical systems.

Nathaniel Osgood has now made a graphical user interface for StockFlow. This software runs in your browser, so teams can collaborate to build stock-flow diagrams.



For more try our papers:

- John Baez, Xiaoyan Li, Sophie Libkind, Nathaniel Osgood and Evan Patterson, Compositional modeling with stock and flow diagrams.
- Sophie Libkind, Andrew Baas, Micah Halter, Evan Patterson and James Fairbanks, An algebraic framework for structured epidemic modeling.