On 2D QFT - from **A**rrows to **D**isks

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December 22, 2006

Quantum field theory can be regarded as the study of **representations of geometric categories**.

Parallel transport in a vector bundle $E \to X$ with connection ∇ is a functor

$$\mathrm{tra}_\nabla : \mathcal{P}_1(X) \to \mathrm{Vect}\,.$$

This can be quantized. **Propagation in the quantum theory** is a functor

$$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Vect}\,.$$

Propagation in 2-dimensional field theory has been conceived in terms of functors

$$U : 2\mathrm{Cob}_S \to \mathrm{Vect}\,.$$

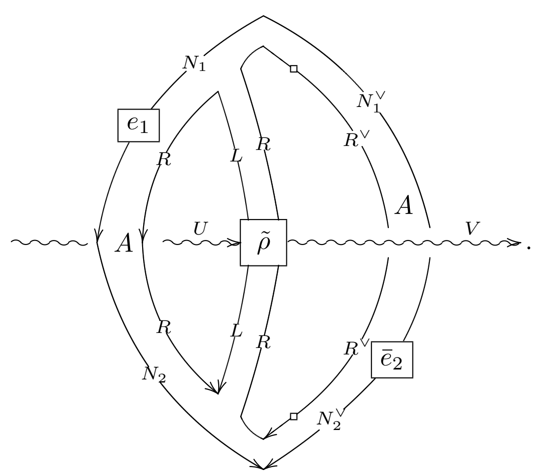
The local structure used to build such functors gives rise to **2-vector transport** 2-functors

$$\mathrm{tra} : \mathcal{P}_2 \to 2\mathrm{Vect}\,.$$

We discuss this for topological and conformal 2-dimensional field theory.

Our aim here is to say what a **1-point disk correlator** in a 2-dimensional quantum field theory is, and how it looks like this:

$$\langle \bar{e}_2 | I_\rho e_1 \rangle \quad = $$



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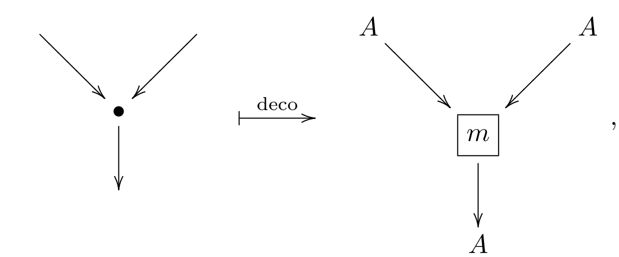
Our strategy is **internalization**: we identify the arrow theory of 1-dimensional quantum field theory, known as quantum mechanics. Categorifying this, we obtain 2-dimensional quantum field theory.

Our imagery is the **charged 2-particle**. A 2-vector transport on target space describes a background field to which a 2-particle couples. The quantization of this system gives rise to a 2-vector transport on the parameter space of the 2-particle.

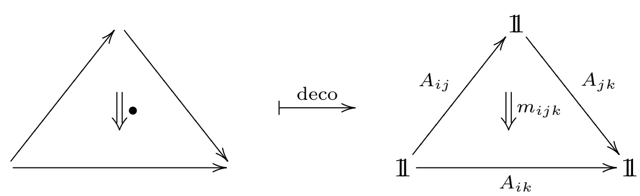
Our **motivation** are structural similarities between

- the formula for surface holonomy of a gerbe in local data;
- the state sum formula for propagation along a surface in topological 2-dimensional field theory;
- the ribbon diagram formula for propagation along a surface in conformal 2-dimensional field theory.

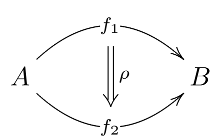
In all three of these cases the quantity associcated to a given surface is obtained, basically, by decorating a dual triangulation of the surface with objects and morphisms of a Frobenius algebroid.



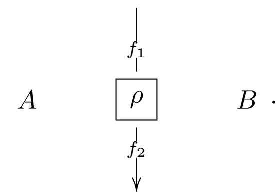
Our **claim** is that all these formulas are special cases of those describing a **locally trivialized 2-transport**.



We simply have to replace globular diagrams



by string diagrams



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global transport	reps of geometric categories	
by		
local data	n-anafunctors on geometric n-categories	
with		
gluing/descent data	by equivalences	by special ambijunctions
which is		
realized as	transition functions	Wilson networks
that		
take values in	n-groups (principal) reps thereof (associated)	n-monoids (of reps of configuration space)
such that		
composition yields	patchwise parallel transport	state sum
and		
serves in physics as	phases on target space (classical)	amplitudes on parameter space (quantum)

Table 1: Quantum field theory is, from the functorial point of view, the theory of **representations of "geometric" categories**. Typically, these are categories of cobordisms with extra structure – or n-categorical refinements of these. Depending on the details, this involves various concepts, as indicated in the above table.

1 Introduction

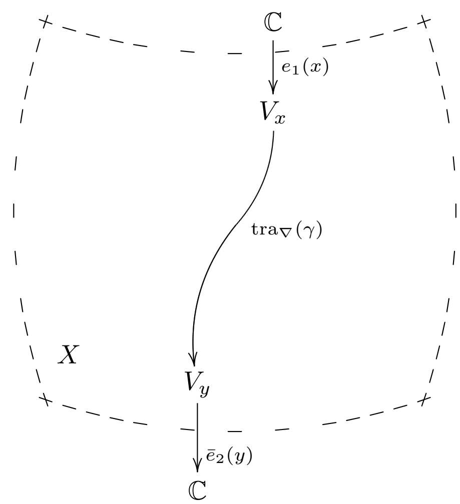
1.1 A map: quantization, categorification and local trivialization

Parallel transport in vector bundles with connection is the model from which we want to understand 2-dimensional quantum field theories and their local state sum description. This involves three orthogonal steps, as indicated in figure 1.

The charged 1-particle. The coupling of a charged particle in a space X to a background field is described by vector bundles $E \to X$ with connection ∇. Parallel transport in the vector bundle is a functor

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$$

that sends paths in base space to morphisms between the fibers over the endpoints.



The parallel transport along the trajectory of the particle models the "phase shift" that the particle suffers due to its charge while traversing its trajectory. This way, any flow v in base space induces an endomorphism

$$U_v : H \to H$$

of the space of sections $H \equiv \Gamma(E)$ of the vector bundle E.

Quantization of the charged 1-particle. For X Riemannian, **quantization** of the charged particle produces a representation of $\mathbb{R}$

$$t \mapsto U(t) : H \to H$$

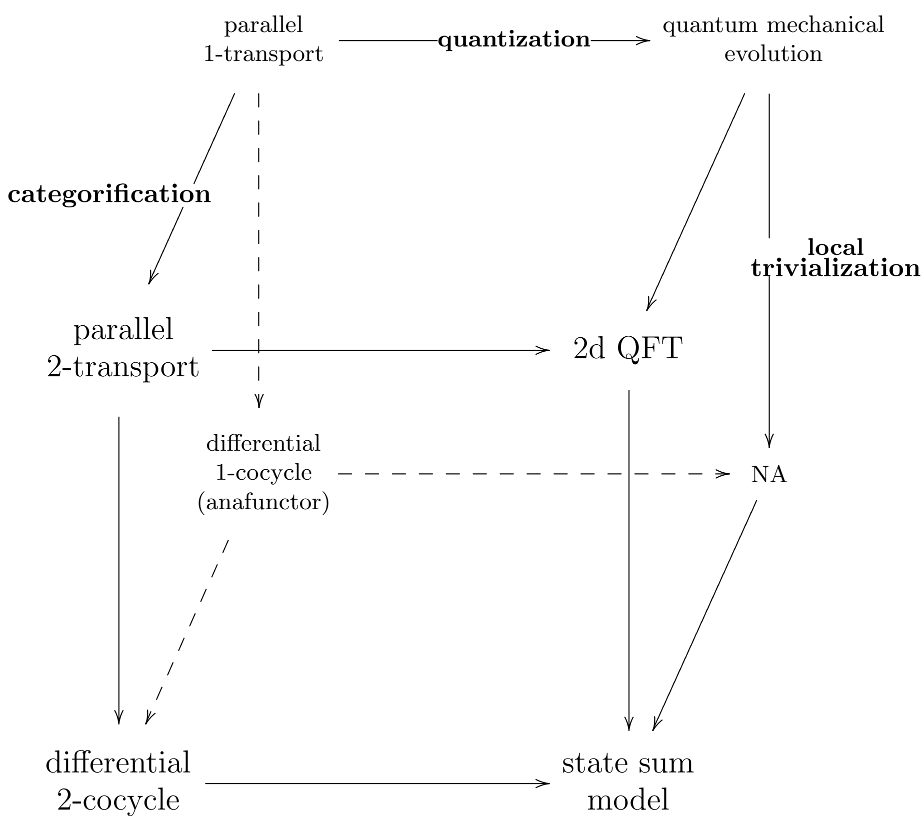


Figure 1: **Quantization, categorification and local trivialization** are the three procedures relating n-vector n-transport that play a role in the local description of n-dimensional quantum field theory. Categorification sends n-transport to $(n+1)$-transport. Quantization sends n-transport on n-paths in configuration space to n-transport on abstract n-paths (parameter space). Local trivialization sends n-transport on globally defined n-paths to n-transport on local n-paths glued by n-transitions.

by unitary operators on H, obtained by the generalized Feynman-Kac formula. $U(t)$ is said to describe **time evolution** or **propagation** of the state of the charged particle.

If we allow ourselves to be slightly more sophisticated, we say that propagation in quantum mechanics is a functor

$$U : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Hilb}$$

from 1-dimensional Riemannian cobordisms to Hilbert spaces.

This way, quantization (of the charged particle) is a procedure that associates to a functor on paths in X with values in vector spaces a functor on abstract 1-dimensional cobordisms.

The charged 2-particle. There is a more or less obvious 2-category

$$\mathcal{P}_2(X)$$

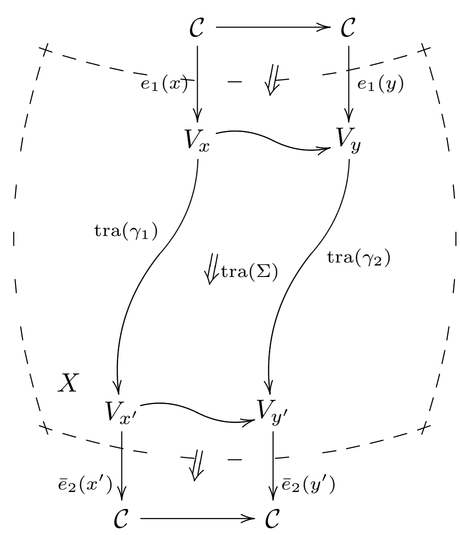
of **2-paths** in X – these are essentially just surfaces cobounding 1-paths in X – and there are several notions of what one might call

$$2\mathrm{Vect}\,,$$

the 2-category of **2-vector spaces**. Fixing any such notion we are lead to consider 2-functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to 2\mathrm{Vect}$$

that describe **parallel 2-transport**



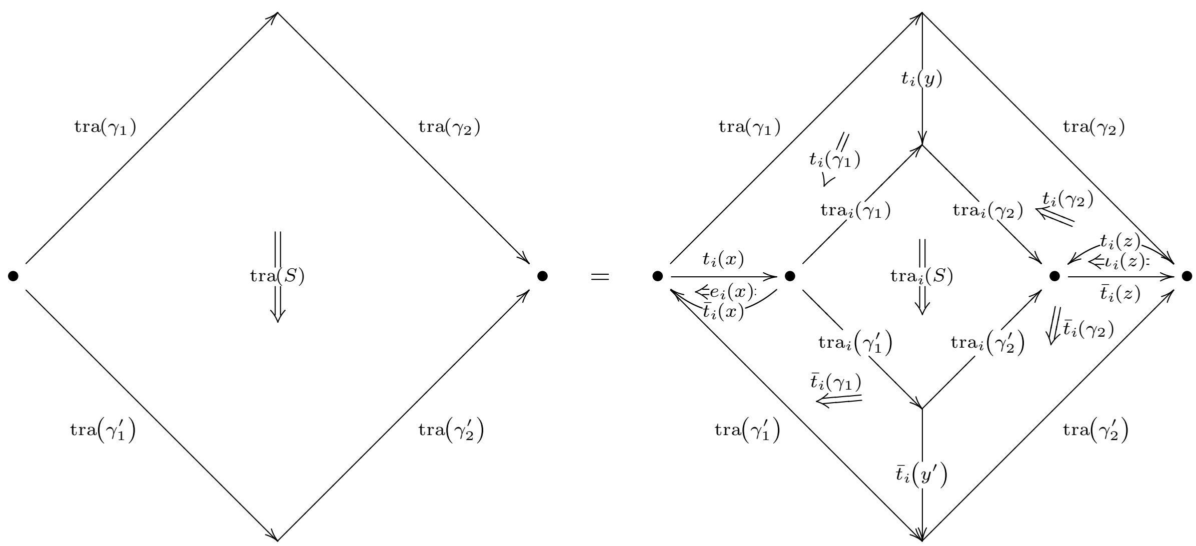
across surfaces Σ.

Categorifcation of quantum propagation. Categorifying propagation 1-functors on 1-dimensional cobordisms with values in vector spaces should lead us to 2-functors on abstract bigons with values in 2-Hilbert spaces.

These should be thought of as refinements of 1-functors on 2-cobordisms with values in Hilbert spaces.

?? provide more details here ?? ?? compare the approach by Stolz-Teichner ??

Local trivialization.



?? provide more details ??

State sum models. *?? see figure ??*

The FRS theorem. The FRS theorem uses state sum internal to modular tensor categories more general than Vect in order to describe not topological, but conformal 2-dimensional field theory.

2-dimensional (rational) conformal field theories are are encoded in special symmetric Frobenius algebra objects internal to a modular tensor category.

The algebra A itself is the space of open string states.

Modules for A descibe boundary conditions, also known as **D-brane**s.

Bimodules for A describe **defect lines**.

Morphisms of twisted modules descibe **boundary field insertions**.

Morphisms of twisted bimodules descibe **bulk field insertions**.

The corresponding topological state sum model computes correlators for the conformal field theory.

The FRS description of disk and annuli correlators. The FRS construction crucially also involves 3-dimensional topological field theory and surgery on the 3-sphere. But all genuinely 2-dimensional ingredients of the formalism already appear in the description of disk correlators.

The disk correlator from locally trivialized 2-transport. The description of the disk correlator in conformal field theory by the FRS theorem can be understood from 2-vector 2-transport with values in twisted bimodules.

?? say more ??

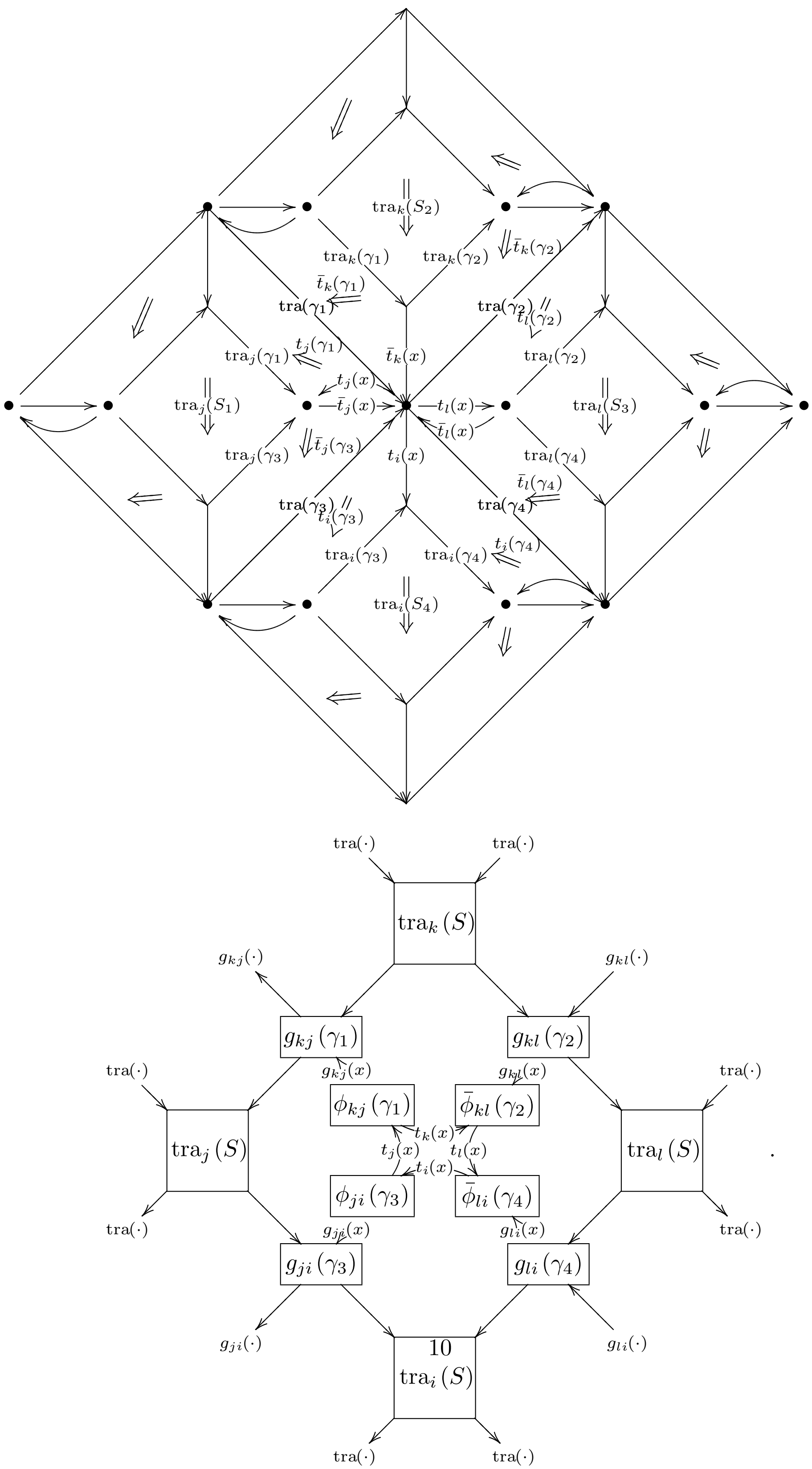
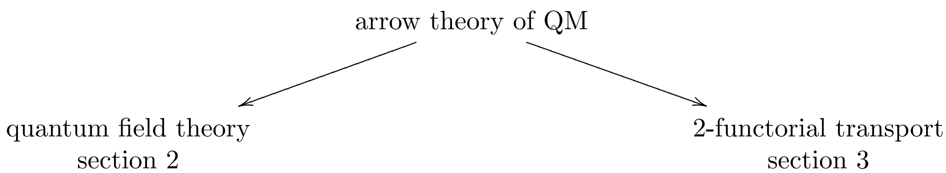


Figure 2: **Local trivialization** of 2-functors induces local decorations by Frobenius algebroids like those appearing in state sum models of 2-dimensional QFT.

1.2 A Rosetta stone: arrow theory of quantum mechanics

This section sets up a correspondence between the physics and the mathematics to follow.



The suggestion is that, according to taste, you start with one of the following sections, then pull yourself back to this one here and see if that helps pushing forward towards the remaining one.

Quantum theory terminology. We find the following concept formation useful and natural.

A **2-transport**, be it the **parallel transport** in a 2-bundle with connection, or the **propagation** in 2-dimensional quantum field theory we write as

$$\mathrm{tra} : \mathcal{P}_2 \to T\,.$$

Here $\mathcal{P}_2$ is a 2-category modelling 2-paths in some space. When we speak just of 2-bundles, this is **base space**. When we think of the 2-bundles as **background field**s in quantum field theory, this is **target space**.

The codomain T is the category of fibers of the 2-bundle. From the point of view of quantum field theory, a morphism in here is a **phase**.

Usually, T is equipped with a monoidal structure. That makes morphisms into T inherit this monoidal structure. We denote by

$$1 \in [\mathcal{P}_2, T]$$

the tensor unit among these morphisms. Physically speaking, 1 is the **vanishing background field**.

The space of generalized objects of a 2-bundle tra with connection

$$\mathrm{sect}_{\mathrm{fl}} \equiv [1, \mathrm{tra}]$$

is the space of **flat sections**. In physics, this is the space of **ground state**s.

n-dimensional quantum field theory describes the propgation of $(n-1)$-dimensional entities. For $n = 1$ these are called **particles**. For $n = 2$ they are sometimes called **strings**. Here we shall call them, more generally, n**-particle**s.

In our language a **2-particle** is a 1-category generated from a single non-trivial morphism. The category

$$\mathrm{par}_{\mathrm{opn}} = \{a \to b\}$$

models the **open 2-particle**. The categories

$$\mathrm{par}_{\mathrm{clsd},1} = \Sigma(\mathbb{Z})$$

and

$$\mathrm{par}_{\mathrm{clsd},2} = \Sigma(\mathbb{N})$$

model the **closed 2-particle**. In the language of σ-models, these categories play the role of **parameter space**.

A morphism

$$\gamma : \mathrm{par} \to \mathcal{P}_2$$

from parameter space to target space is a **field configuration**. This is where **field theory** gets its name from. The morphisms of the category

$$\mathrm{Cob} = [\mathrm{par}, \mathcal{P}_2]$$

are embedded **cobordisms**. Physically they correspond to **trajectories** of the 2-particle in its configuration space. Accordingly, embeddings of subcategories

$$\mathrm{conf} \xrightarrow{\subset} [\mathrm{par}, \mathcal{P}_2]$$

essentially surjective on objects are called **configuration space**s. The isomorphisms in conf are those that relate **gauge equivalent configurations**. The morphisms not in conf are the **physical trajectories**.

By postcomposition, the background field tra **transgresses** to configuration space

$$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, T]\,.$$

A generalized object of this,

$$e : 1_* \to \mathrm{tra}_*\,,$$

is a **section** of the 2-bundle with respect to conf. For the quantum theory, such a section is known as a **state**.

The space of all sections

$$\mathrm{sect}_{\mathrm{conf}} = [1_*, \mathrm{tra}_*]$$

is, accordingly, the **space of states**.

The space of states is naturally acted on by

$$\mathrm{obs} = \mathrm{End}(1_*)\,.$$

This is the monoid of (position) **observable**s. Usually we have

$$\mathrm{obs} = [\mathrm{par}, \mathrm{obs}_{\mathrm{loc}}]$$

in which case $\mathrm{obs}_{\mathrm{loc}}$ are the **local observables**. The space of states is moreover naturally acted on by

$$G = \mathrm{Aut}(\mathrm{tra}_*)\,.$$

$$
\begin{array}{c} \cdot \\ \big\downarrow{\scriptstyle t} \\ \cdot \end{array}
\qquad\qquad
\begin{array}{c} \mathbb{C} \\ \big\downarrow{\scriptstyle e_1} \\ H \\ \big\downarrow{\scriptstyle \exp(it\Delta)} \\ H \\ \big\downarrow{\scriptstyle \bar{e}_2} \\ \mathbb{C} \end{array}
\;=\; \langle e_2 | \exp(it\Delta) e_1 \rangle
$$

Figure 3: The 0-disk and the 0-disk correlator with two boundary insertions in 1-dimensional quantum field theory.

This is the **group of local gauge transformations**.

Similarly, there is the space of **co-sections**

$$\mathrm{cosect}_{\mathrm{conf}} = [\mathrm{tra}_*, 1_*] \,.$$

Physically, these, or rather the natural pairing

$$(\cdot,\cdot) : \mathrm{cosect} \times \mathrm{sect} \to \mathcal{C}$$

corresponds to **measurements**. The pairing should be thought of as being the image of the identity under the Hom

$$(\cdot,\cdot) : \; \mathrm{cosect} \times \mathrm{sect} \xrightarrow{\mathrm{Hom}(\cdot,\cdot)} [\mathrm{End}(\mathrm{tra}_*), \mathrm{End}(1_*)] \xrightarrow{\mathrm{ev}(\cdot,\mathrm{Id})} \mathcal{C} \;.$$

On the space of sections, equipped with the above pairing, physical processes act as linear operators. An operator $T : \mathrm{sect} \to \mathrm{sect}$ has an **adjoint** $T^\dagger : \mathrm{cosect} \to \mathrm{cosect}$ if

$$(\bar{e}_2, T e_1) \simeq (T^\dagger \bar{e}_2, e_1)$$

for all sections e_1 and cosections e_2.

An important example is the translation along a flow in in configuration space.

Flows. We formulate the arrow theory of a **flow along a vector field**.

Let $\mathcal{P}_1$ be a category. Let

$$\mathrm{F}(\mathcal{P}_1) \subset \Sigma(\mathrm{Aut}(\mathcal{P}))$$

be the category whose single object is $\mathcal{P}_1$, and whose morphisms are natural

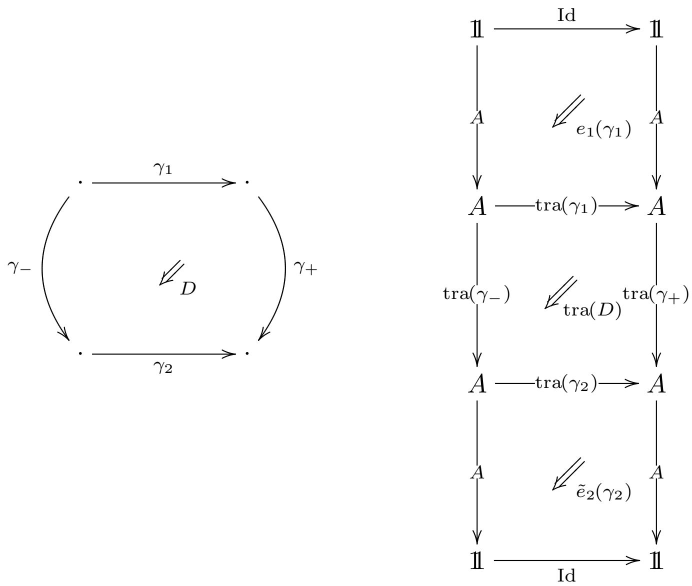
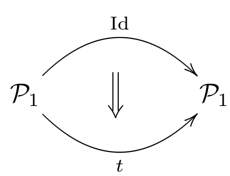


Figure 4: The disk and the disk correlator with two boundary insertions in 2-dimensional quantum field theory.

transformations

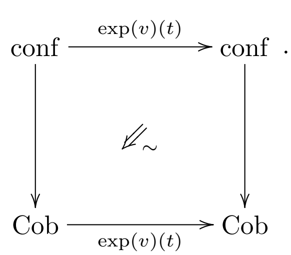


with composition being horizontal composition of natural transformations.

Definition 1 *For R some group, an R-**flow** on $\mathcal{P}_1$ is a functor*

$$\exp(v) : \Sigma(R) \to F(\mathcal{P}_1)\,.$$

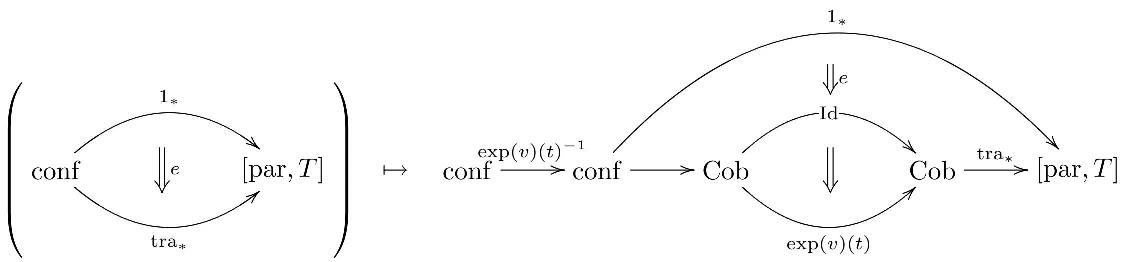
An R-flow on Cob is compatible with the configuration space symmetries if



In that case, the R-flow $\exp(v)$ defines, for any $t \in R$, a **translation operator**

$$\exp(v)(t) : \mathrm{sect} \to \mathrm{sect}$$

on the space of states, which sends any section e to



Given a section e_1 and a cosection $\bar{e}_2$, the expression

$$(\bar{e}_2, \exp(v)(t)e_1)$$

provides us with a measure for the **correlator** after translation along v, with **boundary insertions** e_1 and e_2

For a simple **example** try example 1 below.

2 Background on low-dimensional quantum field theory

Example 1 (quantum mechanics of the charged point particle)

In physics, the study of what is called the quantum mechanics of the charged particle involves the following ingredients.

There is a point, $\{\bullet\}$, supposed to model an elementary particle.

There is a smooth, Riemannian space X, called the **target space** and supposed to model the physical space that the particle propagates in.

There is a hermitean vector bundle $E \to X$ with connection ∇ on X, called a **background field** and supposed to model a physical gauge field, like the electromagnetic field.

A **configuration** of this physical system is a map from the particle to target space, $c : \{\bullet\} \to X$, modelling the idea of a physical state where the particle is found at the point $c(\bullet)$ in configuration space.

Accordingly, the space of maps $[\{\bullet\}, X]$ is called the **configuration space** of the system. For the point particle, the configuration space coincides with target space.

To a path in configuration space, modelling a **trajectory** of the point particle, the connection ∇ associates, by parallel transport, a morphism of hermitean vector spaces. This is called the **phase** associated to the given path.

These ingredients are known as the **classical** aspects of the physical system. From them, one finds the **quantum** aspects, for instance by applying geometric quantization.

The bundle (E, ∇) on target space can be transgressed to a bundle with connection on configuration space. For the point particle this step is empty.

Combining the Riemannian structure on X with the hermitean structure on E, the space of sections $\Gamma(E)$ of the bundle on configuration space inherits a scalar product. Completing with respect to this yields a Hilbert space of sections, called the **space of states** of the quantum particle.

From the phase associated to each path in configuration space we obtain an operator $\Delta = \nabla_E^\dagger \nabla_E$ on the space of states, called the **Hamiltonian**. It gives rise to a 1-parameter familiy of operators, $U(t) = \exp(it\Delta)$, called the **propagator** and modelling the operation of propagating quantum states through time.

For the present case, the integral kernel of this operator can rigorously be expressed as the phase integrated over all paths in configuration space connecting two given configurations.

This setup is called the quantum mechanics of the charged point particle.

Remark. In conclusion, the quantization step sends a parallel vector transport on target space

$$(\, x \xrightarrow{\gamma} y \,) \quad \mapsto \quad (\, E_x \xrightarrow{\mathrm{tra}_\nabla(\gamma)} E_y \,)$$

to a vector transport on parameter space

$$(\bullet \xrightarrow{[0,t]} \bullet) \quad \mapsto \quad (\Gamma(E) \xrightarrow{\quad U(t) \quad} \Gamma(E)) \,.$$

2.1 Functorial QFT

Similar considerations as in example 1 have lead people to a similar characterization of the structures appearing in d-dimensional quantum field theory as follows:

Definition 2 *Let $d\mathrm{Cob}_S$ be a symmetric monoidal category of d-dimensional cobordisms equipped with some extra structure S. Then a* **d-dimensional quantum field theory** *with respect to S is a monoidal functor*

$$U : d\mathrm{Cob}_S \to \mathrm{Hilb}\,.$$

Various obvious slight modifications of this definition can be considered and have been considered. In particular, the codomain is sometimes taken to be not Hilbert spaces, but just vector spaces.

One of the simplest nontrivial and best known examples is 2-dimensional topological field theory.

Example 2 (closed 2-dimensional topological field theory)

Let 2Cob be the category whose objects are disjoint unions of the circle with itself, and whose morphisms are diffeomorphism classes of oriented 2-manifolds cobounding these circles. Since in this category all morphisms are completely characterized by the topology of any manifold representing that morphism, representations of this category are addressed as **topological** field theories.

Proposition 1 *The category of functors*

$$U : 2\mathrm{Cob} \to \mathrm{Vect}$$

is equivalent to that of commutative Frobenius algebras.

Remark. This result can be understood both from a global, as well as from a local perspective.

Global Perspective. Globally, proving this statement amounts to realizing that gluing 3-holed spheres corresponds, under the functor U, to taking associative products and coproducts on the vector space $A = U(S^1)$ associated by U to a single copy of the circle. The disk then maps to a unit and counit on A, and topological invariance implies the compatibility of the product and coproduct with these units as well as the Frobenius property.

Local Perspective. It turns out that we can think of this functor also by triangulating any cobordism and suitably decorating the resulting graph with certain local data.

Observation 1 (Fukuma, Hosono, Kawai) *Choose any special Frobenius algebra A (not necessarily commutative). A 2-dimensional topological field theory is then obtained by choosing on any cobordims an oriented dual triangulation, labelling edges of that with A and trivalent vertices with the product or coproduct in A, as required. The resulting morphisms $A^{\otimes n} \to A^{\otimes m}$ then constitute a functor $U : \mathrm{2Cob} \to \mathrm{Vect}$ that is well defined, and in particular independent of the choices involved in its construction.*

It turns out that the commutative Frobenius algebra A_s of the global picture arises as the *center* of the Frobenius algebra A_l in the local picture. When we generalize the cobordisms in the domain and also admit cobordisms between open intervals, then our functor $U : \mathrm{2Cob}_{\mathrm{oc}} \to \mathrm{Vect}$ will assign A_l to the open interval, A_s to the circle and assign to any open-closed cobordism a morphisms obtained by local data as above.

This statement has been turned into a rigorous theorem, by Lauda and Pfeiffer.

Theorem 1 (Lauda,Pfeiffer) *Open/closed 2-dimensional topological field theories are equivalent to knowledgeable Frobius algebras.*

?? discuss this in more detail ??

There are various ways to think about such decorated graphs. Similar structures are sometimes called **Wilson networks** or **spin networks**. We will reencounter the general mechanism here when we talk about the local description of **parallel surface transport** in vector 2-bundles (or in gerbes).

It turns out that the subset of quantum field theories that are both interesting and at the same time tractable is rather small. This phenomenon has lead to a wide gap between the development of quantum field theory in physics and in mathematics.

To some extent, the only physically interesting quantum field theories that are also mathematically well understood are the topological ones. However, progress has been made in extracting the topological essence of non-topological quantum field theories.

Example 3 (2-dimensional conformal field theory)

The next best thing after topological cobordisms are conformal cobordisms.

There are already technical difficulties with constructing a category $\mathrm{2Cob}_{\mathrm{conf}}$ of conformal 2-dimensional cobordisms. The naive identity morphisms do not exist.

One can either try to deal with this problem, or else be content with working with a notion of category without requiring identity morphisms. Either way, we would then say

Definition 3 (G. Segal) *A 2-dimensional conformal field theory is a functor*

$$U : 2\mathrm{Cob}_{\mathrm{conf}} \to \mathrm{Vect}\,.$$

Here Vect in general denotes topological vector spaces.

Actually, such a functor is, more precisely, a 2-dimensional conformal field theory of vanishing central charge. More generally, one takes the functors U to be just *projective*, involving a multiple of a cocylce, known as the Liouville action, by a factor c, known as the central charge.

It turns out that understanding such functors is hard. A great advance has been obtained by Fuchs, Runkel and Schweigert, Fjelstad and Fröhlich, in the rational case. They noticed that rational conformal field theory is essentially like topological conformal field theory - but *internalized* not in Vect, but in some modular tensor category $\mathcal{C}$.

2.2 The FRS theorem solving rational conformal field theory.

Theorem 2 (FFRS) *Let V be a vertex operator algebra, such that $\mathcal{C} = \mathrm{Rep}(V)$ is a modular tensor category.*

Then any (special symmetric) Frobenius algebra object A internal to $\mathcal{C}$ defines a 2-dimensional rational conformal field theory $U_A : 2\mathrm{Cob}_{\mathrm{conf}} \to \mathrm{Vect}$.

The FFRS theorem allows us to split, schematically,

$$\begin{aligned}
\text{(R)CFT} &= \text{complex analytic data} + \text{topological data} \\
&= \text{chiral data} + \text{sewing constraints} \\
&= \mathrm{Rep}(V) + (A \in \mathrm{Obj}(\mathrm{Rep}(V)))\,.
\end{aligned}$$

It is known how knowledge of V alone allows to compute spaces of "pre-correlators", or "conformal blocks" associated to each extended conformal surface. These are spaces of functions that potentially encode the value of the quantum field theory functor on that surface, obtained by taking into account just the local symmetries.

The construction of a full conformal field theory then amounts to picking, in a consistent fashion, for each extended conformal surface one of its asscociated pre-correlators, such that this assignment conspires to form a functor on $2\mathrm{Cob}_{\mathrm{conf}}$.

The FRS theorem tells us that this last step is purely topological in nature, and that there exists a topological field theory which computes consistent choices of conformal blocks.

This topological field theory is constructed essentially in the same local way as described by Fukuma, Hosono, Kawai and Lauda, Pfeiffer, the only difference being that where before we decorated graphs with algebra objects internal to Vect, we now decorate them with algebra objects internal to a potentially more general modular tensor category $\mathcal{C}$.

We say this again, slightly more detailed:

given a **conformal cobordism** (X, g), find in the vector space

$$\mathrm{Hom}(\partial_{\mathrm{in}}(X,g), \partial_{\mathrm{out}}(X,g)) \simeq (\partial_{\mathrm{in}}(X,g) \otimes \partial_{\mathrm{out}}(X,g)^*)^*$$

the **correlator** of a 2d CFT

↓

given the data of a **chiral CFT** in terms of a **vertex operator algebra** V, it is sufficent to look at the subspace

$$B_V(X,g) \subset (\partial_{\mathrm{in}}(X,g) \otimes \partial_{\mathrm{out}}(X,g)^*)^*$$

of **conformal blocks**

↓

the $B_V(X,g)$ form a projective vector bundle with flat connection over the moduli space of conformal structures on X;
it is sufficient to consider the space

$$V(X)$$

of **flat sections** of this vector bundle

↓

FFRS theorem:
the true correlator, regarded as an element of $V(X)$, is the correlator of a 3d TFT on an extended 3-manifold with boundary X

&

this extended 3-manifold is a fattened version of a Wilson network of a 2d TFT internal to $\mathrm{Rep}(V)$

Table 2: **The main idea of the FFRS theorem**. Imposing chiral symmetries on a 2-dimensional conformal field theory allows to decouple the dependence on the conformal structure from the global behaviour under gluing of cobordisms.

Correlator. The image of any cobordism X under the QFT functor is a morphism $V_{\mathrm{in}} \xrightarrow{f(X)} V_{\mathrm{out}}$ of vector spaces. The image of this morphism under the isomorphism

$$\mathrm{Hom}(V_{\mathrm{in}}, V_{\mathrm{out}}) \simeq V_{\mathrm{in}}^* \otimes V_{\mathrm{out}} \simeq (V_{\mathrm{in}} \otimes V_{\mathrm{out}}^*)^*$$

is called the correlator of X.

Sewing and Factorization. For certain choices of extra structure S, cobordisms with that extra structure do not provide naive identity morphisms and hence do not form categories with identities in the obvious way. One way to reformulate the desired functoriality property without having to use identity morphisms is this:

Given any cobordism X, and given a way to cut it such as two obtain two new boundary components, one incoming and one outgoing, factorization is the demand that the morphism associated to the full cobordism is the obvious trace of the morphism associated to the cobordism obtained after cutting.

2-dimensional Conformal Field Theory. A representation of the category of 2-dimensional oriented cobordisms with conformal structure and collared boundary components is called a conformal field theory of vanishing central charge.

More generally, one is interested in functors that respect conformal rescalings only projectively. A conformal field theory of **central charge** c is a representation of Riemannian cobordisms such that the correlators of two Riemannian surfaces whose metrics differs by by a conformal factor e^σ differ by the factor $e^{cS[\sigma]}$, where S is the **Liouville action functional**.

Chiral 2d Conformal Field Theory. A chiral conformal field theory is one for which the vector spaces assigned to boundary components are modules of a **vertex operator algebra** V and whose correlators take values in the space of **conformal blocks**

$$B(\ \partial_{\mathrm{in}}(X,g) \xrightarrow{X} \partial_{\mathrm{out}}(X,g)\ ) \subset (V_{\mathrm{in}} \otimes V_{\mathrm{out}}^*)^*\,,$$

of V. This is a space of invariants of the action of V on its modules, with respect to X.

Conformal blocks can be thought of as **pre-correlators** that are compatible with the **local symmetries** of the conformal field theory, but from which the true correlators compatible with the global factorization property still need to be picked.

The spaces of conformal blocks form a projective vector bundle over the moduli space of conformal structures on a given X. This vector bundle naturally carries a flat connection, the **Knizhnik-Zamolodchikov connection**. The space

$$V(X)$$

of *flat sections* of the bundle of spaces of conformal blocks with respect to this connection is hence a vector space we may associate to a topological cobordism X.

The insight underlying the FFRS theorem is: *picking the true correlators of a full conformal field theory from the space of conformal blocks of a chiral conformal field theory is equivalent to constructing a certain topological field theory that assigns to each topological cobordism X an element in $V(X)$.*

2.3 n-functorial quantum field theory?

The fact that field theories conceived as representations of cobordism categories can have a local description, in which data is assigned to pieces of cobordisms, is a first indication that we may want to find a refinement of that definition.

Excision for elliptic objects. One of the intended applications of Segal's definition of conformal field theory was a geometric description of elliptic cohomology. In that context, one considers conformal cobordisms equipped with the extra structure of a map from the cobordism into some fixed space X.

For this to have a chance of being applicable to a generalized cohomology theory like elliptic cohomology, one needs to have a notion of locality with respect to X. This, however, is necessarily violated by functors $2\mathrm{Cob}^X_{\mathrm{conf}} \to \mathrm{Vect}$.

Observation 2 (Stolz,Teichner) *In order for Segal's definition of conformal field theory to be useful for the description of elliptic cohomology, one needs to refine 1-functors on cobordisms to 2-functors on a 2-category of surface elements.*

All this motivates

Definition 4 *Let* 2Vect *be some flavor of a 2-category of 2-vector spaces and let $\mathcal{P}_2$ be a 2-category that models 2-dimensional geometric structures. Then a* **2-vector 2-transport** *on $\mathcal{P}_2$ is a 2-functor*

$$\mathrm{tra} : \mathcal{P}_2 \to 2\mathrm{Vect}\,.$$

	$\mathrm{tra} : \mathcal{P}_2 \to 2\mathrm{Vect}$	
local trivialization	adjoint equivalence	special ambijunction
description	parallel surface transport	propagation in 2d QFT
domain	target space	parameter space
as morphism of 3-transport	transition gerbe	with field insertions

Table 3: **2-Vector transport** describes parallel surface transport in a 2-vector bundle (a gerbe) with connection; but also evolution (propagation) in 2-dimensional quantum field theory.

3 2-Functorial Quantum Field Theory

We have said that topological 2-dimensional field theory can be constructed from dual triangulations decorated with Frobenius algebras in Vect.

Rational conformal 2-dimensional field theories can be constructed from dual triangulations decorated with Frobenius algebras internal to a modular tensor category.

Line bundle gerbes with connection can be constructed from dual triangulations decorated in something like Frobenius algebroids.

All three of these are examples of locally trivializable 2-transport.

3.1 2-Transport

Definition 5 *Let X be some smooth space and let $p : U \to X$ be a surjective submersion. The 2-category*

$$\mathcal{P}_2(U^\bullet)$$

of **2-paths in the transition 2-groupoid** *is generated from 2-paths in U, 1-paths in $U^{[2]}$ and 0-path in $U^{[3]}$, subject to relations which make $U^{[2]}$ a Frobenius algebroid.*

Proposition 2 (KW,usc) *2-paths in X are equivalent to 2-paths in the transition 2-groupoid*

$$\mathcal{P}_2(X) \simeq \mathcal{P}_2(U^\bullet)\,.$$

Definition 6 *Let* $T' \xrightarrow{\;i\;} T$ *be a morphism of 2-categories and let* $\mathrm{tra} : \mathcal{P}_2(X) \to T$ *be a 2-functor. We say that* tra *is* **p-locally i-trivializable** *if there exists*

$$\begin{array}{ccc} \mathcal{P}_2(U) & \xrightarrow{\;p\;} & \mathcal{P}_2(X) \\ {\scriptstyle \mathrm{tra}_U}\downarrow & \overset{\sim}{\swarrow}{\scriptstyle t} & \downarrow{\scriptstyle \mathrm{tra}} \\ T' & \xrightarrow[\;i\;]{} & T \end{array}$$

such that t fits into a special ambidextrous adjunction.

Proposition 3 *Every p-locally i-trivializable 2-functor* tra *on $\mathcal{P}_2(X)$ gives rise to a 2-functor on $\mathcal{P}_2(U^\bullet)$ that coincides with $i_*\mathrm{tra}_U$ on $\mathcal{P}_2(U)$.*

Remark. More is true. There is an equivalence of locally trivializable 2-functors on $\mathcal{P}_2(X)$ with suitable 2-functors on $\mathcal{P}_2(U^\bullet)$. This can be understood as saying that locally trivializable 2-functors form a 2-stack.

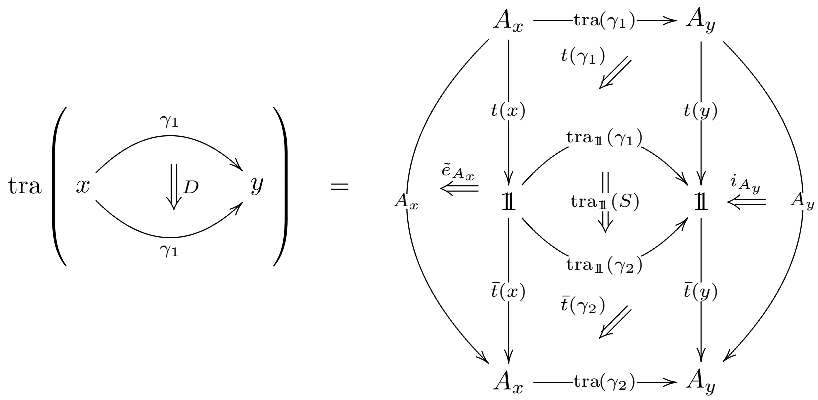


Figure 5: If the **trivialization** tra $\xrightarrow{t}$ $\mathrm{tra}_{\mathbb{1}}$ is by a ***special*** **ambidextrous adjunction** we can express tra entirely in terms of $\mathrm{tra}_{\mathbb{1}}$ and the trivialization data.

3.2 Zoo of 2-Bundles with connection: parallel surface transport

Several kinds of 2-bundles ($\sim$ "gerbes") with connection arise from 2-transport that is locally trivializable not just by some special ambijunction – but by an adjoint equivalence.

Heuristically, the fact that the local trivialization is an equivalence implies that the global 2-transport is obtained from locally gluing **typical fibers**.

We shall adopt the slightly abusive but convenient terminology of addressing the very 2-functor

$$\mathrm{tra} : \mathcal{P}_2 \to T$$

as a **2-bundle with connection** if it has local trivializations by adjoint equivalences. We do not explicitly consider the **total space** of such a 2-bundle, whatever that might be.

Simply by choosing different morphisms $i : T' \longrightarrow T$ we obtain various kinds of 2-bundles with connection.

Principal 2-transport.

Example 4 (bibundle gerbe) *Let* $G_2 = \mathrm{AUT}(H)$ *be the automorphism 2-group of a group* H*. Transition data of parallel 2-transport with respect to the canonical embedding*

$$i : \Sigma(\mathrm{AUT}(H)) \to \Sigma(H\mathrm{BiTor})$$

is equivalent to Aschieri-Jurčo principal **bibundle gerbes** *with fake-flat connection.*

Remark. This has an immediate generalization to arbitrary strict 2-groups.

Remark. The equivalence is actually a canonical isomorphism. A local i-trivialization as above *is* a bibundle gerbe with fake-flat connection. For instance, the transition $g : p_1^*\mathrm{tra} \to p_2^*\mathrm{tra}$ *is* a transition bibundle equipped with the special kind of twisted connection that is described by Aschieri-Jurčo. Analogous remarks apply to the following examples.

As a special case we get

Example 5 ($U(1)$-principal bundle gerbe) *Let $G_2 = \Sigma(\Sigma(U(1)))$ be the double suspension of $U(1)$. Transition data of parallel 2-transport with respect to the canonical embedding*

$$i : \Sigma(\Sigma(U(1))) \to \Sigma(U(1)\mathrm{Tor})$$

*is equivalent to principal $U(1)$***-bundle gerbes** *with connection.*

Remark. We say "bundle gerbe with connection" where one sometimes sees "with connection and curving". There is no place in this world for a bundle gerbe with connection but without a notion of "curving".

Remark. The *fake*-flatness condition disappears in the abelian case.

Notice that we did not use the nontrivial automorphism of $U(1)$ in the above example. In fact

Example 6 ($U(1)$-principal bundle gerbe over unoriented surfaces) *Let $G_2 = \Sigma(\mathrm{AUT}(U(1))) = (U(1) \to \mathbb{Z})$ be the automorphism 2-group of $U(1)$. Parallel 2-transport locally trivial with respect to the canonical embedding*

$$i : \Sigma(\mathrm{AUT}(U(1))) \to \Sigma(U(1)\mathrm{Tor})$$

admits certain $\mathbb{Z}_2$-equivariant structures – known as **Jandl structures** *– that allow to define holonomy on unoriented surfaces.*

A bundle gerbe is a transition bundle. We may further trivialize these transition bundles to obtain full cocycle data.

Example 7 (JB,usc: nonabelian differential cocycle data) *Let G_2 be any strict 2-group. Transition data of 2-transport with respect to $i = \mathrm{Id}_{\Sigma(G_2)}$ is equivalent to the local nonabelian cocycle data of a G_2-gerbe with fake flat connection.*

The **fake** flatness we encounter everywhere is a phenomenon not visible in ordinary parallel transport along paths. It is a manifestation of the respect of the transport 2-functor for the vertical composition in the target 2-category. For some applications, fake flatness is just what we want:

Example 8 (BF-theory) *Solutions of the equations of motion of G-BF-theory are are fake flat G_2-transport functors for $G_2 = (G \to G)$.*

Relaxing the fake flatness constraint amounts to passing from the transport codomain locally being a strict 2-group to higher categorical groups.

Example 9 (Breen-Messing data) *Let $G_3 = \mathrm{INN}(G_2)$ be the 3-group of inner automorphisms of a strict 2-group G_2. Transition data of 2-transport with respect to $i = \mathrm{Id}_{\Sigma(G_3)}$ is equivalent to the local nonabelian cocycle data of gerbes with connection as given by Breen-Messing.*

Remark. At the infinitesimal level, where groupoids and their morphisms are replaced by algebroids and their morphisms, this has been noticed by Danny Stevenson. He relates it to **higher Schreier theory**. Ordinary Schreier theory says that extensions of groupoids

$$K \to G \to B$$

are classified by *pseudo*-functors from the 1-groupoid B to the 2-groupoid $\mathrm{AUT}(K)$. Recall that a principal G-bundle $P \to X$ can be conceived in terms of its exponentiated **Atiyah sequence** of groupoids

$$\begin{array}{ccccc}
\mathrm{AUT}(\mathrm{Ad}P) & & & & \\
& & \nwarrow^{(\nabla, F_\nabla)} & & \\
\mathrm{Ad}P & \longrightarrow & \mathrm{Trans}(P) & \longrightarrow & X \times X \\
\| & & \| & & \| \\
P \times_G G & \longrightarrow & P \times_G P & \longrightarrow & X \times X
\end{array} .$$

A section ∇ on P with curvature F_∇ is a pseudofunctor from the pair groupoid $X \times X$ to $\mathrm{AUT}(\mathrm{Ad}(P))$, but locally taking values only in inner automorphisms. The curvature F_∇ of ∇ provides the compositor for this pseudofunctor. The Bianchi identity corresponds to the coherence for the compositor.

Associated 2-vector transport We obtain associated 2-transport by choosing the morphism $i : T' \to T$ to be a representation of a 2-group on 2-vector spaces. There are several notions of 2-vector spaces. For the moment, let a 2-vector space be a $\mathrm{Vect}_{\mathbb{C}}$-module category. We will only be interested in the image of the canonical embedding $\mathrm{Bim}(\mathrm{Vect}_C) \to {}_{\mathrm{Vect}_C}\mathrm{Mod}$.

Proposition 4 (the canonical 2-representation) *For $G_2 = (H \to G)$ any strict 2-group, and $\rho : \Sigma(H) \to \mathrm{Vect}_{\mathbb{C}}$ any ordinary representation, there is a canonical 2-representation*

$$\tilde{\rho} : \Sigma(G_2) \to \mathrm{Bim}(\mathrm{Vect})\,.$$

This $\tilde{\rho}$ represents G_2 on the category of modules of the algebra spanned by the image of ρ.

Example 10 (line-2-bundle) *Let $E \to X$ be a $\mathrm{PU}(H)$-bundle on X. Using $PU(H) \simeq \mathrm{Aut}(K(H))$, we canonically associate to it a bundle $A \to X$ of algebras of compact operators. A connection on that bundle gives rise to a transport 1-functor*

$$\mathcal{P}_1(X) \to \mathrm{Bim}(\mathrm{Vect}_{\mathbb{C}})\,.$$

Extending this to a 2-functor

$$\mathcal{P}_2(X) \to \mathrm{Bim}(\mathrm{Vect}_{\mathbb{C}})$$

yields a line-2-bundle with connection. This is associated to a principal $U(1)$-bundle gerbe by the canonical rep of $\Sigma(U(1))$. Locally i-trivializing this we obtain the line bundle gerbe with connection classified by the original $PU(H)$-bundle.

Example 11 (line bundle gerbe with connection) *Parallel 2-transport locally trivialized with respect to the canonical embedding*

$$i : \Sigma(\Sigma(U(1))) \to \Sigma(\mathrm{Vect}_{\mathbb{C}})$$

is equivalent to line bundle gerbes with connection.

Example 12 (string bundle) *Let $G_2 = \mathrm{String}_G = (\hat{\Omega}_k G \to PG)$ be the strict version of the String 2-group, for G a compact simple and simply connected Lie group and $k \in \mathbb{Z}$ a level. For any positive energy rep of $\hat{\Omega}_k G$ the above construction of the canonical 2-rep should go through. As a result, we would get a notion of a connection on a String_G-bundle.*

We can also consider locally trivializable 2-transport with values in higher dimensional vector spaces, but the local trivialization will now just be a special ambijunction, essentially expressing the duality between a vector space V and its dual V^*.

Example 13 (FHK from locally trivialized 2-transport) *The FHK decoration prescription is that of a p-locally i-trivialized 2-vector transport for*

$$i : \{\bullet\} \to \Sigma(\mathrm{Vect})\,.$$

3.3 Classical theory: sections, phases and holonomy

Example 14 (sections of 1-vector bundles)

Let $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ be a vector bundle with parallel transport. Let $1 : \mathcal{P}_1(X) \to \mathrm{Vect}$ be the tensor unit in the category of all such functors, i.e. the functor wich sends every path to the identity on the ground field. Then morphisms

$$1 \to \mathrm{tra}$$

are in bijection with flat sections of the vector bundle.

We can restrict both 1 and tra to the discrete category on the collection of objects of $\mathcal{P}_1(X)$ to obtain 1_* and tra_*. The morphisms

$$1_* \to \mathrm{tra}_*$$

are in bijection with general sections of the underlying vector bundle.

This example motivates

Definition 7 *Let* $1 : \mathcal{P}_2(X) \to \mathrm{Bim}(\mathcal{C})$ *be the tensor unit, i.e. the 2-functor that sends everything to the identity on the tensor unit in* $\mathcal{C}$*. Then, for any 2-vector transport* $\mathrm{tra} : \mathcal{P}_2(X) \to \mathrm{Bim}(\mathcal{C})$ *we say that*

$$[1, \mathrm{tra}]$$

is the **space of flat sections** *of* tra.

Often we are interested in more than the flat sections. Let par be any 1-category, fix

$$\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_2(X)]$$

and denote by

$$\mathrm{tra}_* : \mathrm{conf} \to [\mathrm{par}, \mathrm{Bim}(\mathcal{C})]$$

the 2-functor obtained from post-coposition with tra. Then

Definition 8 *The* **space of sections** *of* tra *with respect to* conf *is*

$$\mathrm{sect} \equiv [1_*, \mathrm{tra}_*] \,.$$

Proposition 5 *The space of sections is a module category over the monoidal category*

$$\mathcal{C} = \mathrm{End}(1_*) \,.$$

Example 15 (ordinary sections of a 1-bundle)

Let $\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}$ be an ordinary vector bundle. Let $\mathrm{par} = \{\bullet\}$ be the discrete category on a single element. Let $\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_1(X)]$ be the discrete category on the objects of $\mathcal{P}_1(X)$. Then the objects of $\mathrm{sect} = [1_*, \mathrm{tra}_*]$ are the ordinary sections of that vector bundle. Morphisms in sect are morphisms induced on sections from bundle endomorphisms that leave the base space invariant.

Moreover, $\mathcal{C} = \mathrm{End}(1_*)$ in this example is the monoid of $\mathbb{C}$-valued functions on X, acting on the space of sections in the usual fashion.

Example 16 (gerbe modules from 2-sections)

Let

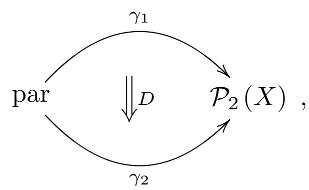
$$\mathrm{tra} : \mathcal{P}_2(U^\bullet) \longrightarrow \Sigma(1d\mathrm{Vect}) \xrightarrow{\subset} \mathrm{Bim}(\mathrm{Vect})$$

be a line bundle gerbe with connection. Let $\mathrm{par} = \{a \to b\}$ be a model for the open interval. Let $\mathrm{conf} \subset [\mathrm{par}, \mathcal{P}_2(U^\bullet)]$ be the sub-2-category whose morphisms are only those coming from 1-paths in $U^{[2]}$. Then

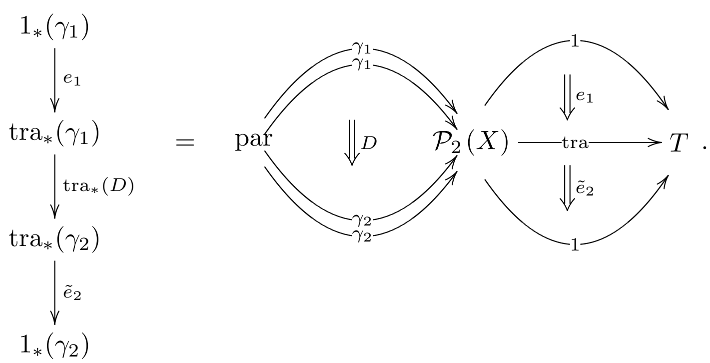
Proposition 6 *A section* $[1_*, \mathrm{tra}_*]$ *in this case is in each connected component of* conf *a choice of gerbe module* E_a *over the endpoint* a*, a choice of gerbe module* E_b *over the endpoint* b*, connected by a morphism of gerbe modules.*

Remark. In physics, gerbe modules are known as **Chan-Paton bundles on D-branes**. In this language the above proposition says that the endpoints of an open string couple to a Chan-Paton bundle on a D-brane.

Definition 9 *A* **disk transport** *associated to a cobordism*



as well as to a section e_1 and a cosection $\tilde{e}_2$ is the morphism is the correlator of e_1 with $\bar{e}_2$ after translation along D:



This describes a state e_1 coming in, propagating along D, and being projected on a state $\bar{e}_2$ coming out. The result is the **two-point disk holonomy** of D under the surface transport tra.

Example 17 (general form of 2-point disk holonomy)

We want to restrict attention to the case where tra takes values in right-induced bimodules

$$\mathrm{tra} : \mathcal{P}_2 \to \mathrm{RIBim}(\mathcal{C}) \subset \mathrm{Bim}(\mathcal{C}) \,,$$

and that the sections involved are such that

$$(\ \mathbb{1} \xrightarrow{e_1(x)} A_x\ ) = (\ \mathbb{1} \xrightarrow{A_x} A_x\ )\,,$$

as well as

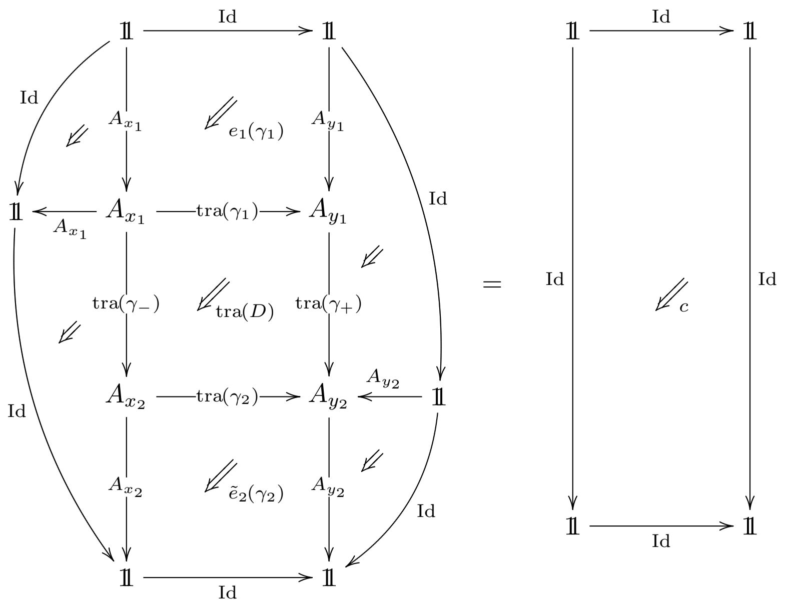
$$(\ A_x \xrightarrow{\tilde{e}_2(x)} \mathbb{1}\ ) = (\ A_x \xrightarrow{A_x} \mathbb{1}\ )\,,$$

for all $x \in X$. Moreover, let tra be such that

$$(\ A_x \xrightarrow{\mathrm{tra}(\gamma)} A_y\ ) = (\ A_x \xrightarrow{(A_x,\phi(\gamma))} A_y\ )$$

for $\phi(\gamma)$ and algebra homomorphism.

The 2-point disk holonomy then comes from a 2-morphism in Bim of the form



Example 18 (2-point disk holonomy of a line bundle gerbe)

Regard a line bundle gerbe with connection as a 2-functor to Bim as in example 10. Take the cobordism to be a disk by setting $\gamma_1 = \mathrm{Id}$ and $\gamma_2 = \mathrm{Id}$. Assume there exists a complex vector bundle $V \to \partial D$ with connection (V, ∇) over the boundary, such that

$$A_p \equiv \mathrm{tra}\,(p) = \mathrm{End}_{V_p}$$

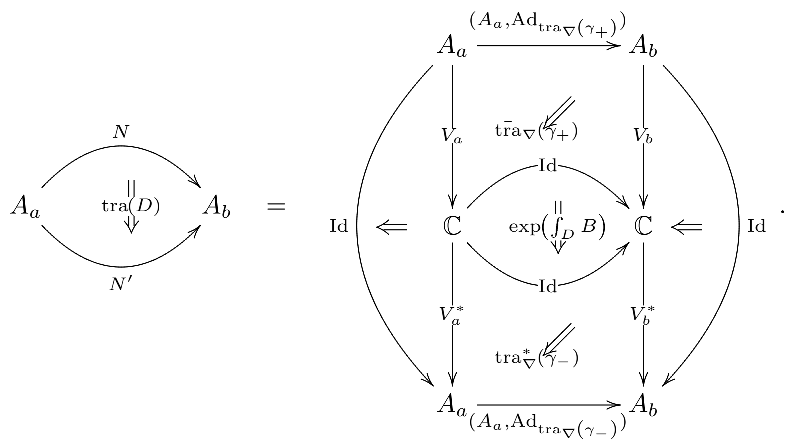
for all $p \in \partial D$. In the language of bundle gerbes, this says that the gerbe module descends over the boundary to an untwisted vector bundle.

Furthermore, let

$$B \in \Omega^2\left(D^1\right)$$

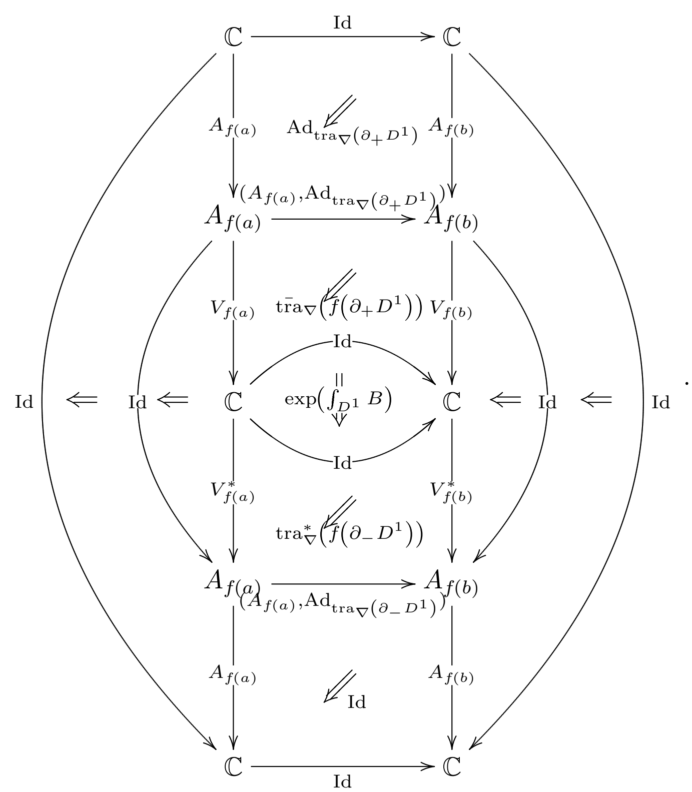
be the globally defined curving 2-form of the 2-transport trivialized over the disk.

Then we have the equality



Inserting this into the general equation from example 17 yields

$\mathrm{hol}_{\mathrm{tra}}(D) \quad \equiv$



Proposition 7 *The right hand side is a complex number, whose value is*

$$\mathrm{hol}_{\mathrm{tra}}(D) = \exp\left(\int_D B\right) \mathrm{Tr}\left(\mathrm{tra}_\nabla(\partial D)\right) .$$

3.4 Quantum transport

We will not solve the mystery of quantization here. But we shall illuminate some aspects.

Assume the quantization of a charged 2-particle has been performed, resulting in a 2-vector transport on parameter space with values in twisted bimodules

$$\mathrm{TwBim}(\mathcal{C}) \subset \mathrm{Cyl}(\Sigma(\mathrm{Bim}(\mathcal{C})))$$

internal to a given abelian braided monoidal category $\mathcal{C}$.

We will indicate how the correlator

$$\langle \bar{e}_2 | T_\rho e_1 \rangle$$

of a state e_1 with a costate $\bar{e}_2$ across a disk which is assigned a given 2-morphism $\rho \in \mathrm{Mor}_2(\mathrm{TwBim})$ has the form indicated in the introduction.

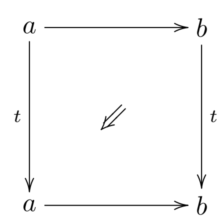
First, consider the category of abstract par-cobordisms, modelling the **world-volume** of our 2-particle.

Definition 10 *For a given parameter space* par *let*

$$\mathrm{Cob}_{\mathrm{par}}$$

be the 2-category coming from the double category that is generated from the category of horizontal morphism being par *and that of vertical morphisms being* $\Sigma(\mathbb{R})$.

A 2-morphism in $\mathrm{Cob}_{\mathrm{par}}$ for $\mathrm{par} = \{a \to b\}$ the open 2-particle is

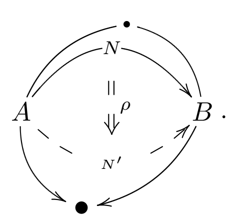


for $t \in \mathbb{R}$.

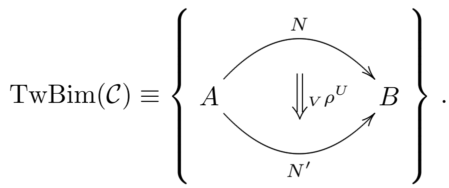
Definition 11 *For* $\mathcal{C}$ *a braided monoidal category,* $\mathrm{Bim}(\mathcal{C})$ *is monoidal and we denote by*

$$\Sigma(\mathrm{Bim}(\mathcal{C}))$$

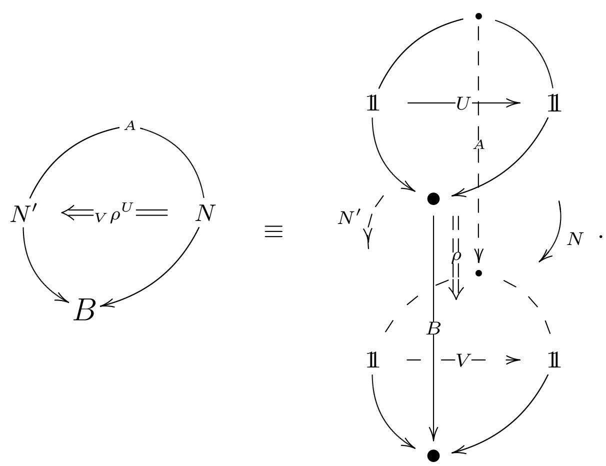
its suspension. A 3-morphism in there we draw as



Definition 12 *The 2-category* $\mathrm{TwBim}(\mathcal{C})$ *of* **twisted bimodules** *is the 2-category of tin cans in* $\Sigma(\mathrm{Bim}(\mathcal{C}))$ *whose top and bottom are* $1\!\!1$*-*$1\!\!1$*-bimodules,*



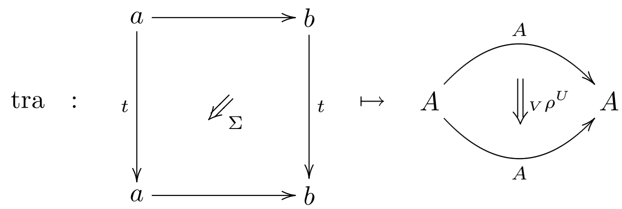
Here



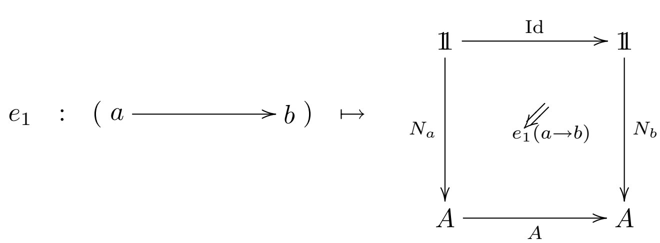
In this context

Example 19 (1-point disk correlator)

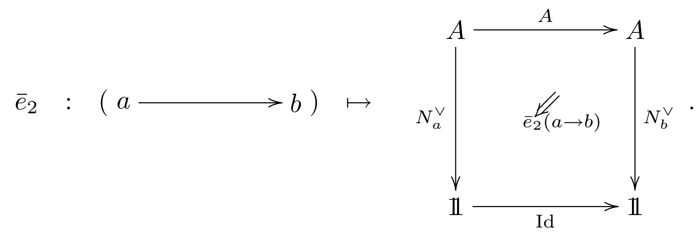
where $\mathrm{tra} : \mathrm{Cob}_{\mathrm{par}} \to \mathrm{TwBim}(\mathcal{C})$ is a 2-vector transport such that



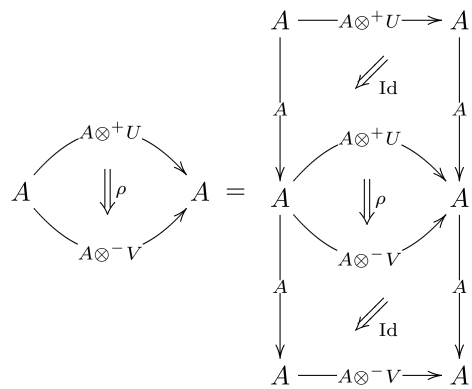
for given t, the following examples describe the disk correlator over Σ for given section



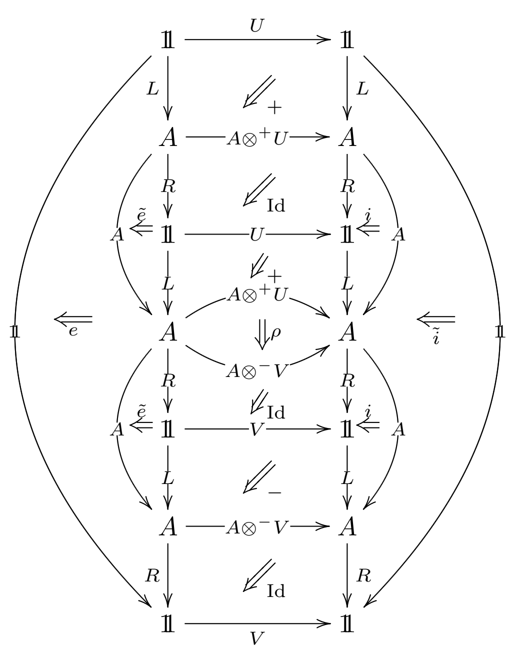
and cosection



First let $N_1 = A$ and $N_2 = A$ and let $e_1(a \to b)$ and $\bar{e}_2(a \to b)$ be identity morphisms. This corresponds to the trivial boundary field insertion. By writing

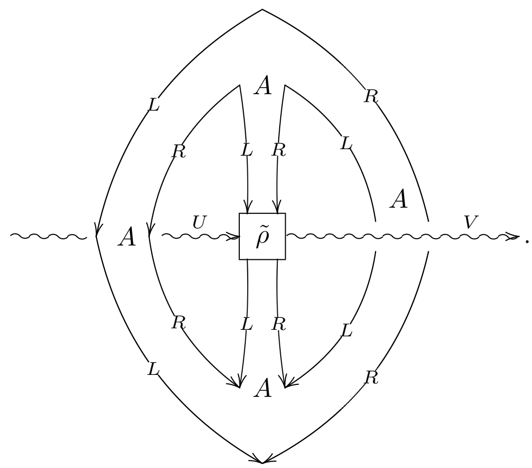


we find the corresponding disk correlator to be

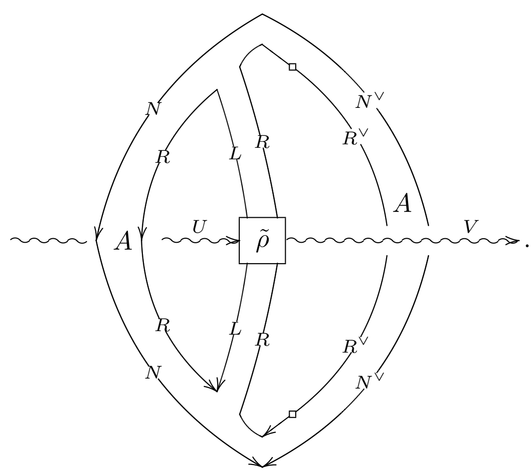


Here R and L denote A, regarded as, respectively, a left or right module over itself.

Proposition 8 *The Poincaré-dual string diagram in $\mathcal{C}$ of this globular diagram is*



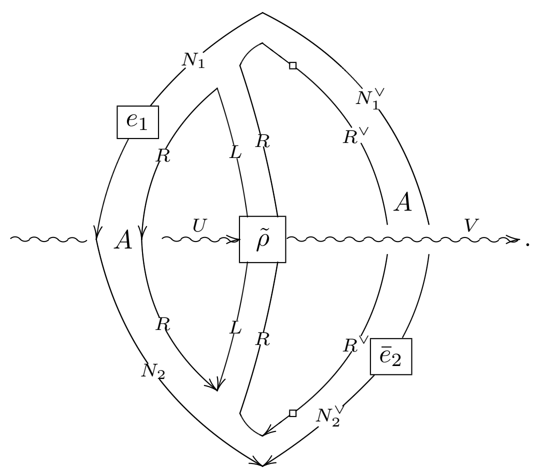
If N_a and N_b are allowed to be arbitrary, but $e_1(a \to b)$ and $\bar{e}_2(a \to b)$ still identity morphisms, this becomes



	physics	arrow theory	FHK/FRS
tra	2d QFT	2-transport	decoration prescription
A	space of open string states	2-vector space	Frobenius algebra
e	boundary field insertion	section of 2-transport	morphism of one-sided modules
ρ	bulk field insertion	image of bulk under 2-transport	morphism of (twisted) bimodules
N	boundary condition (D-brane)	value of section on objects	one-sided module

Table 4: Part of the dictionary that indicates how concepts in quantum field theory are captured by local "state sum" prescriptions, like those of Fukuma-Hosono-Kawai and Fuchs-Runkel-Schweigert, which in turn are realized here in terms of locally trivialized 2-vector transport.

Finally, for nontrivial morphisms $e_1 := e_1(a \to b)$ and $\bar{e}_2 := \bar{e}_2(a \to b)$ we get



Here we have slightly deformed the diagram and inserted canonical isomorphisms $L \simeq R^\vee$.

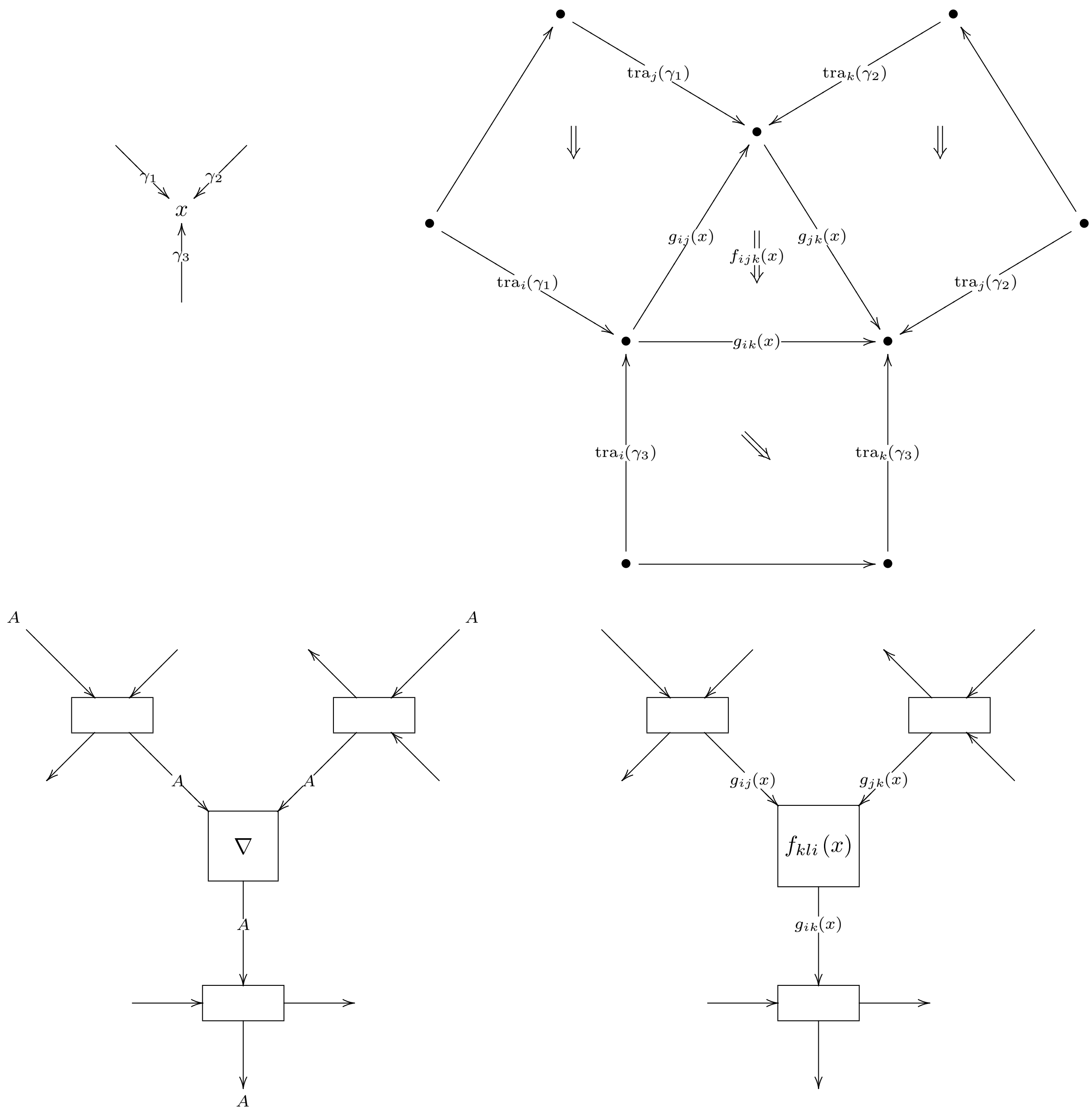
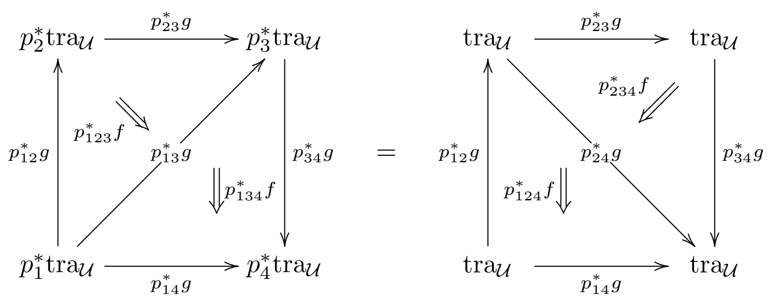
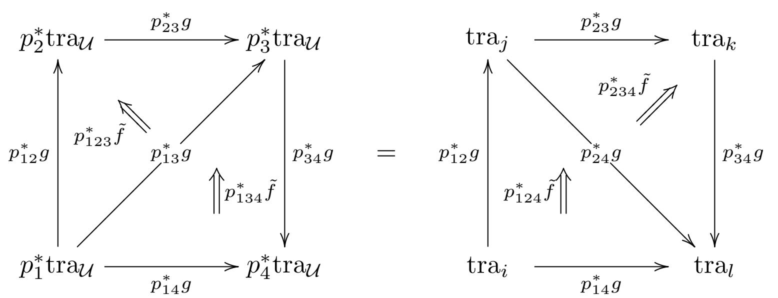


Figure 6: **2-Anafunctors** decorate dual triangulations subordinate to a cover of base space with gluing 2-morphism, shown in the top row in globular notation. In the Poincaré-dual string diagram notation one manifestly recognizes the decoration structure of gerbe surface holonomy (bottom right) or, alternatively, of the state sum prescription in 2-dimensional quantum field theory (bottom left).

1. *associativity of the product*



2. *associativity of the coproduct*



3. *Frobenius property*

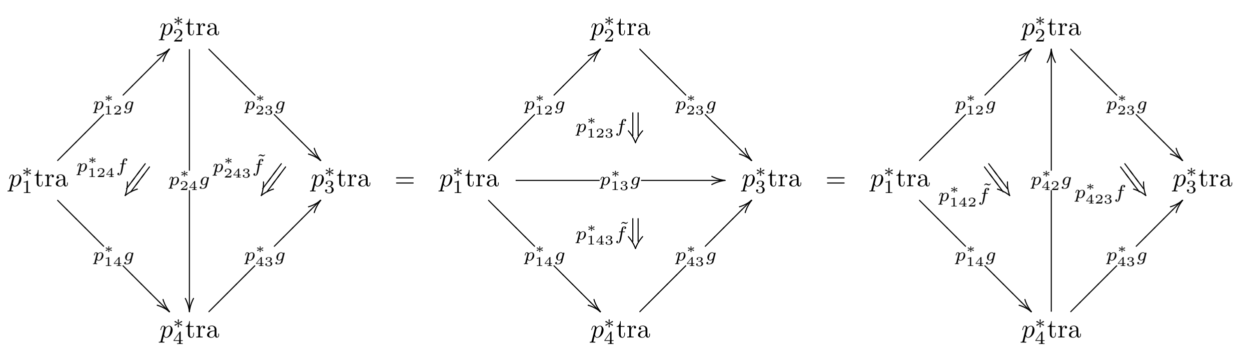


Figure 7: A local trivialization $p^*\mathrm{tra} \xrightarrow{t} \mathrm{tra}_{\mathcal{U}}$ of the 2-functor tra by a special ambidextrous adjunction implies that the **transition data** satisfies **relations** expressing the idea of a special **Frobenius algebroid**.

4 Interlude: transition gerbes, bulk fields and a kind of holography

A state in an n-dimensional quantum field theory, being a morphism of transport n-functors, is itself an $(n-1)$-transport with values in an $(n-1)$-category of cylinders in an n-category.

Hence to a state in n-dimensional quantum field theory we may try to associated a correlator in an $(n-1)$-dimensional quantum field theory.

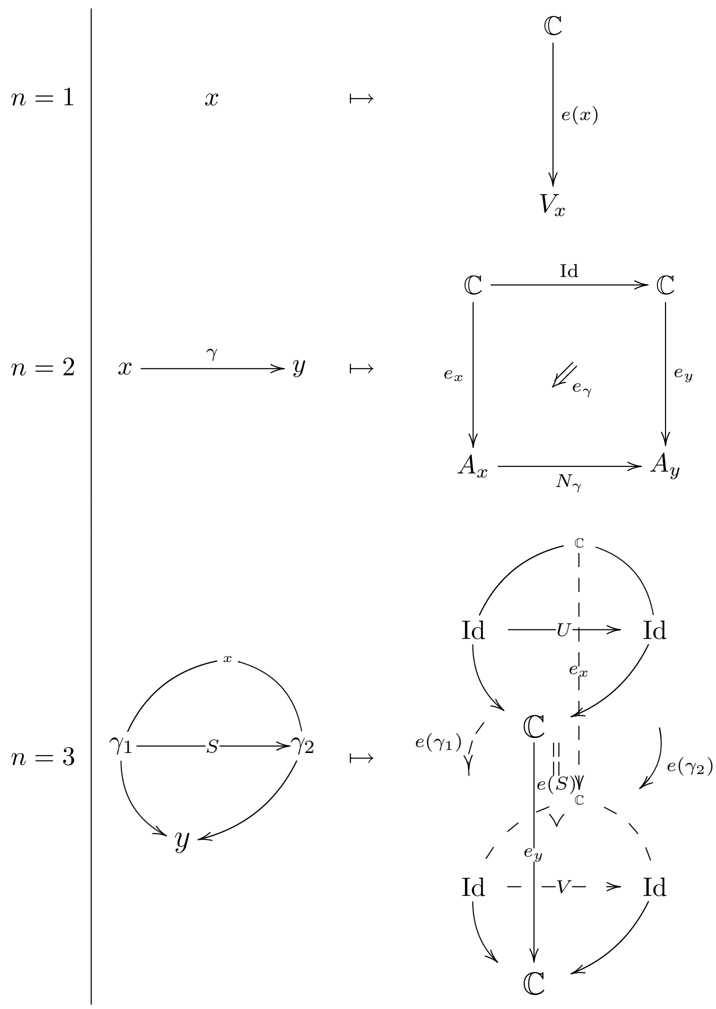


Figure 8: A **state of the charged n-particle** is a morphisms of n-functors $e : 1_* \to \mathrm{tra}_*$, hence itself an $(n-1)$-functor with values in **cylinders** in the codomain of tra.

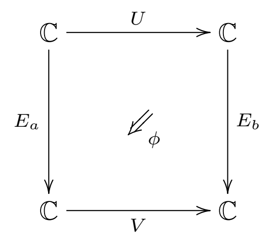
Example 20 (quantum mechanical correlator with bulk insertion)

Consider a quantum mechanical 1-transport, but not with values in Vect, but with values in cylinders in Bim(Vect) $\subset$ 2Vect:

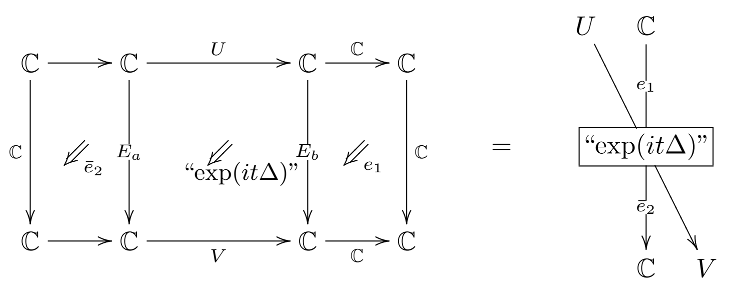
$$\mathrm{tra} : \{\, a \xrightarrow{t} b \,\} \to \mathrm{Cyl}(\mathrm{Bim}(\mathrm{Vect}))\,.$$

For instance

$$\mathrm{tra} : \{\, a \xrightarrow{t} b \,\} \quad \mapsto$$



As for other 1-transport, we may consider the 2-point disk correlator obtained from that. It would read



Here we denoted by "$\exp(it\Delta)$" the linear map defined by this procedure, in order to emphasize how it plays the same role as the quantum mechanical propagator, but twisted by the presence of incoming bulk insertions in U and outgoing bulk insertions in V.

Remark. Bulk field insertions in 2-dimensional quantum field theory follow the same general mechanism, now for $n = 3$.

?? say how simimlar remarks apply to transition $(n-1)-$bundles: their multiplicative structure is a consequence of them taking values in cylinders ??

5 Application: On WZW and Chern-Simons

An especially rich and well understood class of 2-dimensional conformal field theories are those whose target space is a Lie group manifold. These are the **Wess-Zumino-Witten** models.

For given Lie group G and given central extension $\hat{L}_k G$ of the corresponding loop group, these models are controlled by the modular tensor category

$$\mathcal{C} = \mathrm{Rep}(\hat{L}_k G)\,.$$

This means in particular that boundary conditions in these theories are encoded by $\mathrm{Rep}(\hat{L}_k G)$-module categories.

This can be derived by considering the following target space for 2-functorial field theory.

5.1 Target and configuration space of Chern-Simons theory

Let G be a compact simple and simply connected group. G-Chern-Simons theory is a 3-dimensional quantum field theory that associates to a 3-dimensional surface X a quantity obtained from summing, over all trivial G-bundles with connection 1-form A on X, the integral

$$\int_X \mathrm{CS}(A)$$

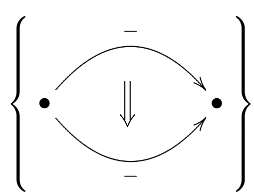
of the Chern-Simons 3-form

$$\mathrm{CS}(A) = \langle A \wedge dA \rangle + \frac{1}{3}\langle A \wedge [A \wedge A]\rangle\,.$$

In order to better understand what this means, we will now cast this setup in the general form of our arrow theory of quantum mechanics.

Notice that, by the above, a Chern-Simons 3-form connection on X is a **field configuration** in Chern-Simons theory along the **trajectory** X.

For simplicity, restrict attention to propagation along trajectories whose incoming and outgoing boundaries are 2-spheres. We can then model the **parameter space** of our Chern-Simons theory by the 2-groupoid

par = 

freely generated from one nontrivial 2-morphisms as indicated.

We will argue in a moment that the relevant **target space** is

$$\mathrm{tar} = \Sigma(\mathrm{INN}(\mathrm{String}_G))\,,$$

the suspension of the 3-group of inner automorphisms of the String_G-2-group.

But in our present context, we are interested only in the space of states of Chern-Simons theory, and its relation to 2-dimensional conformal field theory, not in the dynamics of Chern-Simons theory itself. Therefore it suffices for us to know the **configuration space**

$$\mathrm{conf} \subset [\mathrm{par}, \Sigma(\mathrm{INN}(\mathrm{String}_G))] \,.$$

The choice of morphisms in conf determines which configurations are to be considered gauge equivalent. We shall take a semi-skeletal version of configuration space and set

$$\mathrm{conf} = [\mathrm{par}, \Sigma(\mathrm{String}_G)] \subset [\mathrm{par}, \Sigma(\mathrm{INN}(\mathrm{String}_G))] \,.$$

The codomain for Chern-Simons parallel 3-transport.

Proposition 9 *For any Lie algebra g, there is a semistrict Lie-3-algebra*

$$\mathrm{cs}(g)$$

such that a 3-connection with values in this Lie-3-algebra

$$d\mathrm{tra} : \mathrm{Lie}(\mathcal{P}_1(X)) \to \mathrm{cs}(g)$$

is in degree 1 a g-valued 1-form $A \in \Omega^1(X, g)$, in degree 2 the curvature F_A of A as well as a 2-form $B \in \Omega_2(X)$, and in degree 3 the 3-form

$$C = \mathrm{CS}(A) + dB \,.$$

At the infinitesimal level $\mathrm{cs}(g)$ is the right target space for Chern-Simons theory. The true target space should therefore be the 3-group that integrates the Chern-Simons Lie-3-algebra. To obtain this, first consider

Definition 13 *For $(\delta : h \to g)$ any strict Lie 2-algebra coming from a differential crossed module, we may form the associated Lie-3-algebra*

$$\mathrm{inn}(h \to g)$$

of **inner derivations** *of $(\delta : h \to g)$.*

Proposition 10 $\mathrm{inn}(h \to g)$ *is the Lie 3-algebra characterized by the fact that a 3-connection with values in it is a 1-form $A \in \Omega(X, g)$ and a 2-form $B \in \Omega^2(X, h)$ such that with*

$$\beta = F_A + \delta(B)$$

and

$$H = d_A B$$

we have

$$d_A \beta = \delta(H)$$

and

$$d_A H + \beta \wedge B = 0 \,.$$

Proposition 11 *The Chern-Simons Lie-3-algebra is a sub-Lie-3-algebra of the inner derivations of the string Lie-2-algebra*

$$\mathrm{cs}(g) \xrightarrow{\subset} \mathrm{inn}(\mathrm{string}_k(g)) \qquad (k = -1)\,.$$

Remark. I expect that this inclusion is in fact an equivalence.

This means that as a Lie-3-group integrating the Chern-Simons Lie-3-algebra we should take the 3-group of inner automorphisms of the String 2-group,

$$G_3 = \mathrm{INN}(\mathrm{String}_K)\,.$$

In other parts of the literature the 2-gerbe relevant for Chern-Simons theory is usually characterized in terms of its transition 1-gerbes, which are required to be WZW gerbes.

Definition 14 *A G-WZW-gerbe on a space X at level k is a gerbe on X which is obtained by pullback along a map*

$$g : X \to G$$

of the canonical gerbe on G.

In the existing literature, the status of this definition for gerbes with connection remains inconclusive. The result above seems to indicate that the transition gerbe for a Chern-Simons 2-gerbe with connection should be a nonabelian gerbe with structure 2-group the 2-group

$$\mathrm{Cyl}(\mathrm{INN}(\mathrm{String}_G))$$

of cylinders in inner autmorphisms of the String 2-group.

Proposition 12 *Let $\tilde{G}_3 = (U(1) \to \hat{\Omega}_k G \to PG)$ be the strict Lie-3-group inside* $\mathrm{INN}(\mathrm{String}_G)$*. A transition 2-bundle for a $\tilde{G}_3$-3-bundle is a 2-functor from the 2-groupoid of the covering of the given double intersection U_{ij} to the 2-group*

$$\mathrm{Cyl}^{\mathrm{Id}}(\tilde{G}_3)$$

of cylinders in $\tilde{G}_3$ with trivial top and bottom. Such a 2-functor a pullback of the canonical $U(1)$-bundle gerbe

$$\begin{array}{ccc} L & & \\ \downarrow & & \\ PG \times \Omega G & \rightrightarrows & PG \\ & & \downarrow \\ & & G \end{array}$$

along a map

$$g : U_{ij} \to G\,,$$

together with a choice of section.

Remark. Notice that this essentially yet another way of saying that the String_G-2-group *is* the multiplicative bundle gerbe on G.

5.2 States of Chern-Simons and Correlators of WZW

As before, we take G to be a simple, simply connected and compact Lie group, and let $k \in H^3(G, \mathbb{Z})$ be a level. From the centrally extended loop group, $\hat{\Omega}_k G$, we can form the groupoid $\mathrm{String}_G \equiv \; PG \ltimes \hat{\Omega}_k G \rightrightarrows PG$ over based paths in G.

This groupoid can be regarded from two points of view. As a centrally extended groupoid, it is the canonical bundle gerbe with class k over G. The groupoid has a strict monoidal structure, with strict monoidal inverses. Therefore it can also be regarded as a strict 2-group.

Being monoidal, we can form the suspension $\Sigma(\mathrm{String}_G)$, which is a 2-category with a single object.

Here we discuss the

5.2.1 $\mathrm{Rep}(L_k G)$ and states of the 2-particle on $\Sigma(\mathrm{String}_G)$

Before studying the states of the 3-particle on $\Sigma(\mathrm{String}_G)$, it is of interest to consider just the 2-particle obtained as the boundary of that 3-particle.

So let

$$\mathrm{par} = \Sigma(\mathbb{Z})$$

and consider $1 : \Sigma(\mathrm{String}_G) \to \mathrm{Bim}$ to be the trivial 2-vector bundle on $\Sigma(\mathrm{String}_G)$ (instead of the trivial 3-vector bundle that will be relevant for the 3-particle).

Proposition 13 *The groupoid*

$$\Lambda\mathrm{String}_G \equiv [\Sigma(\mathbb{Z}), \Sigma(\mathrm{String}_G)]_{/\sim}$$

obtained by identifying isomorphic 1-morphisms in configuration space is a central extension of the the loop groupoid

$$\Lambda G \equiv [\Sigma(\mathbb{Z}), \Sigma(G)]$$

of G*.*

Proposition 14 *The monoidal category* $\mathcal{C} = \mathrm{End}(1_*)$ *is*

$$\mathcal{C} = [\Sigma(\mathbb{Z}), \mathrm{Rep}(\Lambda\mathrm{String}_G)]\,.$$

Proposition 15 *The category* $\mathrm{Rep}(\mathrm{String}_G)$ *is a category of equivariant gerbe modules on* G*.*

Remark. Simon Willerton has shown that, for G a finite group, $\Lambda G = [\Sigma(\mathbb{Z}), \Sigma(G)]$ plays the role of the loop group of G, in that

Proposition 16 (Willerton) *For G a finite group we have*

$$B\Lambda G \simeq LBG\,.$$

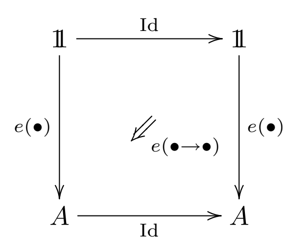
In as far as this statement for finite groups generalizes to Lie groups, the above proposition is apparently analogous to the **Freed-Hopkins-Teleman theorem**. This identifies the representation ring of the loop group with the twisted equivariant K-theory of the group.

Proposition 17 *For 2-transport on $\mathcal{P}_{\mathrm{cyl}}$ with values in $T = \mathrm{Bim}(\mathcal{C})$ we have on $\Sigma(\mathbb{Z})$*

$$\mathrm{End}(1_*) = \Lambda\mathcal{C}$$

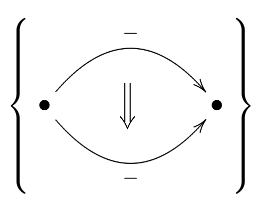
Remark. Proposition 14 says that the space of states is a module category for $\Lambda\mathrm{Rep}(\Lambda\mathrm{String}_G)$. This follows indepdently of which kind of 2-bundle we choose on target space. But modules for loops in $\mathcal{C}$ are in particular obtained from loops in modules of $\mathcal{C}$.

Thence let Mod_A be a $\mathcal{C}$-module category, with A an algebra internal to $\mathcal{C}$ and take a section e to be an object in $\Lambda\mathrm{Mod}_A$:



5.2.2 Algebras internal to $\mathrm{Rep}(L_kG)$ and states of the 3-particle on $\Sigma(\mathrm{String}_G)$

Let

$\mathrm{par} =$ 

model the 3-particle. Consider a trivial 3-vector bundle on $\Sigma(\mathrm{String}_G)$ and its space of sections relative to the configuration space

$$\mathrm{conf} = [\mathrm{par}, \Sigma(\mathrm{String}_G)]\,.$$

Proposition 18 *The space of 3-states on* conf *is something like*

$$[\mathrm{par}, \mathrm{Bim}(\mathrm{Rep}(\Lambda_2\mathrm{String}_G))]\,.$$

?? give more details ??

?? point out how such a 3-state is a 2-transport with values in twisted bimodules ??