Simplicial interpretation of bigroupoid principal 2-bundles

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Abstract

We describe bigroupoid principal 2-bundles, and we give a complete classification of these objects in terms of newly defined nonabelian cohomology. We also show that bigroupoid principal 2-bundles gives Glenn's simplicial 2-torsors after the application of the Duskin nerve functor for bicategories.

Contents

1 Bicategories

Bicategories were defined by Benabou [Be], and from the modern perspective, we could call them weak 2-categories. Instead of stating their original definition we will use Batanin's approach to weak n-categories given in [Bt]. In this approach a bicategory $\mathcal{B}$, given by the reflexive 2-graph

$$\mathcal{B} \equiv (B_2 \underset{d_0^1}{\overset{d_1^1}{\rightleftarrows}} B_1 \underset{d_0^0}{\overset{d_1^0}{\rightleftarrows}} B_0)$$

is a 1-skeletal monoidal globular category, given by the diagram of categories and functors

$$\mathcal{B}_1 \underset{D_0}{\overset{D_1}{\rightleftarrows}} \mathcal{B}_0$$

where the category $\mathcal{B}_1$ is the category of morphisms of the bicategory $\mathcal{B}$ and the category $\mathcal{B}_0$ is the image $\mathcal{D}(B_0)$ of the discrete functor $\mathcal{D}\colon \mathrm{Set} \to \mathrm{Cat}$ which just turns an object of $\mathcal{E}$ into a discrete internal category in $\mathcal{E}$. Source functor D_1 is defined by $D_1 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_1^0 d_1^1 = d_1^0 d_0^1\colon B_2 \to B_0$, and a target functor D_0 is defined by $D_0 := d_1^0\colon B_1 \to B_0$ and $D_1 := d_0^0 d_1^1 = d_0^0 d_1^1\colon B_2 \to B_0$, where we used the same notation for objects and morphisms parts of the functor. Also, the unit functor $I\colon B_0 \to B_1$ is defined by $I := s_0\colon B_0 \to B_1$ on the level of objects, and $I := s_1\colon B_1 \to B_2$ on the level of morphisms, where $s_0\colon B_0 \to B_1$ and $s_1\colon B_1 \to B_2$ are section morphisms in the above 2-graph from left to right, which we didn't label to avoid too much indices.

In the lower definition of a bicategory we will denote the vertex $\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ of the following pullback of functors

$$\begin{array}{ccc} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{Pr_2} & \mathcal{B}_1 \\ {\scriptstyle Pr_1}\downarrow & & \downarrow{\scriptstyle D_0} \\ \mathcal{B}_1 & \xrightarrow[D_1]{} & \mathcal{B}_0 \end{array}$$

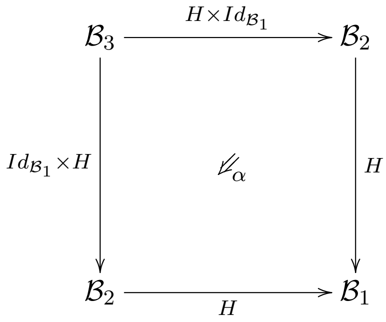
by $\mathcal{B}_2 := \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ and likewise $\mathcal{B}_3 := \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$, and so on. Thus we will adopt the following convention: for any functor $P\colon \mathcal{E} \to B_0$, the first of the symbols

$$\mathcal{E} \times_{\mathcal{B}_0} \mathcal{B}_1 \;\; and \;\; \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{E}$$

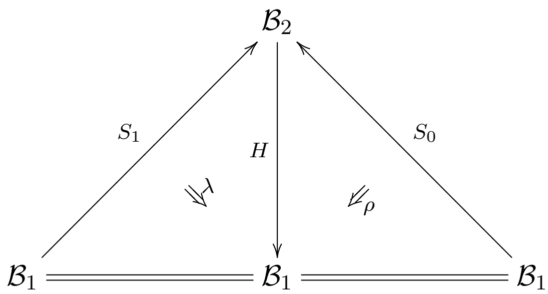
will denote the pullback of P and D_0, and the second one that of D_1 and P.

Definition 1.1. *A weak 2-category (or a* bicategory*) $\mathcal{B}$ consists of:*

- *two categories, a discrete category $\mathcal{B}_0$ of objects, and a category $\mathcal{B}_1$ of morphisms of the weak 2-category $\mathcal{B}$,*
- *functors $D_0, D_1 \colon \mathcal{B}_1 \to \mathcal{B}_0$, called target and source functors, respectively, a functor $I \colon \mathcal{B}_0 \to \mathcal{B}_1$, called unit functor, and a functor $H \colon \mathcal{B}_2 \to \mathcal{B}_1$, called the horizontal composition functor,*
- *natural isomorphism*



- *natural isomorphisms*



  where the functor $S_0 \colon \mathcal{B}_1 \to \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(D_0, Id_{\mathcal{B}_1})} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_0 \xrightarrow{I \times Id_{\mathcal{B}_1}} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

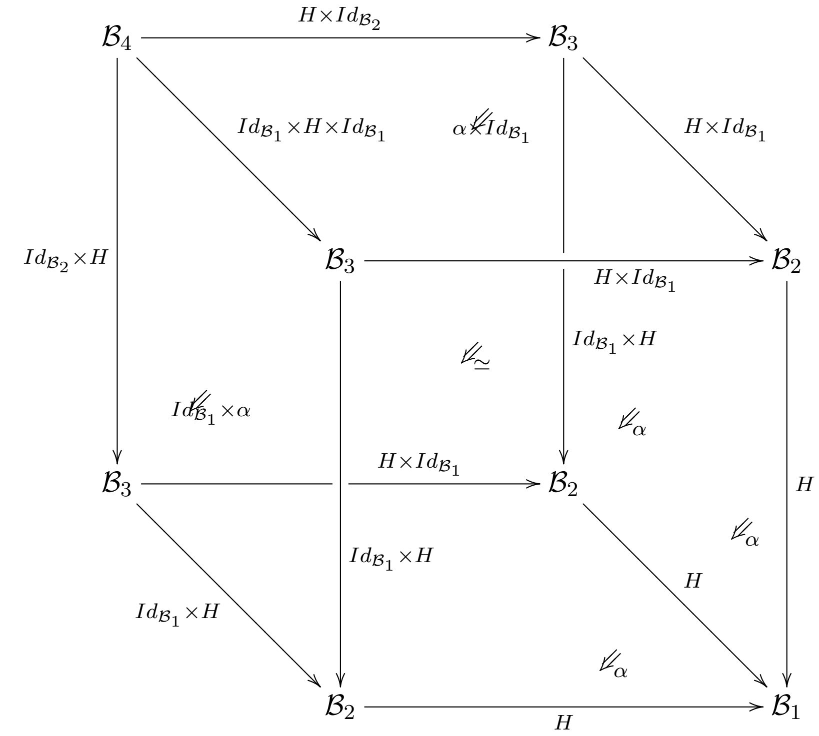
  and the functor $S_1 \colon \mathcal{B}_1 \to \mathcal{B}_2$ is defined by the composition

$$\mathcal{B}_1 \xrightarrow{(Id_{\mathcal{B}_1}, D_1)} \mathcal{B}_0 \times_{\mathcal{B}_0} \mathcal{B}_1 \xrightarrow{Id_{\mathcal{B}_1} \times I} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1,$$

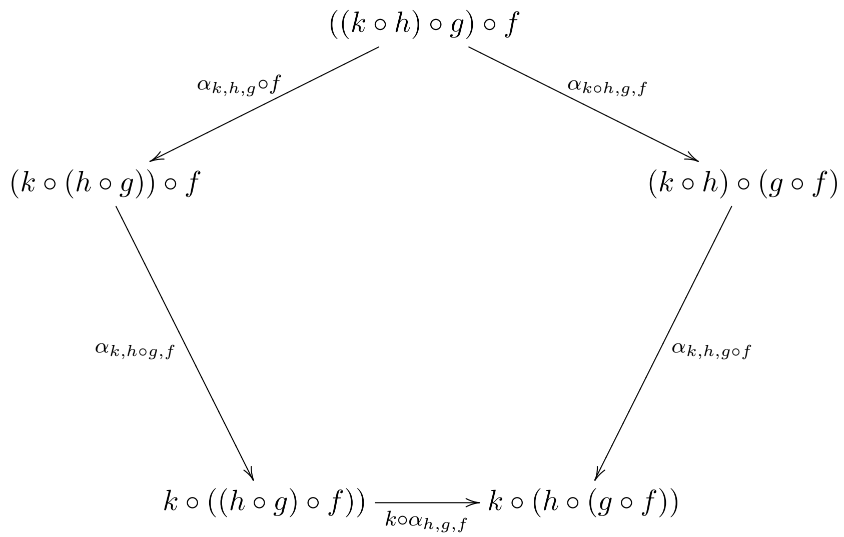
  or more explicitly for any 1-morphism $f \colon x \to y$ in $\mathcal{B}$ (i.e. object in $\mathcal{B}_1$) we have $S_0(f) = (f, i_x)$ and $S_1(f) = (i_y, f)$,

such that following axioms are satisfied:

- *associativity 3-cocycle*

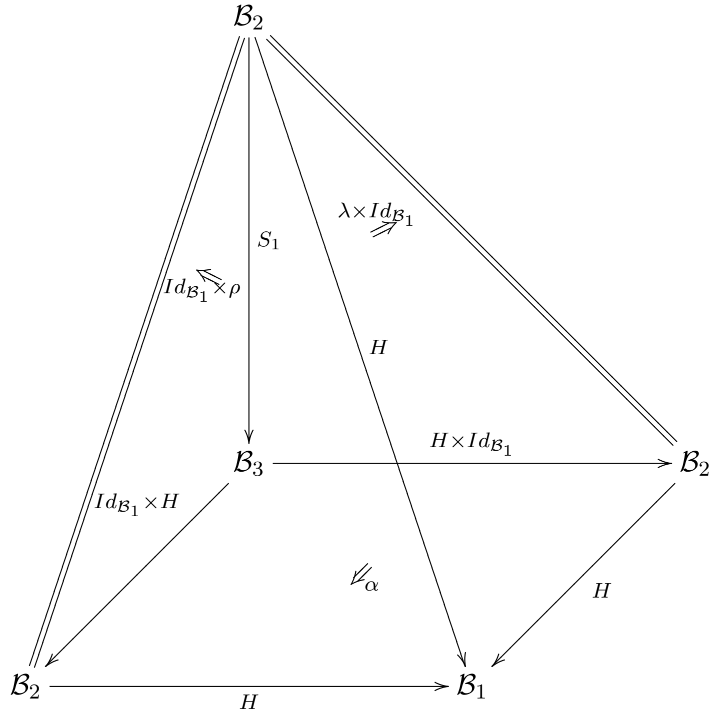


which for any object (k, h, g, f) *in* $\mathcal{B}_4$ *becomes the commutative pentagon*

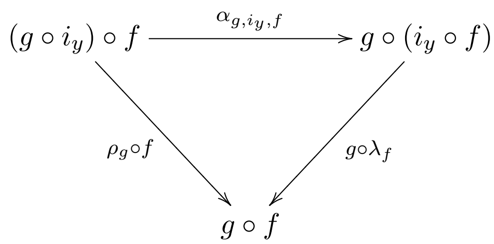


of components of natural transformations

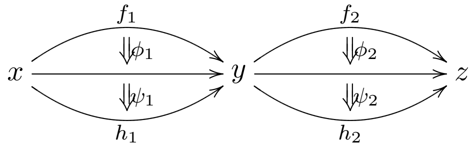
- *the commutative pyramid*



which for any object (g, f) *in* $\mathcal{B}_2$ *becomes the triangle diagram*



Remark 1.1. *Note that in the above definition of the horizontal composition functor* $H\colon \mathcal{B}_2 \to \mathcal{B}_1$*, for any diagram of 2-arrows (i.e. a morphism in a category* $\mathcal{B}_2 \times_{\mathcal{B}_1} \mathcal{B}_2$*)*



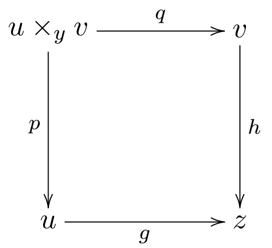
by functoriality we immediately have a Godement interchange law

$$(\psi_2 \circ \psi_1)(\phi_2 \circ \phi_1) = (\psi_2\psi_1) \circ (\phi_2\phi_1).$$

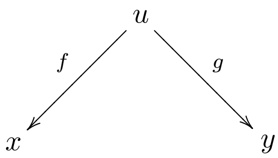
Example 1.1. *(Strict 2-categories) A weak 2-category in which associativity and left and right identity natural isomorphisms are identities is called (strict) 2-category.*

Example 1.2. *(Monoidal categories) Monoidal category is precisely a weak 2-category $\mathcal{B}$ in which $\mathcal{B}_0 = 1$ is terminal discrete category (or one point set). Strict monoidal category is a one object 2-category.*

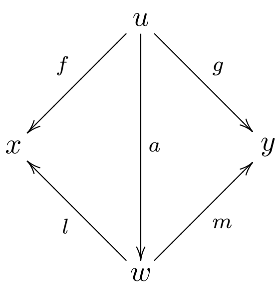
Example 1.3. *(Weak 2-category of spans) Let $\mathcal{C}$ be a cartesian category (that is a category with pullbacks). First we make a choice of the pullback*



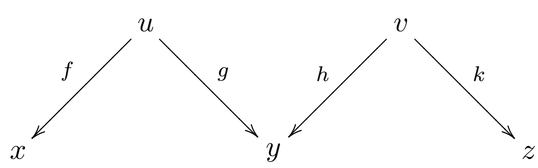
for any such diagram $x \xrightarrow{f} z \xleftarrow{g} y$ in a category $\mathcal{C}$. We construct the weak 2-category $Span(\mathcal{C})$ of spans in the category $\mathcal{C}$. The objects of $Span(\mathcal{C})$ are the same as objects of $\mathcal{C}$. For any two objects x, y in $Span(\mathcal{C})$, a 1-morphism $u\colon x \nrightarrow y$ is a span



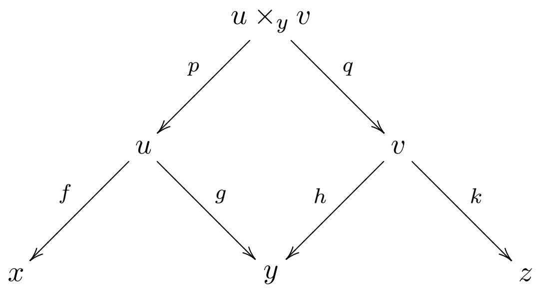
and a 2-morphism $a\colon z \nRightarrow w$ is given by the commutative diagram



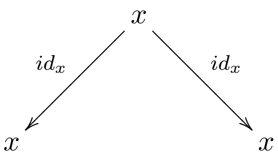
from which we easily see that vertical composition of 2-morphisms is given by the composition in $\mathcal{C}$. Horizontal composition of composable 1-morphisms



is given by the pullback

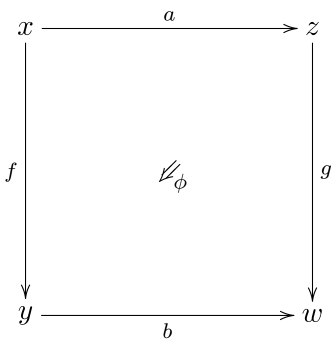


and from here we have obvious horizontal identity $i_x \colon x \nrightarrow x$



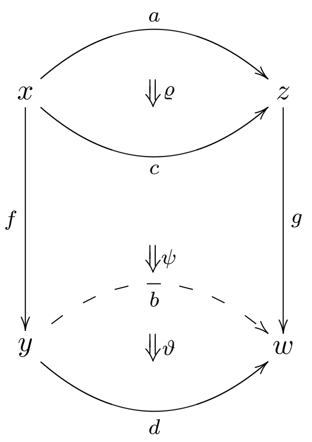
Example 1.4. *(Bimodules) Let* Bim *denote the weak 2-category whose objects are rings with identity. For any two rings* A *and* B*,* $Bim(A, B)$ *will be a category of* $A-B$ *bimodules and their homomorphisms. Horizontal composition is given by the tensor product, and associativity and identity constraints are the usual ones for the tensor product.*

Example 1.5. *(Weak 2-categories of 1-morphisms) Let* $\mathcal{B}$ *be a weak 2-category. The weak 2-category* $\mathcal{B}^{\rightarrow}$ *of 1-morphisms, associated to* $\mathcal{B}$ *has 1-morphisms of* $\mathcal{B}$ *for objects, thus* $\mathcal{B}_0^{\rightarrow} = B_1$*. A 1-morphism from* $f \colon x \to y$ *to* $g \colon z \to w$ *is a triple* (a, ϕ, b) *consisting of 1-morphisms* $a \colon x \to z$*,* $b \colon y \to w$ *and a 2-morphism* $\phi \colon g \circ a \Rightarrow b \circ f$ *as in the diagram*



and a 2-morphism from (a, ϕ, b) *to* (c, ψ, d) *is a pair* (ϱ, ϑ) *of 2-morphisms* $\varrho \colon a \Rightarrow c$ *and*

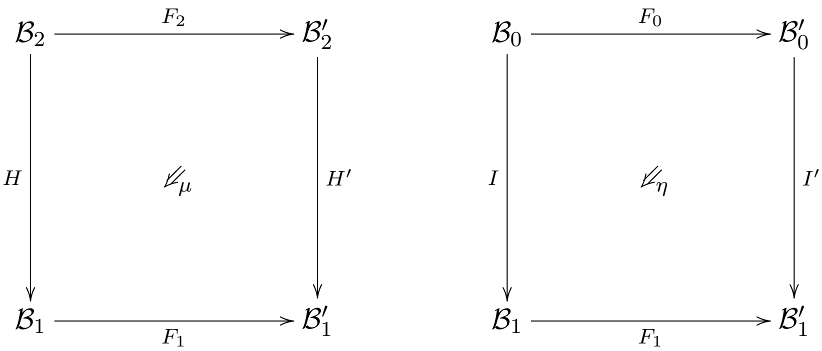
$\vartheta\colon b \Rightarrow d$ *such that the diagram*



(in which we omitted ϕ*) commutes. Associativity and identity constraints for* $\mathcal{B}^{\to}$ *are naturally induced from those of* $\mathcal{B}$*.*

Definition 1.2. *A weak 2-functor* $F\colon \mathcal{B} \to \mathcal{B}'$ *between weak 2-categories consists of the following data:*

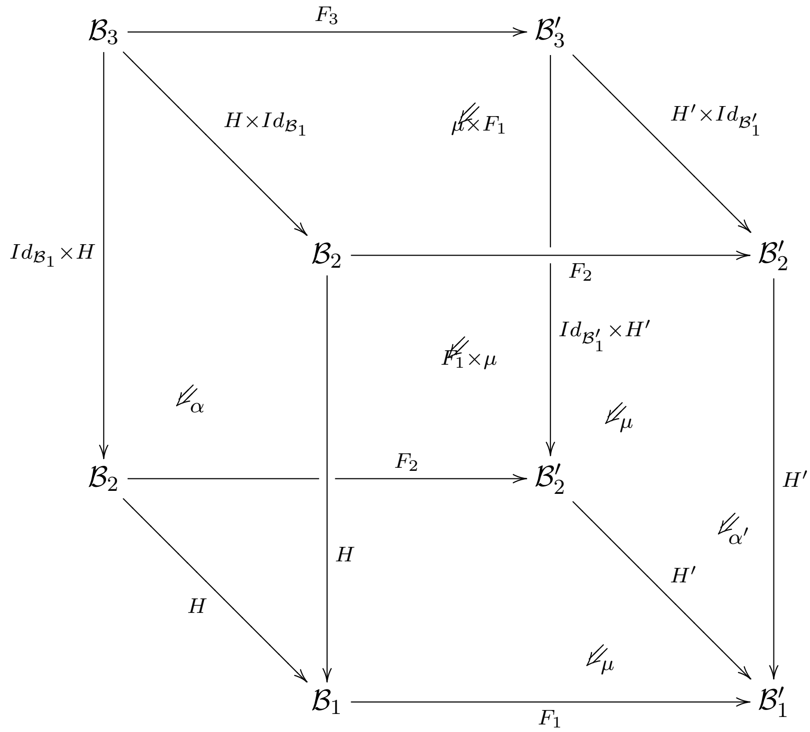
- *a (discrete) functor* $F_0\colon \mathcal{B}_0 \to \mathcal{B}'_0$*, and a functor* $F_1\colon \mathcal{B}_1 \to \mathcal{B}'_1$*,*
- *natural transformations*



given by components $\mu_{g,f}\colon F(g) \circ F(f) \to F(g \circ f)$ *and* $\eta_x\colon i'_{F(x)} \to F(i_x)$*, respectively (in which we omitted the subscripts on functor signs in order to avoid too much indices),*

such that following axioms are satisfied:

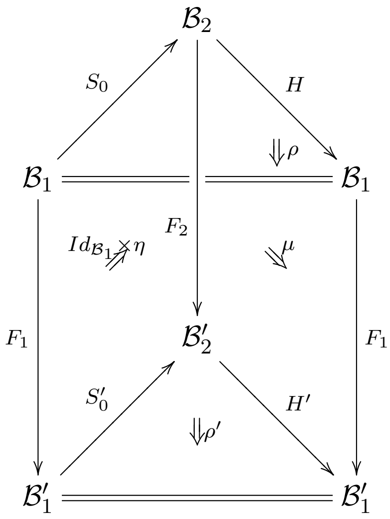
- *commutative cube*



which when evaluated at the object (h, g, f) *in* $\mathcal{B}_3$ *becomes a commutative diagram*

$$\begin{array}{ccccc}
(F(h) \circ F(g)) \circ F(f) & \xrightarrow{\mu_{h,g} \circ F(f)} & F(h \circ g) \circ F(f) & \xrightarrow{\mu_{h \circ g, f}} & F((h \circ g) \circ f) \\
\Big\downarrow {\scriptstyle a'_{F(h),F(g),F(f)}} & & & & \Big\downarrow {\scriptstyle F(a_{h,g,f})} \\
F(h) \circ (F(g) \circ F(f)) & \xrightarrow[F(h) \circ \mu_{g,f}]{} & F(h) \circ F(g \circ f) & \xrightarrow[\mu_{h, g \circ f}]{} & F(h \circ (g \circ f))
\end{array}$$

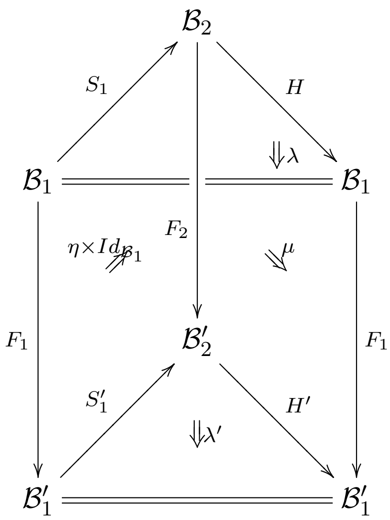
- *a commutative diagram*



which when evaluated at the object f in $\mathcal{B}_1$ becomes a commutative diagram

$$\begin{array}{ccccc} F(f)\circ i'_{F(x)} & \xrightarrow{F(f)\circ\eta_x} & F(f)\circ F(i_x) & \xrightarrow{\mu_{f,i_x}} & F(f\circ i_x) \\ {\scriptstyle \rho'_{F(f)}}\downarrow & & & & \downarrow{\scriptstyle F(\rho_f)} \\ F(f) & = & = & = & F(f)\end{array}$$

- *a commutative diagram*



which when evaluated at the object f in $\mathcal{B}_1$ becomes a commutative diagram

$$\begin{array}{ccccc}
i'_{F(y)} \circ F(f) & \xrightarrow{\eta_y \circ F(f)} & F(i_y) \circ F(f) & \xrightarrow{\mu_{i_y,f}} & F(i_y \circ f) \\
{\scriptstyle \lambda'_{F(f)}} \downarrow & & & & \downarrow {\scriptstyle F(\lambda_f)} \\
F(f) & & = & & F(f)
\end{array}$$

Remark 1.2. *If both $\mathcal{B}$ and $\mathcal{B}'$ are strict 2-categories then the coherence for composition becomes*

$$\begin{array}{ccc}
F(h) \circ F(g) \circ F(f) & \xrightarrow{\mu_{h,g} \circ F(f)} & F(h \circ g) \circ F(f) \\
{\scriptstyle F(h) \circ \mu_{g,f}} \downarrow & & \downarrow {\scriptstyle \mu_{h \circ g,f}} \\
F(h) \circ F(g \circ f) & \xrightarrow[\mu_{h,g \circ f}]{} & F(h \circ g \circ f)
\end{array}$$

and the coherence for identities become two commutative triangles

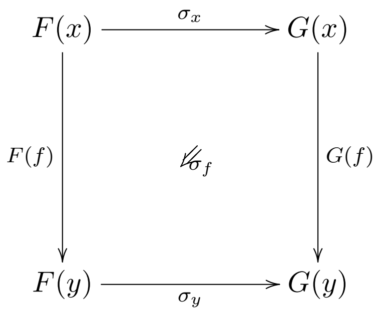
$$\begin{array}{ccc}
 & F(f) \circ F(i_x) & \\
{\scriptstyle F(f) \circ \eta_x} \nearrow & & \searrow {\scriptstyle \mu_{f,i_x}} \\
F(f) & = & F(f)
\end{array}
\qquad
\begin{array}{ccc}
 & F(i_y) \circ F(f) & \\
{\scriptstyle \eta_y \circ F(f)} \nearrow & & \searrow {\scriptstyle \mu_{i_y,f}} \\
F(f) & = & F(f)
\end{array}$$

Definition 1.3. *A (left) lax natural transformation $\sigma \colon F \Longrightarrow G$ is defined by the following data:*

- *a natural transformation $\sigma_0 \colon F_0 \to G_0$ between (discrete) functors (which just amounts to the family of morphisms $\sigma_x \colon F(x) \to G(x)$),*
- *natural transformation*

$$\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{G_1} & \mathcal{B}'_1 \\
{\scriptstyle F_1} \downarrow & \swarrow_{\sigma_1} & \downarrow {\scriptstyle \sigma_0^*} \\
\mathcal{B}'_1 & \xrightarrow[\sigma_{0*}]{} & \mathcal{B}'_1
\end{array}$$

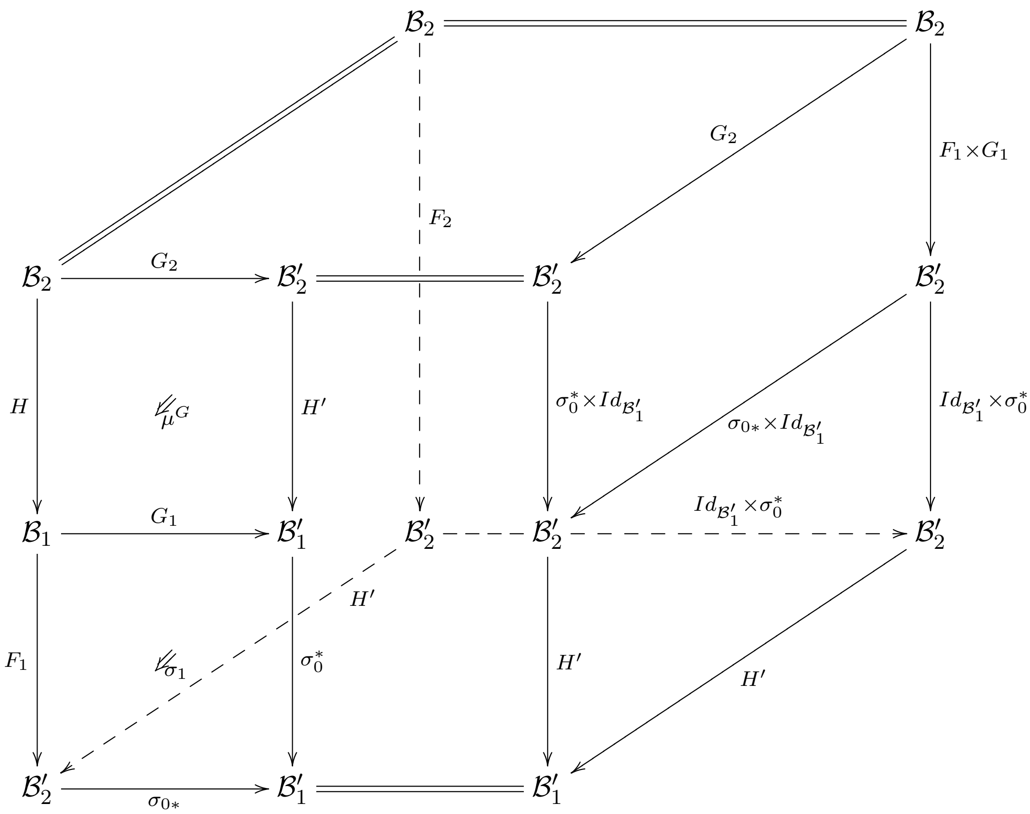
whose component at the object $f\colon x \to y$ in $\mathcal{B}_1$ is given by the square



which is a 2-morphism $\sigma_f\colon G(f)\circ\sigma_x \Longrightarrow \sigma_y \circ F(f)$,

such that the following axioms are satisfied:

- *the following cube of functors and natural transformations*



commutes, which becomes a commutative diagram of natural transformations

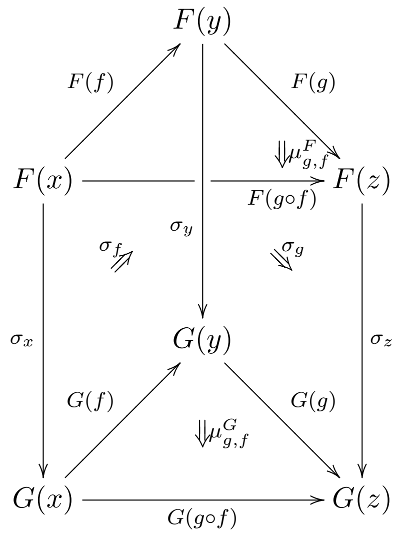
$$
\begin{array}{ccccc}
(G(g)\circ G(f))\circ\sigma_x & \xrightarrow{\alpha'_{G(g),G(f),\sigma_x}} & G(g)\circ(G(f)\circ\sigma_x) & \xrightarrow{G(g)\circ\sigma_f} & G(g)\circ(\sigma_y\circ F(f)) \\
\downarrow{\scriptstyle \mu^G_{g,f}\circ\sigma_x} & & & & \downarrow{\scriptstyle \alpha'^{-1}_{G(g),\sigma_y,F(f)}} \\
G(g\circ f)\circ\sigma_x & & & & (G(g)\circ\sigma_y)\circ F(f) \\
\downarrow{\scriptstyle \sigma_{g\circ f}} & & & & \downarrow{\scriptstyle \sigma_g\circ F(f)} \\
\sigma_z\circ F(g\circ f) & \xleftarrow[\sigma_z\circ\mu^F_{g,f}]{} & \sigma_z\circ(F(g)\circ F(f)) & \xleftarrow[\alpha'_{\sigma_z,F(g),F(f)}]{} & (\sigma_z\circ F(g))\circ F(f)
\end{array}
$$

when it is evaluated at the object (g, f) in $\mathcal{B}_2$,

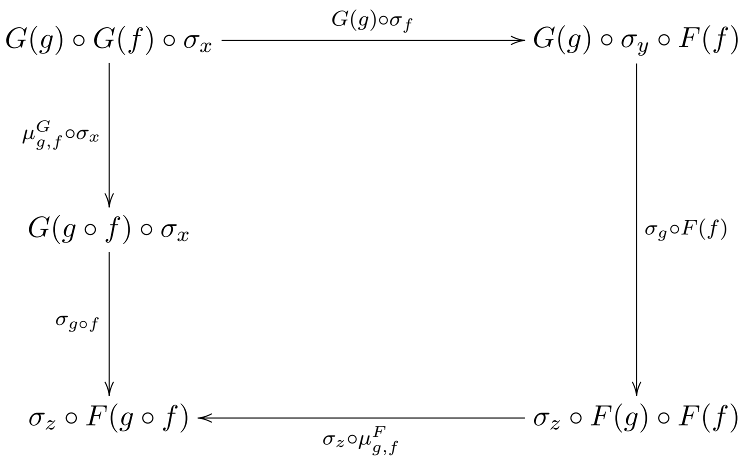
- *a commutative diagram*

$$
\begin{array}{ccccc}
i'_{G(x)}\circ\sigma_x & \xrightarrow{\lambda'_{\sigma_x}} & \sigma_x & \xrightarrow{\rho'^{-1}_{\sigma_x}} & \sigma_x\circ i'_{F(x)} \\
\downarrow{\scriptstyle \eta^G_x\circ\sigma_x} & & & & \downarrow{\scriptstyle \sigma_x\circ\eta^F_x} \\
G(i_x)\circ\sigma_x & & \xrightarrow[\sigma_{i_x}]{} & & \sigma_x\circ F(i_x)
\end{array}
$$

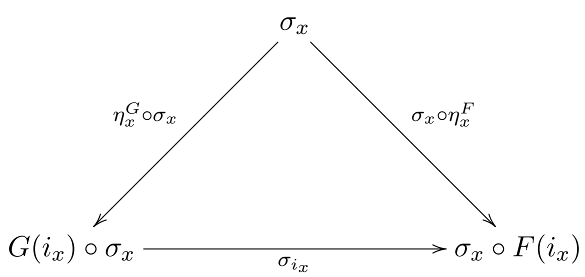
Remark 1.3. *If both $\mathcal{B}$ and $\mathcal{B}'$ are strict 2-categories then the above coherence becomes*



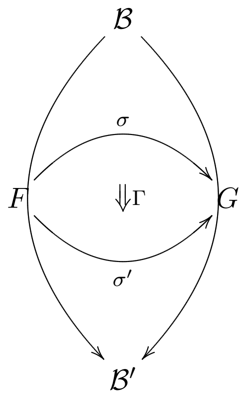
which is equivalent to the commutative diagram



The second coherence becomes the commutative diagram



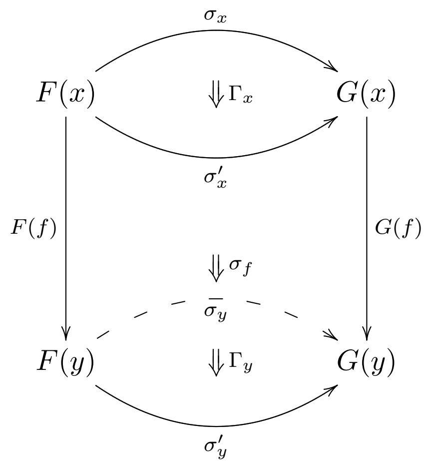
Definition 1.4. *A modification* $\Gamma\colon \sigma \to \sigma'$



consists of the following data:

- *a 2-morphism* $\Gamma_x\colon \sigma_x \to \sigma'_x$ *for each object* x *in* $\mathcal{B}$

such that the following diagram



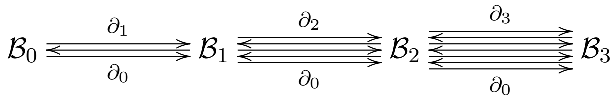
which becomes a diagram

$$\begin{array}{ccc} G(f)\circ\sigma_x & \xrightarrow{G(f)\circ\Gamma_x} & G(f)\circ\sigma'_x \\ \Big\downarrow{\scriptstyle \sigma_f} & & \Big\downarrow{\scriptstyle \sigma'_f} \\ \sigma_y\circ F(f) & \xrightarrow[\Gamma_y\circ F(f)]{} & \sigma'_y\circ F(f) \end{array}$$

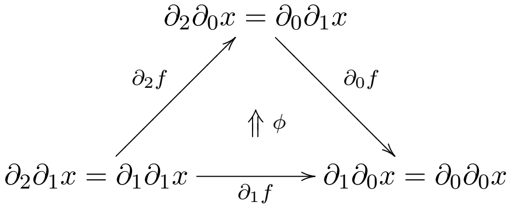
commutes.

2 The second nonabelian cohomology

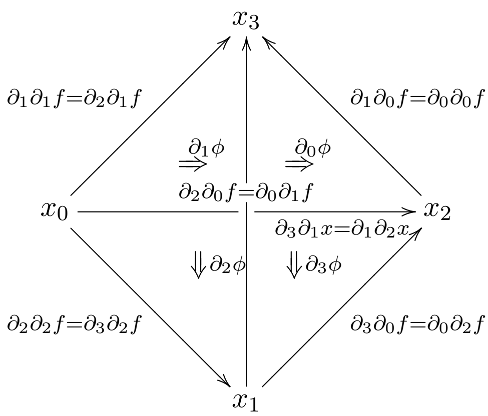
For any 3-truncated cosimplicial bicategory



we define a bicategory $Desc_2(\mathcal{B})$ of 2-descent data in the following way. The objects are triples (x, f, ϕ) consisting of an object x in $\mathcal{B}_0$, a 1-morphism $f\colon \partial_1(x) \to \partial_0(x)$ in $\mathcal{B}_1$, and a 2-morphism

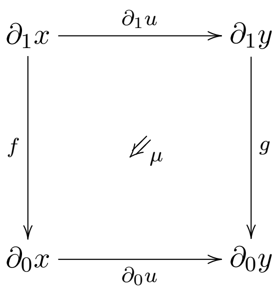


in $\mathcal{B}_2$ such that the diagram

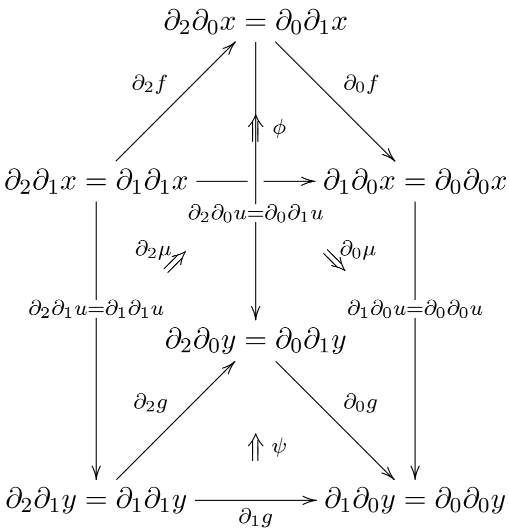


commutes, where the vertices x_i are defined by $x_0 = \partial_3\partial_2\partial_1 x$, $x_1 = \partial_3\partial_2\partial_0 x$, and so on (just by omitting the i-th coface operator from the string).

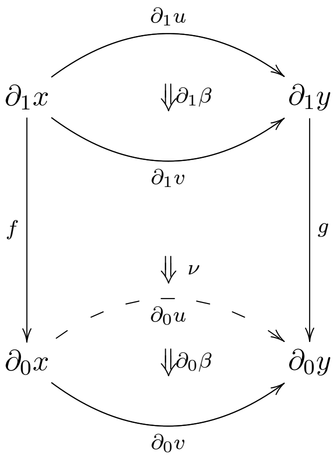
The 1-morphism $(u, \mu)\colon (x, f, \phi) \to (y, g, \psi)$ in $Desc_2(\mathcal{B})$ consists of a 1-morphism $u\colon x \to y$ in $\mathcal{B}_0$, together with the 2-morphism



in $\mathcal{B}_1$, such that the diagram



commutes. The 2-morphism $\beta\colon (u,\mu) \Rightarrow (v,\nu)$ in $Desc_2(\mathcal{B})$ is a 2-morphism $\beta\colon u \Rightarrow v$ in $\mathcal{B}_0$, such that the diagram



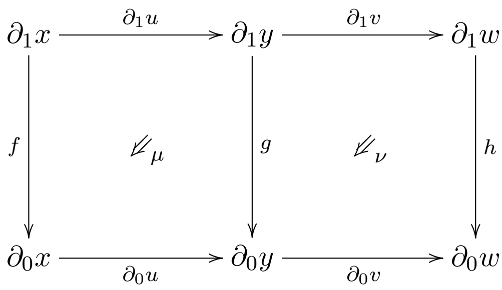
commutes.

Proposition 2.1. *The descent bicategory $Desc_2(\mathcal{B})$ is a bicategory.*

Proof. For any two composable 1-morphisms in $Desc_2(\mathcal{B})$

$$(x,f,\phi) \xrightarrow{(u,\mu)} (y,g,\psi) \xrightarrow{(v,\nu)} (w,h,\xi)$$

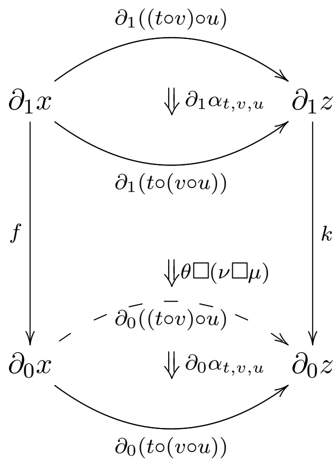
we define the composition by $(v, \nu) \circ (u, \mu) = (v \circ u, \nu \Box \mu)$ where $\nu \Box \mu$ is a 2-morphism obtained by the pasting of the diagram



in the bicategory $\mathcal{B}_1$. The horizontal and vertical compositions of 2-morphisms in $Desc_2(\mathcal{B})$ are inherited from the bicategory $\mathcal{B}_0$. So the associativity and left and right identity coherence are also inherited from the bicategory $\mathcal{B}_0$, and we see that for any three composable 1-morphisms in $Desc_2(\mathcal{B})$

$$(x, f, \phi) \xrightarrow{(u,\mu)} (y, g, \psi) \xrightarrow{(v,\nu)} (w, h, \xi) \xrightarrow{(t,\theta)} (z, k, \zeta)$$

the component $\alpha_{t,v,u} \colon [(t, \theta) \circ (v, \nu)] \circ (u, \mu) \Rightarrow (t, \theta) \circ [(v, \nu) \circ (u, \mu)]$ of the associativity isomorphism satisfy



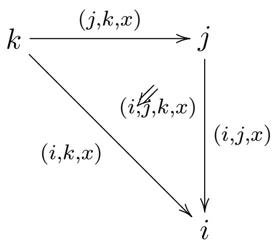
directly from the definition of the composition in $Desc_2(\mathcal{B})$. □

The second Čech nonabelian cohomology is defined with respect to the covering $\mathcal{U} = \{U_i\}_{i\in I}$ of the topological space X. The epimorphism $e = (e_i)_{i\in I} \colon U = \coprod_{i\in I} U_i \to X$, induced by the family of embeddings $e_i \colon U_i \to X$, gives a 3-truncation of the simplicial resolution $U_\bullet$

$$U_3 \overset{d_0}{\underset{d_3}{\rightrightarrows}} U_2 \overset{d_0}{\underset{d_2}{\rightrightarrows}} U_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} U_0 \xrightarrow{\ e\ } X$$

where $U_0 = \coprod_{i\in I} U_i$, $U_1 = \coprod_{i,j\in I} U_{ij}$, $U_2 = \coprod_{i,j,k\in I} U_{ijk}$ and $U_3 = \coprod_{i,j,k,l\in I} U_{ijkl}$ (where U_{ij} denotes the double intersection $U_{ij} = U_i \cap U_j$ and so on).

This is just the 3-truncation of the nerve of the Čech groupoid associated to the covering $e \colon U \to X$, whose objects are given by the elements (i,x) of U, and for which there exists a unique morphism $(i,j,x) \colon (j,x) \to (i,x)$ for any element $x \in U_{ij}$. Thus, target and source morphisms defines face operators $d_0^1, d_1^1 \colon U_1 \to U_0$ which are given by the first and the second projection, respectively. The 2-simplex (i,j,k,x) in U_2 may be seen as the diagram



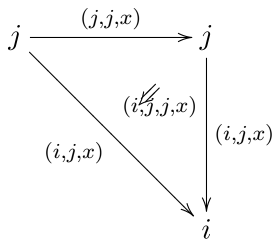
from which we see that the face operators $d_0^2, d_1^2, d_2^2 \colon U_2 \to U_1$ are defined by

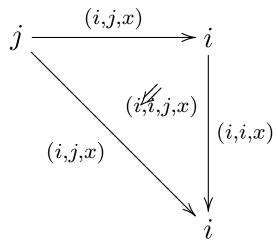
$$\begin{aligned} d_0^2(i,j,k,x) &= (i,j,x) \\ d_1^2(i,j,k,x) &= (i,k,x) \\ d_2^2(i,j,k,x) &= (j,k,x) \end{aligned}$$

and they are just three possible inclusions of triple intersections into double intersections. The degeneracy operators $s_0^2, s_1^2 \colon U_1 \to U_2$ are given by

$$\begin{aligned} s_0^2(i,j,x) &= (i,j,j,x) \\ s_1^2(i,j,x) &= (i,i,j,x) \end{aligned}$$

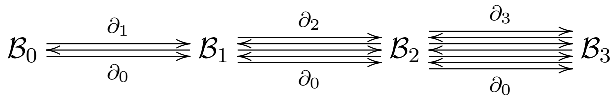
and these two degenerate 2-simplices may be seen as the two diagrams





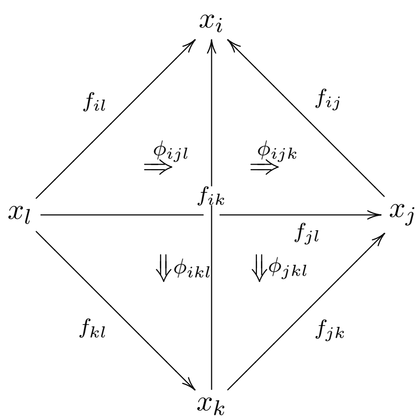
respectively.

The 3-truncation of the simplicial resolution of the covering $\mathcal{U} = \{U_i\}_{i \in I}$ defines a cosimplicial bicategory



where each bicategory $\mathcal{B}_n$ has objects given by the discrete category $(\mathcal{B}_i)_0$ defined by the set $Hom_{\mathcal{E}}(U_n, B_0)$, and whose category of 1-morphisms and 2-morphisms is given by the fiber of the small fibration $\mathcal{F}_{\mathcal{B}} U_n$ over the object U_n in $\mathcal{E}$. On the level of objects, coface operators are defined by the precomposition $\partial_i^n(f) = f d_i^n$ for any object $f \colon U_{n-1} \to B_0$ of the bicategory $\mathcal{B}_{n-1}$, from where we see that these are the strict homomorphisms of bicategories.

Thus the 2-cocycle in the second Čech nonabelian cohomology is given by the triple $(\mathbf{x}, \mathbf{f}, \phi)$, where $\mathbf{x} = (x_i)_{i \in I}$ is the family of morphisms $x_i \colon U_i \to B_0$ together with the family $\mathbf{f} = (f_{ij})_{i,j \in I}$ of morphisms $f_{ij} \colon U_{ij} \to B_1$ such that $s_0 f_{ij} = x_j$ and $t_0 f_{ij} = x_i$. The family $\phi = (\phi_{ijk})_{i,j,k \in I}$ is given by morphisms $\phi_{ijk} \colon U_{ijk} \to B_2$ which satisfy $s_1 \phi_{ijk} = f_{ik}$ and $t_1 \phi_{ijk} = f_{ij} \circ f_{jk}$ and we can view it as the 2-simplex



commutes, which means that we have the identity

$$(f_{ij} \circ \phi_{jkl})\phi_{ijl} = \alpha_{ijkl}(\phi_{ijk} \circ f_{kl})\phi_{ikl}$$

for the nonabelian 2-cocycle $(x_i, f_{ij}, \phi_{ijk})$ with values in the bicategory $\mathcal{B}$.

3 Actions of bicategories

Let $\mathcal{B}$ be a bicategory. There is a weak 2-monad $\mathcal{T}$ on a comma 2-category $\mathrm{Cat} \downarrow \mathcal{B}_0$ naturally induced by $\mathcal{B}$ as following. It is a weak 2-functor $\mathcal{T} : \mathrm{Cat} \downarrow \mathcal{B}_0 \to \mathrm{Cat} \downarrow \mathcal{B}_0$, whose image for each object $\Lambda : \mathcal{C} \to \mathcal{B}_0$ of $\mathrm{Cat} \downarrow \mathcal{B}_0$, is defined by $\mathcal{T}(\Lambda) := \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$.

Definition 3.1. *A right action of a bicategory $\mathcal{B}$ on a category $\mathcal{C}$ is given by by the following data:*

- *a functor $\Lambda : \mathcal{C} \to \mathcal{B}_0$ from the category $\mathcal{C}$ to the discrete category of objects $\mathcal{B}_0$ of the weak 2-category $\mathcal{B}$, called the momentum functor,*
- *a functor $\Phi : \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{C}$, called the action functor, and we usually write $\Phi(p, f) := p \triangleleft f$, for any object (p, f) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$, and $\Phi(a, \phi) := a \triangleleft \phi$ for any morphism $(a, \phi) : (p, f) \to (q, g)$ in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$,*
- *a natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{\Phi \times Id_{\mathcal{B}_1}} & \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \\ {\scriptstyle Id_{\mathcal{C}} \times \mathcal{H}} \downarrow & \swarrow_{\kappa} & \downarrow {\scriptstyle \Phi} \\ \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow[\Phi]{} & \mathcal{C} \end{array}$$

whose component for any object (p, f, g) in $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ is written as

$$\kappa_{p,f,g} : (p \triangleleft f) \triangleleft g \to p \triangleleft (f \circ g),$$

- *a natural isomorphism*

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_0 & \xrightarrow{Id_{\mathcal{C}} \times I} & \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \\ {\scriptstyle (Id_{\mathcal{C}}, \Lambda)} \uparrow & \swarrow_{\iota} & \downarrow {\scriptstyle \Phi} \\ \mathcal{C} & = & \mathcal{C} \end{array}$$

where we write for each object p in $\mathcal{C}$

$$\iota_p : p \triangleleft i_{\Lambda(p)} \to p$$

such that following axioms are satisfied:

- *equivariance of the action*

$$\begin{array}{ccc} \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{\Phi} & \mathcal{C} \\ {\scriptstyle Pr_2}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ \mathcal{B}_1 & \xrightarrow[D_1]{} & \mathcal{B}_0 \end{array}$$

which means that for any object (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$, *we have* $\Lambda(p \triangleleft f) = D_1(f)$, *and for any morphism* $(a, \phi)\colon (p, f) \to (q, g)$ *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$, *we have* $\Lambda(a \triangleleft \phi) = D_1(\phi)$,

- *for any object* (p, f, g, h) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1$ *the following diagram*

$$\begin{array}{ccc} & ((p \triangleleft f) \triangleleft g) \triangleleft h & \\ {\scriptstyle \kappa_{p,f,g} \triangleleft h}\swarrow & & \searrow{\scriptstyle \kappa_{p \triangleleft f,g,h}} \\ (p \triangleleft (f \circ g)) \triangleleft h & & (p \triangleleft f) \triangleleft (g \circ h) \\ {\scriptstyle \kappa_{p,f \circ g,h}}\downarrow & & \downarrow{\scriptstyle \kappa_{p,f,g \circ h}} \\ p \triangleleft ((f \circ g) \circ h) & \xrightarrow[p \triangleleft \alpha_{f,g,h}]{} & p \triangleleft (f \circ (g \circ h)) \end{array}$$

commutes,

- *for any object* (p, f) *in* $\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1$ *following diagrams*

$$\begin{array}{ccc} (p \triangleleft i_{\Lambda_0(p)}) \triangleleft f & \xrightarrow{\kappa_{p,i_{\Lambda_0(p)},f}} & p \triangleleft (i_{\Lambda_0(p)} \circ f) \\ {\scriptstyle \iota_p \triangleleft f}\searrow & & \swarrow{\scriptstyle p \triangleleft \lambda_f} \\ & p \triangleleft f & \end{array} \qquad \begin{array}{ccc} (p \triangleleft f) \triangleleft i_{s_0(f)} & \xrightarrow{\kappa_{p,f,i_{s_0(f)}}} & p \triangleleft (f \circ i_{s_0(f)}) \\ {\scriptstyle \iota_{p \triangleleft f} \triangleleft i_{s_0(f)}}\searrow & & \swarrow{\scriptstyle p \triangleleft \rho_f} \\ & p \triangleleft f & \end{array}$$

commute.

Remark 3.1. *Note the fact that $\Phi\colon \mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{C}$ is a functor, immediately implies an interchange law*

$$(b \triangleleft \psi)(a \triangleleft \phi) = (ba) \triangleleft (\psi\phi)$$

Definition 3.2. *Let $\pi\colon \mathcal{C} \to M$ be a bundle of categories over an object M in $\mathcal{E}$. A (fiberwise) right action of a bicategory $\mathcal{B}$ on a bundle of categories $\pi\colon \mathcal{C} \to M$ is given by the action of the bicategory $\mathcal{B}$ on a category $\mathcal{C}$ for which the diagram*

$$\begin{array}{ccc}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{\Phi} & \mathcal{C} \\
{\scriptstyle Pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{B}_1 & \xrightarrow[\pi]{} & M
\end{array}$$

commute. We call a bundle $\pi\colon \mathcal{C} \to M$, a $\mathcal{B}$-2-bundle over M.

Definition 3.3. *Let $(\mathcal{C}, \Lambda, \Phi, \alpha, \iota)$ and $(\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ be two $\mathcal{B}$-categories. A $\mathcal{B}$-equivariant functor from $(\mathcal{C}, \Lambda, \Phi, \alpha, \iota)$ to $(\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ is a pair (F, θ) consisting of*

- *a functor $F\colon \mathcal{C} \to \mathcal{D}$*
- *a natural transformation $\theta\colon F \circ \Psi \to \Phi \circ (F \times Id_{\mathcal{B}_1})$*

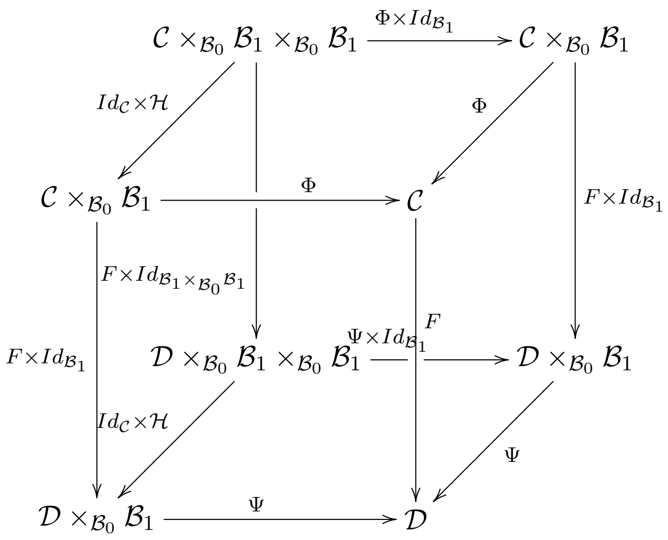
$$\begin{array}{ccc}
\mathcal{C} \times_{\mathcal{B}_0} \mathcal{B}_1 & \xrightarrow{F \times Id_{\mathcal{B}_1}} & \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1 \\
{\scriptstyle \Phi}\downarrow & \Swarrow_{\phi} & \downarrow{\scriptstyle \Psi} \\
\mathcal{C} & \xrightarrow[F]{} & \mathcal{D}
\end{array}$$

such that following conditions are satisfied

- $\Omega \circ F = \Lambda$

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
& {\scriptstyle \Lambda}\searrow \quad \swarrow{\scriptstyle \Omega} & \\
& \mathcal{B}_0 &
\end{array}$$

- *the diagram of natural transformations*



commutes, which means that two natural transformations

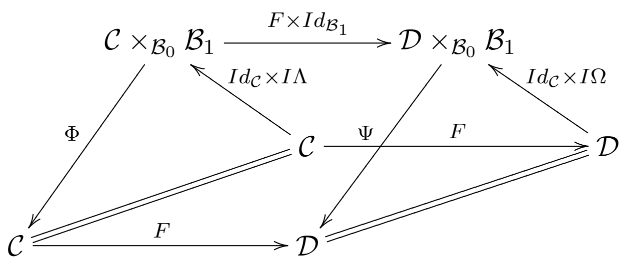
$$(\theta \circ (Id_{\mathcal{C}} \times \mathcal{H}))(\Psi \circ (\theta \times Id_{\mathcal{B}_1}))(\beta \circ (F \times Id_{\mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1}))$$

and

$$(\Psi \circ (\theta \times Id_{\mathcal{B}_1}))(\theta \circ (\Phi \times Id_{\mathcal{B}_1}))\alpha$$

obtained by pasting, are equal.

- *the diagram of natural transformations, which fill the faces*

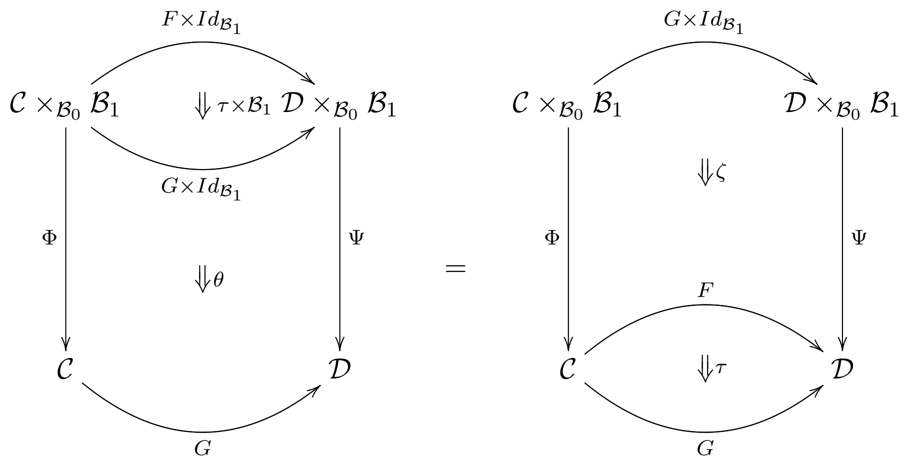


commutes, meaning that we have an equation

$$(\Psi * id_L) \cdot (\theta * \Phi) \cdot (\kappa * F) = id_F * \iota$$

where $L\colon \mathcal{C} \to \mathcal{D} \times_{\mathcal{B}_0} \mathcal{B}_1$ *is a functor given by the equality of functors* $(F \times Id_{\mathcal{B}_1}) \circ (Id_{\mathcal{C}} \times I\Lambda) = I\Omega \circ F$*, which, in turn follows from the equality* $\Omega F = \Lambda$*.*

Definition 3.4. *A $\mathcal{B}$-equivariant natural transformation between $\mathcal{B}$-covariant functors $(F, \theta), (G, \zeta)\colon (\mathcal{C}, \Lambda, \Phi, \alpha, \iota) \to (\mathcal{D}, \Psi, \Omega, \beta, \kappa)$ is a natural transformation $\tau\colon F \to G$ such that following equality*



of natural transformations is satisfied.

The above construction gives rise to the 2-category in an obvious way, so we have a following theorem.

Theorem 3.1. *The class of $\mathcal{B}$-categories, $\mathcal{B}$ equivariant functors and their natural transformations form a 2-category.*

Proof. The vertical and horizontal composition in a 2-category is induced from the composition in Cat. □

4 Bigroupoid principal 2-bundles

Definition 4.1. *A right action of a bigroupoid $\mathcal{B}$ on a groupoid $\mathcal{P}$ is given by the action of the underlying bicategory $\mathcal{B}$ on a category $\mathcal{P}$ given as previously by $(\mathcal{P}, \mathcal{B}, \Lambda, \Phi, \alpha, \iota)$.*

Definition 4.2. *Let $\mathcal{B}$ be an internal bigroupoid in $\mathcal{E}$, and $\pi\colon \mathcal{P} \to X$ a right $\mathcal{B}$-2-bundle of groupoids over X in $\mathcal{E}$. We say that $(\mathcal{P}, \pi, \Lambda, \mathcal{A}, X)$ is a right $\mathcal{B}$-principal-2-bundle (or a right $\mathcal{B}$-torsor) over X if the following conditions are satisfied:*

- *two canonical terminal morphisms $\pi_0\colon P_0 \to X$ and $\pi_1\colon P_1 \to X$ are epimorphisms,*
- *two canonical action morphisms $\lambda_0\colon P_0 \to B_0$ and $\lambda_1\colon P_1 \to B_0$ are epimorphisms,*
- *the induced internal functor*

$$(Pr_1, \Phi)\colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \longrightarrow \mathcal{P} \times_X \mathcal{P}$$

  is a (strong) equivalence of internal groupoids over $\mathcal{P}$ (where both groupoids are seen as objects over $\mathcal{P}$ by the first projection functor).

Example 4.1. *(The unit principal 2-bundle) The unit $\mathcal{B}$-bundle is given by the triple $(\mathcal{B}_1, T, S, \mathcal{H}, B_0)$ where the momentum is given by the source functor $S\colon \mathcal{B}_1 \to \mathcal{B}_0$, and the action is given by the horizontal composition $\mathcal{H}\colon \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{B}_1$.*

Example 4.2. *(The pullback principal 2-bundle) For any principal $\mathcal{B}$-bundle $(\mathcal{P}, \pi, \Lambda, \Phi, X)$ over X, and any morphism $f\colon M \to B_0$, we have a pullback $\mathcal{B}$-principal bundle over M, defined as the quadruple $(f^*(\mathcal{P}), Pr_1, \Lambda \circ Pr_2, f^*(\Phi), M)$.*

5 Cocyclic description of principal 2-bundles

Since we assumed that the functor $(Pr_1, \Phi)\colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is an equivalence, we choose its weak inverse

$$(\mathrm{Pr}_1, \mathcal{D})\colon \mathcal{P} \times_X \mathcal{P} \longrightarrow \mathcal{P} \times_{B_0} \mathcal{B}_1,$$

together with natural isomorphisms

$$(\mathrm{Pr}_1, \mu)\colon Id_{\mathcal{P}\times_{B_0}\mathcal{B}_1} \Longrightarrow (\mathrm{Pr}_1, \mathcal{D}) \circ (Pr_1, \Phi), \qquad (\mathrm{Pr}_1, \nu)\colon (Pr_1, \Phi) \circ (\mathrm{Pr}_1, \mathcal{D}) \Longrightarrow Id_{\mathcal{P}\times_X\mathcal{P}},$$

The second component of the above weak inverse is a functor $D\colon \mathcal{P} \times_X \mathcal{P} \longrightarrow \mathcal{B}_1$, which we we call a division functor, for reasons that we will soon explain. The component of the natural isomorphism $\nu\colon (Pr_1, \Phi) \circ (\mathrm{Pr}_1, \mathcal{D}) \Longrightarrow Id_{\mathcal{P}\times_X\mathcal{P}}$, indexed by the object $(p, q) \in \mathcal{P} \times_X \mathcal{P}$, is an isomorphism $\nu_{p,q}\colon p \triangleleft p^*q \to q$, where we use an abbreviation $p^*q := \mathcal{D}(p, q)\colon \lambda_0(q) \to \lambda_0(p)$, for the 1-morphism in $\mathcal{B}_1$. The natural isomorphism $\mu\colon Id_{\mathcal{P}\times_{B_0}\mathcal{B}_1} \Longrightarrow (\mathrm{Pr}_1, \mathcal{D}) \circ (Pr_1, \Phi)$, is indexed by the object $(p, f) \in \mathcal{P} \times_{B_0} \mathcal{B}_1$, by an isomorphism $\mu_{p,f}\colon p \to p^*(p \triangleleft f)$.

Let's now give a cocyclic description of the principal $\mathcal{B}$-2-bundle $\mathcal{P}$. Since the map $\pi\colon P_0 \to M$ is a surjective submersion, we can find an open cover $M = \bigcup U_i$ of the base manifold M together with local sections $\sigma_i\colon U_i \to P_0$ of the map π. The corresponding statement in the topos $\mathcal{E}$ is that epimorphism $\pi\colon P_0 \to M$ in $\mathcal{E}$ locally splits, since the diagonal morphism $\Delta\colon P_0 \to P_0 \times_M P_0$ is a splitting in $\mathcal{E}/P_0$ of the pullback bundle $\pi^*(\pi)\colon P_0 \times_M P_0 \to P_0$ which is given by $pr_1\colon P_0 \times_M P_0 \to P_0$.

We use the division functor to define $g_{ij} = \mathcal{D}(\sigma_i, \sigma_j)\colon U_{ij} \to B_1$, and a local sections $f^\alpha_{ij}\colon U^\alpha_{ij} \to P_1$ of $\pi s = \pi t$ over some covering U^α_{ij} of U_{ij} such that

$$f_{ij}\colon \sigma_j \to \sigma_i \triangleleft g_{ij}.$$

The following diagram

$$\begin{array}{ccc}
\sigma_k & \xrightarrow{\quad f_{jk} \quad} & \sigma_j \triangleleft g_{jk} \\
 & & \downarrow{\scriptstyle f_{ij}\triangleleft g_{jk}} \\
\downarrow{\scriptstyle f_{ik}} & & (\sigma_i \triangleleft g_{ij}) \triangleleft g_{jk} \\
 & & \downarrow{\scriptstyle \kappa_{ijk}} \\
\sigma_i \triangleleft g_{ik} & \xrightarrow[\sigma_i \triangleleft \beta_{ijk}]{} & \sigma_i \triangleleft (g_{ij} \circ g_{jk})
\end{array}$$

defines a morphism in $\psi \in Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \triangleleft g_{ik}, \sigma_i \triangleleft (g_{ij} \circ g_{jk}))$ by the composition

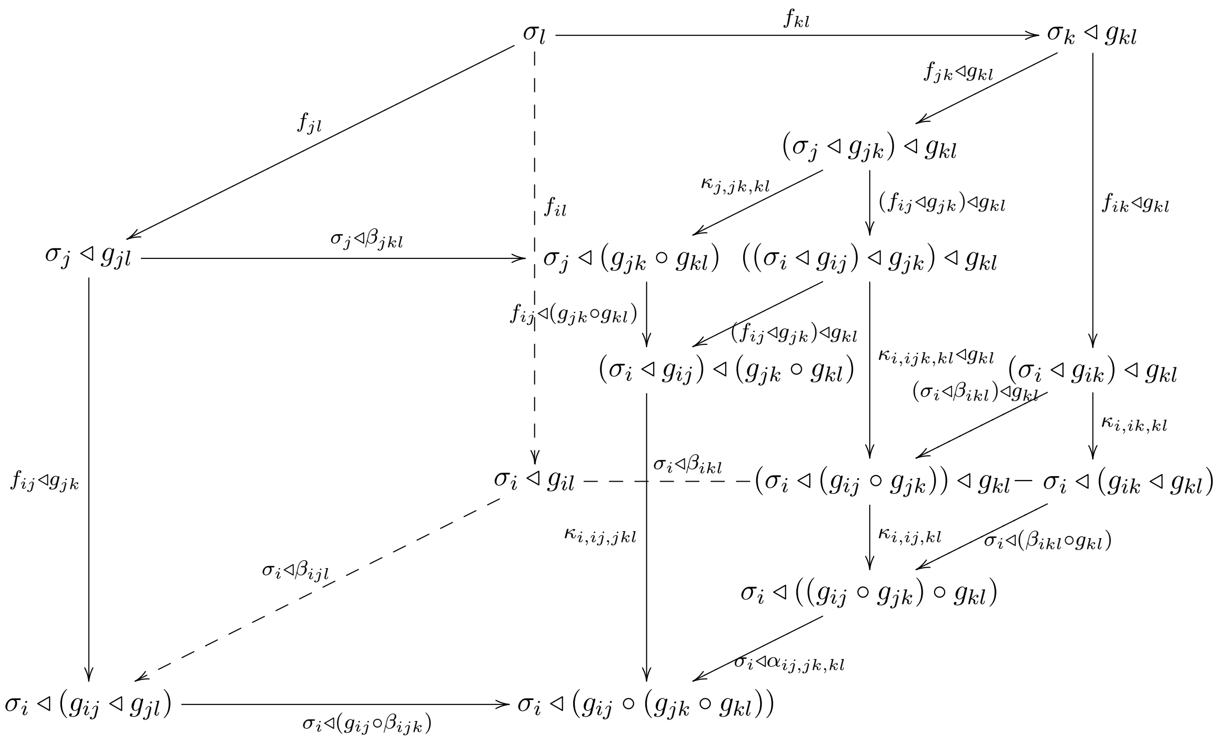
$$\sigma_i \triangleleft g_{ik} \xrightarrow{f_{ik}^{-1}} \sigma_k \xrightarrow{f_{jk}} \sigma_j \triangleleft g_{jk} \xrightarrow{f_{ij} \triangleleft g_{jk}} (\sigma_i \triangleleft g_{ij}) \triangleleft g_{jk} \xrightarrow{\kappa_{ijk}} \sigma_i \triangleleft (g_{ij} \circ g_{jk})$$

and since the set $Hom_{\mathcal{P}\times_X\mathcal{P}}(\sigma_i \triangleleft g_{ik}, \sigma_i \triangleleft (g_{ij} \circ g_{jk}))$ is an image of the induced functor (Pr_1, Φ) which defines a bijective correspondence with the set $Hom_{\mathcal{P}\times_{\mathcal{B}_0}\mathcal{B}_1}((\sigma_i, g_{ik}), (\sigma_i, g_{ij} \circ g_{jk}))$ the inverse image of ψ defines sections $\beta_{ijk} \colon g_{ik} \to g_{ij} \circ g_{jk}$ in B_2, such that the diagram becomes the identity

$$(\sigma_i \triangleleft \beta_{ijk}) f_{ik} = \kappa_{ijk}(f_{ij} \triangleleft g_{jk}) f_{jk}$$

Theorem 5.1. *Any $\mathcal{B}$-2-torsor $\pi \colon \mathcal{P} \to X$ gives rise to the class $\mathcal{H}^2(X, \mathcal{B})$.*

Proof. Consider the following cube



in which all faces except the bottom and right faces are diagrams which define nonabelian cocycles. The right face consists of one such diagram acted by g_{kl}, two are instances of naturality of the action, and one is coherence for action. Since these five faces of the cube in which all arrows are invertible commute, it follows that the sixth (bottom) face

$$\begin{array}{ccc}
\sigma_i \triangleleft g_{il} & \xrightarrow{\sigma_i \triangleleft \beta_{ikl}} & \sigma_i \triangleleft (g_{ik} \circ g_{kl}) \\
 & & \Big\downarrow {\scriptstyle \sigma_i \triangleleft (\beta_{ijk} \circ g_{kl})} \\
{\scriptstyle \sigma_i \triangleleft \beta_{ijl}} \Big\downarrow & & \sigma_i \triangleleft ((g_{ij} \circ g_{jk}) \circ g_{kl}) \\
 & & \Big\downarrow {\scriptstyle \sigma_i \triangleleft \alpha_{ij,jk,kl}} \\
\sigma_i \triangleleft (g_{ij} \circ g_{jl}) & \xrightarrow[\sigma_i \triangleleft (g_{ij} \circ \beta_{ijk})]{} & \sigma_i \triangleleft (g_{ij} \circ (g_{jk} \circ g_{kl}))
\end{array}$$

also commutes. Since the functor $(Pr_1, \Phi) \colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is fully faithful, the inverse image of the diagonal 2-morphism from $\sigma_i \triangleleft g_{il}$ to $\sigma_i \triangleleft (g_{ij} \circ (g_{jk} \circ g_{kl}))$ in the above diagram, consists of the single 2-morphism between g_{il} and $(g_{ij} \circ (g_{jk} \circ g_{kl}))$ which gives the identity

$$(g_{ij} \circ \beta_{jkl})\beta_{ijl} = \alpha_{ij,jk,kl}(\beta_{ijk} \circ g_{kl})\beta_{ikl}$$

for the nonabelian 2-cocycle (g_{ij}, β_{ijk}) with values in the bigroupoid $\mathcal{B}$. □

Theorem 5.2. *The above correspondence gives a biequivalence*

$$2Tors(X, \mathcal{B}) \sim_{bi} \mathcal{H}^2(X, \mathcal{B})$$

Proof. We take a class in $\mathcal{H}^2(M, \mathcal{B})$ represented by the 2-cocycle $(\tau_i, g_{ij}, \beta_{ijk})$, with respect to some covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M, and a 2-truncation of the simplicial resolution $U_\bullet$

$$U \times_M U \times_M U \underset{d_2}{\overset{d_0}{\Rrightarrow}} U \times_M U \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \xrightarrow{\ e\ } M$$

of the epimorphism $e = (e_i)_{i \in I} \colon U = \coprod_{i \in I} U_i \to M$, induced by a family of embeddings $e_i \colon U_i \to M$. This is just the nerve of the Čech groupoid associated to the covering $e \colon U \to M$, whose objects are given by the elements (i, x) of U, and unique morphisms $(i, j, x) \colon (j, x) \to (i, x)$ between any two elements in the same fiber. Thus, target and source morphisms $d_0, d_1 \colon U \times_M U \to U$ are given by the first and the second projection, respectively.

The construction of the 2-torsor $\mathcal{P}$ is given by the pseudocolimit of the pseudosimplicial category over the simplicial resolution $U_\bullet$ of the covering $e\colon U \to M$

$$\begin{array}{ccccccc}
\mathcal{R}_2 & \overset{D_0}{\underset{D_2}{\rightrightarrows\!\!\!\rightarrow}} & \mathcal{R}_1 & \overset{D_0}{\underset{D_1}{\rightrightarrows}} & \mathcal{R}_0 & \xrightarrow{\ \eta\ } & \mathcal{P} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} & & \downarrow{\scriptstyle \pi} \\
U_2 & \overset{d_0}{\underset{d_2}{\rightrightarrows\!\!\!\rightarrow}} & U_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & U_0 & \xrightarrow{\ e\ } & M
\end{array}$$

where $U_0 = U$, $U_1 = U \times_M U$, $U_2 = U \times_M U \times_M U$, and so on.

Each category $\mathcal{R}_n$ is obtained as the pullback $\mathcal{R}_n = (\tau d^n)^*(\mathcal{B}_1)$ where the morphism $d^n\colon U_n \to U_0$ is defined by $d^n = d_n d_{n-1} ... d_1$ for $n \geq 1$, and $d^0 = id_U$.

Explicitly, on the level of objects, the category $\mathcal{R}_0$ is given by the pullback $\tau^*(\mathcal{B}_1)$ of the trivial $\mathcal{B}$-2-torsor $T\colon \mathcal{B}_1 \to \mathcal{B}_0$. Object of the category $\mathcal{R}_0$ are triples (i, x, f) where $\sigma_i(x) = t_0(f)$, and any morphism is given by a triple $(i, x, \phi)\colon (i, x, f) \to (i, x, f')$ where $\phi\colon f \Rightarrow f'$ is a 2-morphism in B_2, such that $\sigma_i(x) = T(\phi)$. The composition in $\mathcal{R}_0$ is inherited from the vertical composition of 2-morphisms in $\mathcal{B}$, and the functor $\rho_0\colon \mathcal{R}_0 \to U$ is given by the projection on the first two factors.

The category $\rho_1\colon \mathcal{R}_1 \to U \times_M U$ over $U \times_M U$ is defined by the pullback $\mathcal{R} = (\tau d_1)^*(\mathcal{B}_1)$. Objects of the category $\mathcal{R}_1$ are quadruples (i, j, x, g) where $\sigma_j(x) = t_0(g)$, and any morphism is given by a quadruple $(i, j, x, \psi)\colon (i, j, x, g) \to (i, j, x, g')$ where $\psi\colon g \Rightarrow g'$ is again a 2-morphism in B_2, such that $\sigma_j(x) = T(\psi)$.

The category $\rho_2\colon \mathcal{R}_2 \to U \times_M U \times_M U$ over $U \times_M U \times_M U$ is defined by the pullback $\mathcal{R} = (\tau d_2 d_1)^*(\mathcal{B}_1)$, so its objects and morphisms are given by quintuples as above.

Two functors $D_0, D_1\colon \mathcal{R}_1 \to \mathcal{R}_0$ are defined by $D_0(i, j, x, g) = (i, x, g_{ij}(x)g)$ and $D_1(i, j, x, g) = (j, x, g)$ on the level of objects and similarly on the level of morphisms.

Three functors $D_0, D_1, D_2\colon \mathcal{R}_2 \to \mathcal{R}_1$ are defined by $D_0(i, j, k, x, h) = (i, j, x, g_{jk}(x)h)$, $D_1(i, j, k, x, h) = (i, k, x, h)$ and $D_2(i, j, k, x, h) = (j, k, x, h)$ on the level of objects and similarly on the level of morphisms.

The following simplicial identities of functors hold on the nose

$$D_1 D_1(i, j, k, x, h) = D_1(i, k, x, h) = (k, x, h) = D_1(j, k, x, h) = D_1 D_2(i, j, k, x, h)$$
$$D_0 D_2(i, j, k, x, h) = D_0(j, k, x, h) = (j, x, g_{jk}(x)h) = D_0(i, j, x, g_{jk}(x)h) = D_1 D_0(i, j, k, x, h)$$

The nontrivial simplicial identity is given by a natural isomorphism $\beta\colon D_0 D_0 \Rightarrow D_0 D_1$, whose component indexed by an object (i, j, k, x, h) of $\mathcal{R}_2$ is given by a morphism (i, x, β_{ijk}^{-1}) from the object $D_0 D_0(i, j, k, x, h) = D_0(i, j, x, g_{jk}(x)h) = (i, x, g_{ij}(x)g_{jk}(x)h)$ to the object $D_0 D_1(i, j, k, x, h) = D_0(i, k, x, h) = (i, x, g_{ik}(x)h)$.

We construct the category $\mathcal{P}$ as a pseudocolimit of the pseudosimplicial category $\mathcal{R}_\bullet$. It is given by a version of the Grothendieck construction, and it goes as follows.

The objects of $\mathcal{P}$ are given by the union of objects of $\mathcal{R}_n$. We describe morphisms in $\mathcal{P}$ by means of a particular example. A morphism $(m, \phi)\colon (i, x, f) \to (i, j, k, x, g)$ from an object (i, x, f) in $\mathcal{R}_0$ to an object (i, j, k, x, g) in $\mathcal{R}_2$ is given by a pair of morphisms, where $m\colon [0] \to [2]$ is a monotonic map in Δ, whose canonical factorization in Δ is given by $m = \delta_1\delta_0$ (so that we have $U(m)(i, j, k, x) = (i, x)$ in U_1). Then the second component of the above pair is given by a morphism $\phi\colon (i, x, f) \to \mathcal{R}(m)(i, j, k, x, g) = (i, x, g_{ik}(x)g)$ in $\mathcal{R}_0$. For another morphism $(n, \psi)\colon (i, j, k, x, g) \to (i, j, k, l, x, h)$, where $n = \delta_1\colon [2] \to [3]$ and $\psi\colon (i, j, k, x, g) \to \mathcal{R}(n)(i, j, k, l, x, h) = (i, j, k, x, g_{kl}(x)h)$, the composition is defined by a pair $(nm, \psi \circ \phi)\colon (i, x, f) \to (i, j, k, l, x, h)$, where the morphism $\psi \circ \phi\colon (i, x, f) \to (i, k, l, x, h)$ is defined by the composition

$$(i, x, f) \xrightarrow{\phi} \mathcal{R}(m)(i, j, k, x, g) \xrightarrow{\mathcal{R}(m)(\psi)} \mathcal{R}(m)\mathcal{R}(n)(i, j, k, l, x, h) \xrightarrow{\sim} \mathcal{R}(nm)(i, j, k, l, x, h)$$

where the last isomorphism is obtained from the component of the natural isomorphism $\beta\colon D_0D_0 \Rightarrow D_0D_1$.

Obviously, $\rho_\bullet\colon \mathcal{R}_\bullet \to U_\bullet$ is a simplicial functor to a discrete simplicial category $U_\bullet$, so that we have simplicial identities of functors $d_i\rho_n = \rho_{n-1}D_i\colon \mathcal{R}_n \to U_{n-1}$, for all $0 \leq i \leq n$. It follows that the functor $e\rho_0\colon \mathcal{R}_0 \to M$ provides a cocone of the pseudosimplicial category $\mathcal{R}_\bullet$, and from the universal property of the pseudocolimit $\mathcal{P}$, we obtain a unique functor $\pi\colon \mathcal{P} \to M$, providing $\mathcal{P}$ with the structure of a bundle of groupoids over M.

The projection $\pi\colon \mathcal{P} \to M$ is explicitly described by $\pi_0(i, j, k, l, x, h) = x$ on the level of objects. Also we have a momentum functor $\lambda\colon \mathcal{P} \to B_0$, defined by $\pi_0(i, j, k, l, x, h) = s_0(h)$, and the action functor is naturally defined by the horizontal composition,

$$(i, j, k, l, x, h) \triangleleft g = (i, j, k, l, x, h \circ g)$$

where we have $s_0(h) = t_0(g)$. We still need to check that $\pi\colon \mathcal{P} \to M$ is a principal $\mathcal{B}$-2-bundle over M, which means that the induced functor $(Pr_1, \Phi)\colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \to \mathcal{P} \times_X \mathcal{P}$ is an equivalence. We will prove that by explicitly defining the corresponding division functor. For any two elements (i, j, k, x, h) and (l, x, g) in the same fiber (over $x \in U_{ijkl} \subseteq M$), it is defined by □

Theorem 5.3. *There exist a biequivalence*

$$2Tors(X, \mathcal{B}) \sim_{bi} Bun(\mathcal{B})$$

Proof. Let's again choose local sections $\sigma_i\colon U_i \to P_0$ of a surjective submersion $\pi\colon P_0 \to M$, and we consider the morphism $\tau\colon U \to B_0$, defined by $\tau = (\tau_i)_{i\in I}$, where $U = \coprod_{i\in I} U_i$ and $\tau_i := \lambda_0\sigma_i\colon U_i \to B_0$. Than the induced morphism

$$\phi\colon \tau^*(\mathcal{B}_1) \longrightarrow \mathcal{P}|_U$$

defined by $\phi(x_i, g) = \sigma_i(x_i)g$ is an equivalence since it is an equivariant morphism of $\mathcal{B}$-2-torsors, and the 2-category $2Tors(X, \mathcal{B})$ is a bigroupoid. □

6 Simplicial interpretation of bigroupoid principal 2-bundles

For an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$, we define the action bicategory $\mathcal{A}_{\mathcal{B}}\mathcal{P}$. Objects are given by objects P_0 of the category $\mathcal{P}$. For any two objects q and p, a 1-morphism is a pair (ψ, h) which we draw as an arrow

$$q \xrightarrow{(\psi,h)} p$$

where $h\colon \lambda_0(q) \to \lambda_0(p)$ is a 1-morphism in the bicategory $\mathcal{B}$, and $\psi\colon q \to p \triangleleft h$ is a morphism in the category $\mathcal{P}$, thus it is an element of P_1. A 2-morphism $\gamma\colon (\psi, h) \Rightarrow (\xi, l)$

$$q \underset{(\xi,l)}{\overset{(\psi,h)}{\rightrightarrows}} p \qquad \Downarrow\gamma$$

is a 2-morphism $\gamma\colon h \Rightarrow l$ in B_2, such that the diagram of morphisms in $\mathcal{P}$

$$\begin{array}{ccc} q & \xrightarrow{\psi} & p \triangleleft h \\ & {\scriptstyle\xi}\searrow & \downarrow{\scriptstyle p\triangleleft\gamma} \\ & & p \triangleleft l \end{array}$$

commutes. We define the composition for any two composable 1-morphisms

$$r \xrightarrow{(\phi,g)} q \xrightarrow{(\psi,h)} p$$

by $(\psi, h) \circ (\phi, g) = (\psi \circ \phi, h \circ g)\colon r \to p$, where $\psi \circ \phi\colon r \to p \triangleleft (h \circ g)$ is a morphism in $\mathcal{P}$, defined by the composition

$$r \xrightarrow{\phi} q \triangleleft g \xrightarrow{\psi\triangleleft g} (p \triangleleft h) \triangleleft g \xrightarrow{\kappa_{p,h,g}} p \triangleleft (h \circ g)$$

and we will show that this composition is a coherently associative. For any three composable 1-morphisms

$$s \xrightarrow{(\varphi,f)} r \xrightarrow{(\phi,g)} q \xrightarrow{(\psi,h)} p$$

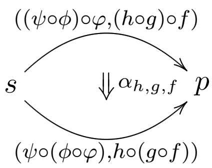
first we have a morphism $((\psi \circ \phi) \circ \varphi, (h \circ g) \circ f)$, where the first term is a composite of

$$s \xrightarrow{\varphi} r \triangleleft f \xrightarrow{(\psi\circ\phi)\triangleleft f} (p \triangleleft (h \circ g)) \triangleleft f \xrightarrow{\kappa_{p,h\circ g,f}} p \triangleleft ((h \circ g) \circ f)$$

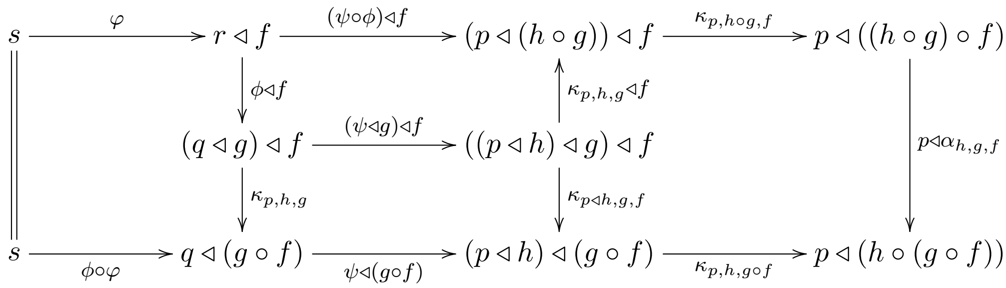
Also we have the composition $(\psi \circ (\phi \circ \varphi), h \circ (g \circ f))$, and the first term is given by a composite

$$s \xrightarrow{\phi\circ\varphi} q \triangleleft (g \circ f) \xrightarrow{\psi\triangleleft(g\circ f)} (p \triangleleft h) \triangleleft (g \circ f) \xrightarrow{\kappa_{p,h,g\circ f}} p \triangleleft (h \circ (g \circ f))$$

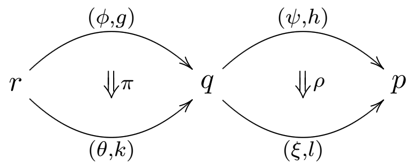
and the component of the associativity $\alpha_{h,g,f} \colon (h \circ g) \circ f \to h \circ (g \circ f)$, defines a 2-morphism



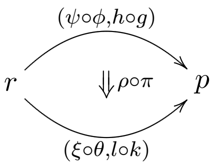
which we see from the commutativity of the diagram



that follows from the definition of the horizontal composition, the naturality and the coherence for quasiassociativity of the action. The horizontal composition of 2-morphisms



is given by the horizontal composition in B_2



since we have a commutative diagram

$$
\begin{array}{ccccccc}
r & \xrightarrow{\phi} & q \triangleleft g & \xrightarrow{\psi \triangleleft g} & (p \triangleleft h) \triangleleft g & \xrightarrow{\kappa_{p,h,g}} & p \triangleleft (h \circ g) \\
& {\scriptstyle \theta} \searrow \quad \swarrow {\scriptstyle q \triangleleft \pi} & & {\scriptstyle \xi \triangleleft g} \searrow \quad \swarrow {\scriptstyle (p \triangleleft \rho) \triangleleft g} & \downarrow {\scriptstyle (p \triangleleft \rho) \triangleleft \pi} & & \downarrow {\scriptstyle p \triangleleft (\rho \circ \pi)} \\
& q \triangleleft k & & (p \triangleleft l) \triangleleft g & & & \\
& {\scriptstyle \xi \triangleleft k} \searrow & {\scriptstyle (p \triangleleft l) \triangleleft \pi} \swarrow & & \searrow {\scriptstyle (p \triangleleft l) \triangleleft \pi} & & \\
& & (p \triangleleft l) \triangleleft k & = & (p \triangleleft l) \triangleleft k & \xrightarrow[\kappa_{p,l,k}]{} & p \triangleleft (l \circ k)
\end{array}
$$

which follows from the interchange law and the naturality of the coherence for the quasi-associativity of the action. The vertical composition of 2-morphisms in $\mathcal{A}_{\mathcal{B}}\mathcal{P}$ is similarly induced from the one in $\mathcal{B}$. Thus we have a following result.

Proposition 6.1. *Let $\mathcal{A}\colon \mathcal{P} \times_{\mathcal{B}_0} \mathcal{B}_1 \to \mathcal{P}$ be an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$. The above construction defines a bicategory $\mathcal{A}_{\mathcal{B}}\mathcal{P}$, which we call the action bicategory (associated to an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$).*

Proof. The coherence of the horizontal composition in $\mathcal{A}_{\mathcal{B}}\mathcal{P}$ is immediately given by the coherence of the horizontal composition in $\mathcal{B}$. □

Let we describe the simplicial set $\mathcal{P}_\bullet$ arising by an application of the Duskin nerve functor

$$\mathcal{N}_2\colon Bicat \to \mathcal{S}Set$$

to the action bicategory $\mathcal{A}_{\mathcal{B}}\mathcal{P}$. The set of 0-simplices is given by P_0. Any 1-simplex is given by an arrow

$$p_j \xrightarrow{(\pi_{ij}, f_{ij})} p_i$$

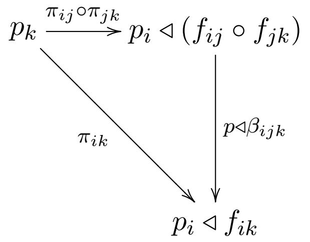
and face operators are defined by $d_0^1(\pi_{ij}, f_{ij}) = p_i$ and $d_1^1(\pi_{ij}, f_{ij}) = p_j$, while the degeneracy is defined by $s_0^1(p_i) = (\iota_{p_i}, i_{p_i})$ and it is given by the arrow

$$p_i \xrightarrow{(\iota_{p_i}, i_{p_i})} p_i$$

where the morphism $\iota_{p_i}\colon p_i \to p_i \triangleleft i_{\Lambda_0(p_i)}$ is an identity coherence of the action. A 2-simplex in $\mathcal{P}_\bullet$ is of the form

$$
\begin{array}{ccc}
p_k & \xrightarrow{(\pi_{ij}, f_{ij})} & p_j \\
& {\scriptstyle (\pi_{ik}, f_{ik})} \searrow \quad \Swarrow \beta_{ijk} & \downarrow {\scriptstyle (\pi_{jk}, f_{jk})} \\
& & p_i
\end{array}
$$

where the diagram



of morphisms in $\mathcal{P}$ commutes, and the morphism $\pi_{ij} \circ \pi_{jk} \colon p_k \to p_i \triangleleft (f_{ij} \circ f_{jk})$ is the composite of

$$p_k \xrightarrow{\pi_{jk}} p_j \triangleleft f_{jk} \xrightarrow{\pi_{ij}\triangleleft f_{jk}} (p_i \triangleleft f_{ij}) \triangleleft f_{jk} \xrightarrow{\kappa_{i,j,k}} p_i \triangleleft (f_{ij} \circ f_{jk})$$

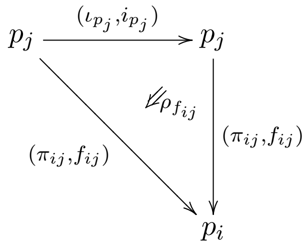
of morphisms in $\mathcal{P}$. Face operators are defined by

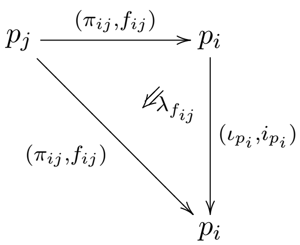
$$d_0^2(\beta_{ijk}) = (\pi_{jk}, f_{jk})$$
$$d_1^2(\beta_{ijk}) = (\pi_{ik}, f_{ik})$$
$$d_2^2(\beta_{ijk}) = (\pi_{ij}, f_{ij})$$

and the degeneracy operators are given by

$$s_0^2(\pi_{ij}, f_{ij}) = \rho_{f_{ij}}$$
$$s_1^2(\pi_{ij}, f_{ij}) = \lambda_{f_{ij}}$$

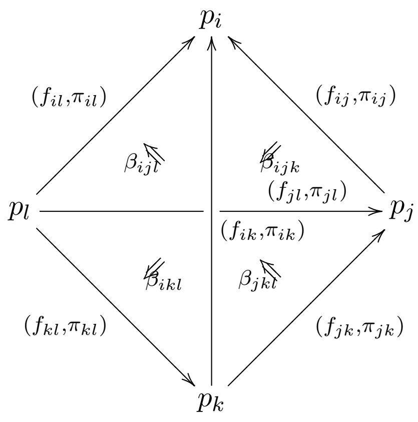
which are the two 2-simplices





respectively, where the 1-morphisms $\rho_{f_{ij}} \colon f_{ij} \circ i_{p_j} \to f_{ij}$ and $\lambda_{f_{ij}} \colon i_{p_i} \circ f_{ij} \to f_{ij}$ are the components of the right and left identity natural isomorphisms in $\mathcal{B}$.

The general 3-simplex is of the form



where we have an identity

$$\beta_{ikl}(\beta_{ijk} \circ f_{kl}) = \alpha_{ijkl}\beta_{ijl}(\beta_{jkl} \circ f_{ij})$$

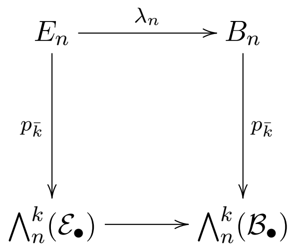
which is just a nonabelian 2-cocycle condition.

Proposition 6.2. *There exists a canonical homomorphism of bicategories $\Lambda\colon \mathcal{A}_{\mathcal{B}}\mathcal{P} \to \mathcal{B}$.*

Proof. A homomorphism $\Lambda\colon \mathcal{A}_{\mathcal{B}}\mathcal{P} \to \mathcal{B}$ is defined by (the component of) the momentum functor $\Lambda_0(p) = \lambda_0(p)$, for any object $p \in \mathcal{A}_{\mathcal{B}}\mathcal{P}$. For any 1-morphism (ψ, h) it is defined by $\Lambda_1(\psi, h) = h$, and for any 2-morphism $\gamma\colon (\psi, h) \Rightarrow (\xi, l)$ in $\mathcal{A}_{\mathcal{B}}\mathcal{P} \to \mathcal{B}$, it is given simply by $\Lambda_2(\gamma) = \gamma$. Since we have $\Lambda((\psi, h) \circ (\phi, g)) = \Lambda(\psi \circ \phi, h \circ g) = h \circ g = \Lambda(\psi, h) \circ \Lambda(\phi, g)$, this homomorphism is strict (it preserves a composition strictly). □

In order to compare our 2-torsors with Glenn's simplicial 2-torsors we will recall some basic definitions from [**?**].

Definition 6.1. *A simplicial map $\Lambda_\bullet\colon \mathcal{E}_\bullet \to \mathcal{B}_\bullet$ is said to be an exact fibration in dimension n, if for all $0 \le k \le n$, the diagrams*



are pullbacks. It is called an exact fibration if it is an exact fibration in all dimensions n.

Using the language of simplicial algebra, Glenn defined actions and n-torsors over n-dimensional hypergroupoids. This objects morally play the role of the n-nerve of weak n-groupoids, and we give their formal definition.

Definition 6.2. *An n-dimensional hypergroupoid is a Kan simplicial object $G_\bullet$ in $\mathcal{E}$ such that the canonical map $G_m \to \bigwedge_m^k(G_\bullet)$ is an isomorphism for all $m > n$ and $0 < k < m$.*

Remark 6.1. *The term n-dimensional hypergroupoid was introduced by Duskin [**?**], for any simplicial object satisfying the above condition without being Kan simplicial object. One of his motivational examples was the standard simplicial model for an Eilenberg-MacLane space $K(A, n)$, for any abelian group object A in $\mathcal{E}$. In [Be], Beke used the term an exact n-type to emphasize the meaning of these objects as algebraic models for homotopy n-types.*

Definition 6.3. *An action of the n-dimensional hypergroupoid is an internal simplicial map $\Lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{E}$ which is an exact fibration for all $m \geq n$.*

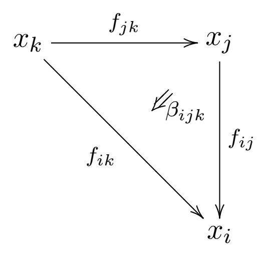
In the case of the bigroupoid $\mathcal{B}$, the Duskin nerve functor is a 2-dimensional hypergroupoid $\mathcal{B}_\bullet = \mathcal{N}_2(\mathcal{B})$ and let $\mathcal{P}_\bullet = \mathcal{N}_2(\mathcal{A}_\mathcal{B}\mathcal{P})$ be the Duskin nerve of an action bigroupoid associated to the action of the bigroupoid $\mathcal{B}$ on the groupoid $\mathcal{P}$. We have a following result.

Theorem 6.1. *Let the bigroupoid $\mathcal{B}$ acts on a groupoid $\mathcal{P}$. Then the simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda) \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ is a (simplicial) action of the bigroupoid $\mathcal{B}$ on the groupoid $\mathcal{P}$, i.e. it is an exact fibration for all $n \geq 2$.*

Proof. We need to show that for any $n \geq 2$ and for any k such that $0 \leq k \leq n$, the diagram

$$\begin{array}{ccc} P_n & \xrightarrow{\lambda_n} & B_n \\ \downarrow{\scriptstyle p_{\bar{k}}} & & \downarrow{\scriptstyle p_{\bar{k}}} \\ \bigwedge_n^k(\mathcal{P}_\bullet) & \xrightarrow[\lambda_n^k]{} & \bigwedge_n^k(\mathcal{B}_\bullet) \end{array}$$

is a pullback. A k-horn $((f_{ij}, \pi_{ij}), ..., (f_{j,k-1}, \pi_{j,k-1}), (f_{k,k+1}, \pi_{k,k+1}), ..., (f_{n-1,n}, \pi_{n-1,n}))$ in $\bigwedge_n^k(\mathcal{P}_\bullet)$ is given by the n-tuple of 1-morphisms in $\mathcal{A}_\mathcal{B}\mathcal{P}$, and its image by $\lambda_2^k \colon \bigwedge_2^k(\mathcal{P}_\bullet) \to \bigwedge_2^k(\mathcal{P}_\bullet)$ is a k-horn in $\bigwedge_n^k(\mathcal{B}_\bullet)$, given by the n-tuple $(f_{ij}, ..., f_{j,k-1}, f_{k,k+1}, ..., f_{n-1,n})$ of 1-morphisms in $\mathcal{B}$. For example, in the case $n = 2$, any filler of a 1-horn $(f_{ij}, -, f_{jk})$ in $\bigwedge_2^1(\mathcal{B}_\bullet)$, is the 2-simplex



in B_2. A 2-simplex in $\mathcal{P}_\bullet$ is a lifting of the previous 2-simplex if it is of the form

$$\begin{array}{ccc} p_k & \xrightarrow{(\pi_{jk},f_{jk})} & p_j \\ & {\scriptstyle(\pi_{ik},f_{ik})}\searrow \quad \Swarrow \beta_{ijk} & \Big\downarrow {\scriptstyle(\pi_{ij},f_{ij})} \\ & & p_i \end{array}$$

where the diagram

$$\begin{array}{ccc} p_k & \xrightarrow{\pi_{ij}\circ\pi_{jk}} & p_i \lhd (f_{ij}\circ f_{jk}) \\ & {\scriptstyle\pi_{ik}}\searrow & \Big\downarrow {\scriptstyle p\lhd\beta_{ijk}} \\ & & p_i \lhd f_{ik} \end{array}$$

of morphisms in $\mathcal{P}$ commutes, and the morphism $\pi_{ij} \circ \pi_{jk} \colon p_k \to p_i \lhd (f_{ij} \circ f_{jk})$ is the composite of

$$p_k \xrightarrow{\pi_{jk}} p_j \lhd f_{jk} \xrightarrow{\pi_{ij}\lhd f_{jk}} (p_i \lhd f_{ij}) \lhd f_{jk} \xrightarrow{\kappa_{i,j,k}} p_i \lhd (f_{ij} \circ f_{jk})$$

so we see that a pair $((f_{ij}, \pi_{ij}), -, (f_{jk}, \pi_{jk}), \beta_{ijk})$ in $\bigwedge_2^1(\mathcal{P}_\bullet) \times_{\bigwedge_2^1(\mathcal{B}_\bullet)} B_2$ uniquely determines above 2-simplex in $\mathcal{P}_2$. Since $\mathcal{P}$ is a groupoid, any pair consisting of a k-horn in $\bigwedge_2^k(\mathcal{B}_\bullet)$, for $k = 0, 2$, and a 2-simplex in $\mathcal{B}_2$ which covers the k-horn, uniquely determines a 2-simplex in $\mathcal{P}_2$, and thus provides a canonical isomorphism $P_2 \simeq \bigwedge_2^k(\mathcal{P}_\bullet) \times_{\bigwedge_2^k(\mathcal{B}_\bullet)} B_2$. Since both simplicial objects are 2-coskeletal, the assertion follows for all $n \geq 2$. □

Observe that even in the case when we just have an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$, the above condition for an exact fibration is still satisfied for inner horns $0 < k < n$. Thus it is sensible to introduce weakened concept of an exact fibration.

Definition 6.4. *A simplicial map $\Lambda_\bullet \colon \mathcal{E}_\bullet \to \mathcal{B}_\bullet$ is said to be a weak exact fibration in dimension n, if for all $0 < k < n$, the diagrams*

$$\begin{array}{ccc} E_n & \xrightarrow{\lambda_n} & B_n \\ {\scriptstyle p_{\bar{k}}}\Big\downarrow & & \Big\downarrow{\scriptstyle p_{\bar{k}}} \\ \bigwedge_n^k(\mathcal{E}_\bullet) & \longrightarrow & \bigwedge_n^k(\mathcal{B}_\bullet) \end{array}$$

are pullbacks. It is called a weak exact fibration if it is a weak exact fibration in all dimensions n.

With respect to this definition we generalize the simplicial actions of n-dimensional hypergroupoids to the case of weak n-dimensional Kan complexes. First we give their formal definition.

Definition 6.5. *A weak n-dimensional Kan complex $G_\bullet$ in $\mathcal{E}$ is a weak Kan complex such that the canonical map $G_m \to \bigwedge_m^k(G_\bullet)$ is an isomorphism for all $m > n$ and $0 < k < m$.*

Now we generalize actions with respect to this simplicial objects.

Definition 6.6. *An action of the n-dimensional Kan complex is an internal simplicial map $\Lambda_\bullet \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{E}$ which is a weak exact fibration for all $m \geq n$.*

This concept provides a following simplicial characterization of an action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$.

Theorem 6.2. *Let the bicategory $\mathcal{B}$ acts on a category $\mathcal{P}$. Then the simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda) \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ is a (simplicial) action of the bicategory $\mathcal{B}$ on the category $\mathcal{P}$, i.e. it is a weak exact fibration for all $n \geq 2$.*

Also, Glenn introduced a simplicial definition of the n-dimensional hypergroupoid n-torsor in $\mathcal{E}$.

Definition 6.7. *An action $\Lambda_\bullet \colon P_\bullet \to \mathcal{B}_\bullet$ is the n-dimensional hypergroupoid n-torsor over X in $\mathcal{E}$ if $P_\bullet$ is augmented over X, aspherical and n-1-coskeletal ($P_\bullet \simeq Cosk^{n-1}(P_\bullet)$).*

In the case of the bigroupoid $\mathcal{B}$, the above definition reduces to the following definition.

Definition 6.8. *A bigroupoid $\mathcal{B}_\bullet$ 2-torsor over an object X in $\mathcal{E}$ is an internal simplicial map $\Lambda_\bullet \colon P_\bullet \to \mathcal{B}_\bullet$ in $\mathcal{S}(\mathcal{E})$, which is an exact fibration for all $n \geq 2$, and where $P_\bullet$ is augmented over X, aspherical and 1-coskeletal ($P_\bullet \simeq Cosk^1(P_\bullet)$).*

Thus in the case when an action of $\mathcal{B}$ on $\mathcal{P}$ is principal, we have the following result.

Theorem 6.3. *Let $\mathcal{P}$ be a principal $\mathcal{B}$-2-bundle over X. Then simplicial map $\Lambda_\bullet = \mathcal{N}_2(\Lambda) \colon \mathcal{P}_\bullet \to \mathcal{B}_\bullet$ is a Glenn's 2-torsor.*

Proof. The simplicial complex $\mathcal{P}_\bullet$ is augmented over X because the action of $\mathcal{B}$ is fiberwise, since for any 1-simplex $(f_{ij}, \pi_{ij}) \colon p_j \to p_i$ in P_0, where $\pi_{ij} \colon p_j \to p_i \triangleleft f_{ij}$ we have

$$\pi_0 d_0(f_{ij}, \pi_{ij}) = \pi_0(p_i) = \pi_0(p_i \triangleleft f_{ij}) = \pi_1(\pi_{ij}) = \pi_0(p_j) = \pi_0 d_1(f_{ij}, \pi_{ij}).$$

The simplicial complex $\mathcal{P}_\bullet$ is obviously aspherical and we prove now that it is also 1-coskeletal. A general 2-simplex in $Cosk^1(P_\bullet)_2$ is a triple $((f_{ij}, \pi_{ij}), (f_{ik}, \pi_{ik}), (f_{jk}, \pi_{jk}))$ which we see as the triangle

$$\begin{array}{ccc} p_k & \xrightarrow{(\pi_{jk}, f_{jk})} & p_j \\ & {\scriptstyle (\pi_{ik}, f_{ik})} \searrow & \downarrow {\scriptstyle (\pi_{ij}, f_{ij})} \\ & & p_i \end{array}$$

from which we have morphisms $\pi_{ij} \circ \pi_{jk} \colon p_k \to p_i \triangleleft (f_{ij} \circ f_{jk})$ and $\pi_{ik} \colon p_k \to p_i \triangleleft f_{ik}$ in $\mathcal{P}$. Now we use the fact that the induced functor

$$(Pr_1, \mathcal{A}) \colon \mathcal{P} \times_{B_0} \mathcal{B}_1 \longrightarrow \mathcal{P} \times_X \mathcal{P}$$

is a (strong) equivalence of internal groupoids over $\mathcal{P}$, and therefore fully faithful. Specially, for the two objects $(p_i, f_{ij} \circ f_{jk})$ and (p_i, f_{ik}) of $\mathcal{P} \times_{B_0} \mathcal{B}_1$, this equivalence induces a bijection

$$Hom_{\mathcal{P} \times_{B_0} \mathcal{B}_1}((p_i, f_{ij} \circ f_{jk}), (p_i, f_{ik})) \simeq Hom_{\mathcal{P} \times_X \mathcal{P}}((p_i, p_i \triangleleft (f_{ij} \circ f_{jk})), (p_i, p_i \triangleleft f_{ik}))$$

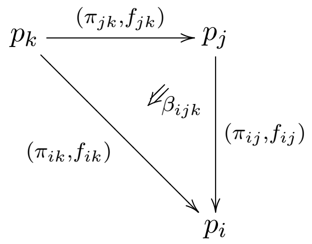
and therefore for a morphism $(id_{p_i}, \pi_{ik} \circ (\pi_{ij} \circ \pi_{jk})^{-1}) \colon (p_i, p_i \triangleleft (f_{ij} \circ f_{jk})) \to (p_i, p_i \triangleleft f_{ik}))$

$$\begin{array}{ccc} p_k & \xleftarrow{(\pi_{ij} \circ \pi_{jk})^{-1}} & p_i \triangleleft (f_{ij} \circ f_{jk}) \\ & {\scriptstyle \pi_{ik}} \searrow & \\ & & p_i \triangleleft f_{ik} \end{array}$$

there exists a unique 2-morphism $\beta_{ijk} \colon f_{ij} \circ f_{jk} \to f_{ik}$ in $\mathcal{B}$, such that the diagram

$$\begin{array}{ccc} p_k & \xrightarrow{\pi_{ij} \circ \pi_{jk}} & p_i \triangleleft (f_{ij} \circ f_{jk}) \\ & {\scriptstyle \pi_{ik}} \searrow & \downarrow {\scriptstyle p \triangleleft \beta_{ijk}} \\ & & p_i \triangleleft f_{ik} \end{array}$$

commutes, and this uniquely determines a 2-simplex



in $\mathcal{P}_2$, which proves that we have a bijection $\mathcal{P}_2 \simeq Cosk^1(P_\bullet)_2$. From here it follows immediately that $\mathcal{P}_\bullet \simeq Cosk^1(P_\bullet)$. □

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