# Parallel Transport in Low Dimensions

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# 1 Parallel 1-Transport: the motivating example

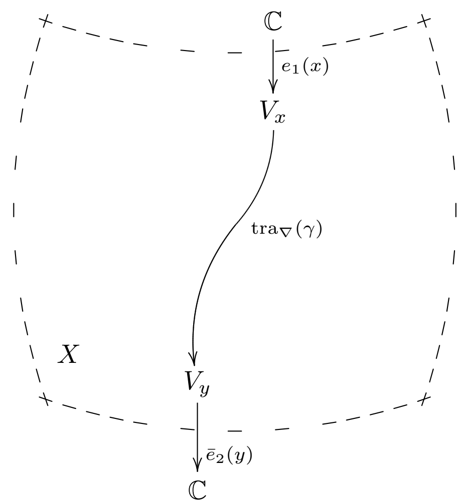
Models of the physics of charged particles are usually formulated in terms of vector bundles

$$V \to X$$

with connection

$$\nabla\,.$$

The part of this formalism most directly connected to what we actually observe in nature is the **parallel transport.**




## 1.1 Connections give rise to parallel transport functors.

One way to think of a connection of a principal bundle is to say that a connection is a prescription that tells us at each point of a principal bundle which tangent vectors are supposed to be **parallel** to the base space.

More precisely:

**Definition 1 (connection in terms of horizontal subspaces)** *Let $p : B \to X$ be a smooth principal $G$-bundle. For each point $b \in B$ of the total space, let*

$$V_b := \ker(p_b^*) \subset T_b B$$

*be the* **vertical subspace** *of the tangent space at that point. $V_b$ is the space of vectors at $b$ that are tangent to the fiber.*

*Then a* **connection** *on the principal bundle is a smooth $G$-invariant choice of complements $H_b$ of $V_b$*

$$T_b B = V_b \oplus H_b$$

$$H_{gb} = g^* H_b$$

*for all $b \in B$. $H_b$ is called the* **horizontal subspace** *of the tangent space at $b$.*

For our purposes, the point of this definition is the following: since $p_b^*$ restricted to $H_b$ is an isomorphism, it follows that a connection allows us to **lift** vectors in base space to **parallel** vectors on the total space.

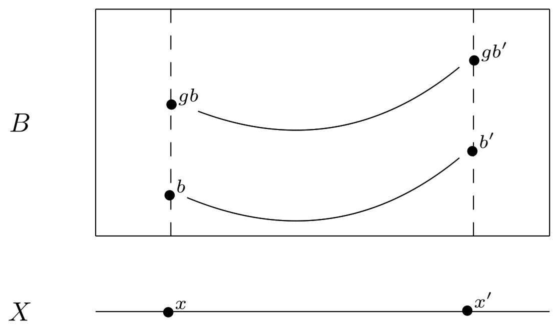
We can integrate this procedure and find for each path

$$x \xrightarrow{\gamma} x'$$

in base space a path

$$b \xrightarrow{\tilde{\gamma}} b'$$

in the total space, which is everywhere parallel to $\gamma$.




We say that $b'$ is obtained from **parallel transporting** $b$ along $\gamma$ from the fiber $B_x$ to the fiber $B_{x'}$.

This way a connection assigns, by parallel transport, to each path $\gamma$ in base space a map

$$\mathrm{tra}(\gamma) : B_x \to B_y$$

between the fibers over the endpoints

This assignment of maps between fibers to paths in base space has some special properties:

- The $G$-invariance of the choice of horizontal subspaces implies that these maps between the fibers commute with the $G$-action on the fibers.
- In particular, this implies that these maps are invertible, since $G$ acts freely and transitively on each fiber.
- The map $\mathrm{tra}(\gamma)$ is independent of the parameterization of $\gamma$.
- If $\bar{\gamma}$ is obtained from $\gamma$ by reversing the direction, then $\mathrm{tra}(\bar{\gamma})$ is the inverse of $\mathrm{tra}(\gamma)$.
- If $\gamma$ is the composition of two paths $\gamma_1$ and $\gamma_2$, then

$$\mathrm{tra}(\gamma) = \mathrm{tra}(\gamma_2) \circ \mathrm{tra}(\gamma_1)\,.$$

| | |
|---|---|
| choice of horizontal subspaces | choice of functor |
| $H_b \subset T_b B$ | $\mathcal{P}_1(X) \to G\mathrm{Tor}$ |
| allows to lift vectors $v \in T_{p(b)}X$ | allows to lift paths $\gamma \in \mathcal{P}_1(X)$ |
| to parallel vectors $\tilde{v} \in T_b B$ | to fiber isomorphisms $\mathrm{tra}(\gamma) : B_x \to B_y$ |
| differential description of connection | integrated description of connection |

Table 1: The ordinary definition of a **connection** on a principal bundle in terms of horizontal subspaces can be understood as the differential description of the concept of parallel transport.

Clearly, all this is trying to tell us that **parallel transport is a functor**

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}$$

that sends paths in base space to morphisms of $G$-torsors.

To make this precise, we need to specify what the groupoid of paths in base space that we are talking about is like.

**Definition 2** *The objects of $\mathcal{P}_1(X)$ are points in $X$. The morphisms*

$$x \xrightarrow{\gamma} y$$

*of $\mathcal{P}_1(X)$ are equivalence classes of smooth maps $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) =$, which are constant in a neighbourhood of 0 and in a neighbourhood of 1, and where two maps are considered equivalent if they are related by an orientation-preserving diffeomorphism. Composition of morphisms is by the obvious concatenation of these maps, modulo the relation that paths related by an orientation reversing diffeomorphism are mutually inverse.*

Given this definition of the groupoid of paths in $X$, our list of properties of parallel transport implies

**Proposition 1** *Given a principal $G$-bundle with connection $B \to X$, parallel transport in that bundle is a functor*

$$\mathrm{tra} : \mathcal{P}_1(X) \to G\mathrm{Tor}\,.$$

The entire discussion generalizes directly to associated bundles.

**Proposition 2** *Given a vector bundle with connection $V \to X$, parallel transport in that bundle is a functor*

$$\mathrm{tra} : \mathcal{P}_1(X) \to \mathrm{Vect}\,.$$

It would be nice if these statements had a converse. We cannot expect every functor from paths to $G$-torsors to define a smooth principal bundle, or from paths to vector spaces to define a smooth vector bundle.

But transport functors that are *smooth* and *locally trivializable* in some suitable sense should do.

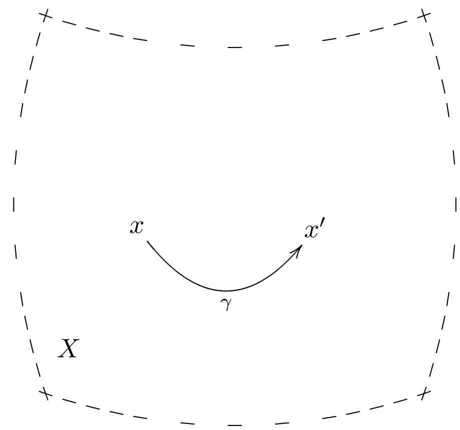



Figure 1: A **morphisms of the path groupoid** of $X$ is an oriented path $\gamma$ cobounding two points $x$ and $x'$ in $X$. Paths that differ by orientation-preserving diffeomorphism are identified. This ensures strict associativity of composition. Paths that differ by orientation-reversing diffeomorphism are taken to represent mutually inverse morphisms.

## 1.2 Locally trivial smooth transport functors

Let us locally trivialize our principal bundle and see what this does to the corresponding parallel transport functor.

So we choose a good cover

$$p : U \to X$$

of base space by open contractible sets.

This allows us to pull back structures over $X$ to $U$, where they may be trivializable.

For the bundle $B \to X$ this means that we can choose a bundle isomorphism

$$t : \ \pi^* B \xrightarrow{\quad \sim \quad} U \times G \ .$$

This amounts to choosing, in a smooth way, for each point $(x,i)$ in $U$ an isomorphism of $G$-torsors

$$t(x,i) : \ B_x \xrightarrow{\sim} G \ .$$

We can translate this into a similar local trivialization of the corresponding

| | |
|---|---|
| local connection 1-form<br>$A \in \Omega^1(U, \mathfrak{g})$ | smooth transport functor<br>$\mathrm{tra}_U : \mathcal{P}_1(U) \to \Sigma(G)$ |
| transition function<br>$g \in \Omega^0(U^{[2]}, G)$ | natural isomorphism<br>$g : p_1^*\mathrm{tra}_U \to p_2^*\mathrm{tra}_U$ |
| $A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1}$ | $\begin{array}{ccc} \bullet & \xrightarrow{g_{ij}(x)} & \bullet \\ {\scriptstyle \mathrm{tra}_i(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathrm{tra}_j(\gamma)} \\ \bullet & \xrightarrow[g_{ij}(y)]{} & \bullet \end{array}$ |
| $g_{ij} g_{jk} = g_{ik}$ | $\begin{array}{ccc} & p_2^*\mathrm{tra}_U & \\ {\scriptstyle p_{12}^*g}\nearrow & & \searrow{\scriptstyle p_{23}^*g} \\ p_1^*\mathrm{tra}_U & \xrightarrow[p_{13}^*g]{} & p_3^*\mathrm{tra}_U \end{array}$ |

Table 2: The **differential cocycle** data describing the local trivialization of a principal bundle with connection is the **descent data** of a local smooth transport functor.

transport functor. Its pullback to $U$ is

$$\begin{tikzcd} \mathcal{P}_1(U) \arrow[r, "p"] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \\ & G\mathrm{Tor} \end{tikzcd} .$$

Which structure on $U$ could this be isomorphic to? Notice that, using the identifications of fibers with copies of $G$ above, we can form a functor

$$\mathrm{tra}_U : \mathcal{P}_1(U) \to \mathrm{Aut}_{G\mathrm{Tor}}(G)$$

such that

$$\begin{tikzcd} B_x \arrow[r, "\mathrm{tra}(\gamma)"] \arrow[d, "t(x,i)"'] & B_y \arrow[d, "t(y,i)"] \\ G \arrow[r, "\mathrm{tra}_U(\gamma)"'] & G \end{tikzcd}$$

for any path $\gamma$ in $U_i$.

An automorphism of $G$ regarded as a $G$-torsor over itself is nothing but an element of $G$. In other words, we have a canonical injection

$$i : \ \Sigma(G) \xrightarrow{\subset} G\mathrm{Tor}$$

of the category $\Sigma(G)$ with a single object and $G$ worth of morphisms into the category of $G$-torsors.

Using this injection, we can think of $\mathrm{tra}_U$ as a functor with values in $G\mathrm{Tor}$ that factors through $i$:

$$\begin{tikzcd}
\mathcal{P}_1(U) \arrow[d, "\mathrm{tra}_U"'] & \\
\Sigma(G) \arrow[r, "\rho"'] & G\mathrm{Tor}
\end{tikzcd}$$

Taken together, we find that the local trivialization $t : p^*B \to U \times G$ of the principal bundle corresponds to a morphism

$$\begin{tikzcd}
\mathcal{P}_1(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\overset{\sim}{\swArrow}_t"] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \\
\Sigma(G) \arrow[r, "i"'] & G\mathrm{Tor}
\end{tikzcd}$$

that relates the corresponding parallel transport functor to a functor on $\mathcal{P}_1(U)$ with values in $\Sigma(G)$.

In contrast to the category $G\mathrm{Tor}$, the category $\Sigma(G)$ is naturally a **smooth category**, namely a category internal to smooth spaces. The same is true for the path groupoid. Since the bundle with connection that we started with was smooth, the functor $\mathrm{tra}_U$ is a smooth functor between smooth categories.

The smoothness of a smooth functor implies that the functor is specified by its derivatives. Functoriality then implies that already the derivatives at all identity morphisms suffice:

**Proposition 3** *Smooth functors*

$$\mathrm{tra}_U : \mathcal{P}_1(U) \to \Sigma(G)$$

*are in bijection with $\mathfrak{g}$-valued 1-forms $A$ on $U$:*

$$\mathrm{tra}_U(\gamma) = \mathrm{P}\exp\Big(\int_0^1 \gamma^* A\Big)\,.$$

On double intersections of the cover, the local trivialization of our bundle yields a smooth natural isomorphism

$$g := p_2^* t \circ p_1^* t : p_1^*\mathrm{tra}_U \to p_2^*\mathrm{tra}_U\,.$$

**Proposition 4** *Such smooth natural isomorphisms between smooth functors coming from 1-forma $A$ and $A'$, respectively, are in bijection with smooth functions $g$ with values in $G$ such that*

$$A = gA'g^{-1} + gdg^{-1}\,.$$

These $G$-valued functions are nothing but the **transition function** describing the local trivialization of our bundle $B$.

The cocycle condition

$$g_{ij}g_{jk} = g_{ik}$$

which they satisfy is an expression of the existence of this triangle:

$$\begin{tikzcd}
 & p_2^*\mathrm{tra}_U \arrow[d, bend left, "p_2^*\bar{t}"] \arrow[ddr, "p_{23}^*g"] & \\
 & p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[dr, "p_3^*t"] & \\
p_1^*\mathrm{tra}_U \arrow[uur, "p_{12}^*g"] \arrow[ur, "p_1^*\bar{t}"] \arrow[rr, "p_{13}^*g"'] & & p_3^*\mathrm{tra}_U
\end{tikzcd} .$$

Using the familiar fact that principal $G$-bundles with connection are equivalent to differential 1-cocycles, we find that principal $G$-bundles with connection are equivalent to descent data

$$\mathrm{Trans}_{i,p}$$

for smooth transport functors taking values in $\Sigma(G)$.

Again, all these considerations go through completely analogously for vector bundles. All we need to do is to replace the injection

$$i : \ \Sigma(G) \longrightarrow G\mathrm{Tor}$$

by a *representation*

$$\rho : \ \Sigma(G) \longrightarrow \mathrm{Vect}\ .$$

## 1.3 Anafunctors

There is an equivalent way to talk about functors on paths of a cover that are related by isomorphisms on double intersections such that a triangle commutes on triple intersections.

As Toby Bartels and John Baez emphasized, we can to think of this situation as characterizing an **anafunctor** - a functor not directly acting on its domain, but on a *cover* of that domain.

Namely, if we let

$$\mathcal{P}_1(U^\bullet)$$

be the category whose morphisms are combinations of paths in $U$ with jumps from one patch into the other, then our locally trivial transport functor $\mathrm{tra}_U$ with transitions $g$ is encoded in the span

$$\begin{tikzcd} \mathcal{P}_1(U^\bullet) \arrow[r, "(\mathrm{tra}_U, g)"] \arrow[d, "p"'] & \Sigma(G) \\ \mathcal{P}_1(X) & \end{tikzcd} .$$

# 2 Parallel 2-Transport: 2-Bundles with Connection

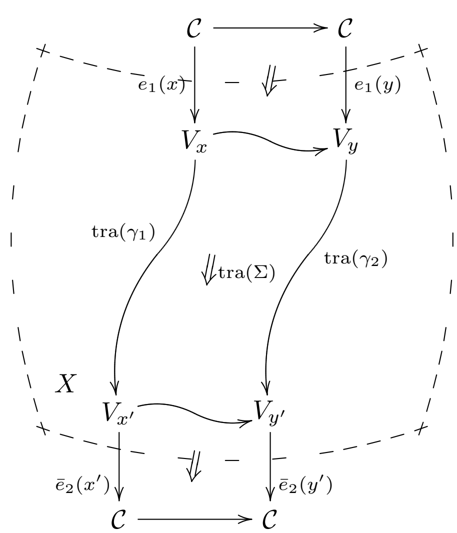
We wish to categorify the description of bundles with connection in terms of descent data of smooth parallel transport functors.

This requires that we

a) find suitable categorifications of the domain $\mathcal{P}_1(X)$ and codomain, $G$Tor or Vect of our parallel transport functors

b) find a suitable categorification of the descent data, i.e. find a suitable notion of **2-anafunctor**.

When such a categorification is available, we can study parallel transport of strings across surfaces:



| **parallel 1-transport** | **parallel 2-transport** |
|---|---|
| domain: path groupoid $\mathcal{P}_1(X)$ | domain: 2-path 2-groupoid $\mathcal{P}_2(X)$ |
| codomain: vector spaces Vect | codomain: 2-vector spaces 2Vect |
| structure group: $G$ | structure 2-group: $G_2$ |
| representation $\rho : \Sigma(G) \to \mathrm{Vect}$ | 2-representation $\rho : \Sigma(G_2) \to 2\mathrm{Vect}$ |
| trivial vector bundles with connection smooth functors: $\mathcal{P}_1(X) \longrightarrow \Sigma(G) \xrightarrow{\rho} \mathrm{Vect}$ | trivial 2-vector bundle with connection smooth 2-functor: $\mathcal{P}_2(X) \longrightarrow \Sigma(G_2) \xrightarrow{\rho} 2\mathrm{Vect}$ |
| vector bundles with connection smooth anafunctors: $\begin{tikzcd} \mathcal{P}_1(U^\bullet) \arrow[r] \arrow[d, "p"'] & \Sigma(G) \arrow[r, "\rho"] & \mathrm{Vect} \\ \mathcal{P}_1(X) \end{tikzcd}$ | 2-vector bundle with connection smooth 2-anafunctor: $\begin{tikzcd} \mathcal{P}_2(U^\bullet) \arrow[r] \arrow[d, "p"'] & \Sigma(G_2) \arrow[r, "\rho"] & 2\mathrm{Vect} \\ \mathcal{P}_2(X) \end{tikzcd}$ |

Table 3: On the left, our description of bundles with connection in terms of parallel transport functors. On the right our categorification of this situation.

## 2.1 2-Paths

We want 2-morphisms in $\mathcal{P}_2(X)$ to look like little surface elements. There are various choices one could make concerning the degree of invertibility and strictness of composition of the 1-morphisms involved. For our purposes, it is useful to make

**Definition 3** *The 2-path 2-groupoid $\mathcal{P}_2(X)$ has as objects the points of $X$, has as morphisms classes of oriented paths in $X$ modulo orientiation preserving diffeomorphism, and 2-morphisms thin homotopy classes of oriented surfaces cobounding such paths.*

The 2-path 2-groupoid is a strict 2-category. Composition is strictly associative. However, it is not a strict 2-groupoid, since a path is not strictly but weakly inverse to its orientation-reversed path.

## 2.2 2-Groups

From the point of view of parallel transport, structure groups $G$ arise as the local trivializations of the transport groupoid. Hence the important characterizing property is that a group is a groupoid with a single object.

This immediately suggests the kind of categorification we need

**Definition 4** *A* **2-group** *is a 2-groupoid with a single object.*

Here we want to work within the 3-category of *strict* 2-categories, strict 2-functors between them, pseudonatural transformations between those and mod-

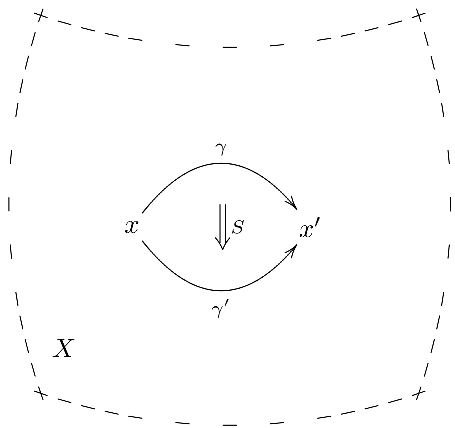



Figure 2: A **2-morphisms of the 2-path 2-groupoid** of $X$ is a thin-homotopy class of a surface $S$ cobounding two diffeomorphism classes $\gamma$ and $\gamma'$ of paths which in turn cobound two points in $x$ and $x'$.

ificatiopn between the latter. For that reason we restrict attention to *strict 2-groups.*

**Definition 5** *A* **strict 2-group** *is a strict 2-groupoid with a single object.*

Strict 2-groups turn out to have a useful description in terms of crossed modules.

**Definition 6** *A* **crossed module** *of groups is a pair* $(G_0, G_1)$ *of groups, together with homomorphisms*

$$G_1 \xrightarrow{t} G_0 \xrightarrow{\alpha} \mathrm{Aut}(G_1)$$

*such that* $t$ *is equivariant with respect to the action induced by* $\alpha$*, i.e. such that*

$$\begin{tikzcd} G_1 \arrow[rr, "\mathrm{Ad}"] \arrow[dr, "t"'] & & \mathrm{Aut}(G_1) \\ & G_0 \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$$\Leftrightarrow \alpha(t(h))(h') = h\, h'\, h^{-1}$$

*and such that*

$$t(\alpha(g)(h)) = g\, t(h)\; g^{-1}\,.$$

Namely we have

**Theorem 1 (classic)** *The 2-category of 2-groups is equivalent to the 2-category of crossed modules.*

This equivalence is induced by identifying $G_0$ with the set of morphisms

$$\mathrm{Mor}_1 = \left\{ \bullet \xrightarrow{g} \bullet \,|g \in G_0 \right\}$$

of the 2-groupoid; $G_1$ with the kernel of the source map, i.e. with those 2-morphisms starting at the identity

$$\left\{ \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} \;\middle|\; h \in G_1 \right\} ;$$

and the set of all morphisms with the semidirect product $G_1 \ltimes G_0$ as

$$\mathrm{Mor}_2 = \left\{ \begin{tikzcd} \bullet \arrow[r, bend left=50, "\mathrm{Id}"] \arrow[r, bend right=50, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd} \;\middle|\; h \in G_1, g \in G_0 \right\} .$$

The main fact to keep in mind, especially for the discussion in section **??**, is the following rule for horizontal and vertical composition of 2-group elements (their precise form depends on some conventions that we chose to fix):

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_1'"'] \arrow[r, phantom, "\Downarrow h_1"] & \bullet \arrow[r, bend left=50, "g_2"] \arrow[r, bend right=50, "g_2'"'] \arrow[r, phantom, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1 \cdot g_2"] \arrow[r, bend right=50, "g_1' \cdot g_2'"'] \arrow[r, phantom, "\Downarrow h_1 \cdot \alpha(g_1)(h_2)"] & \bullet \end{tikzcd}$$

and

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g_1"] \arrow[r, "g_2" description] \arrow[r, bend right=60, "g_3"'] \arrow[r, phantom, shift left=3, "\Downarrow h_1"] \arrow[r, phantom, shift right=3, "\Downarrow h_2"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=50, "g_1"] \arrow[r, bend right=50, "g_3"'] \arrow[r, phantom, "\Downarrow h_2 \cdot h_1"] & \bullet \end{tikzcd} ,$$

where the dot on the right hand side indicates the ordinary product in the respective group.

**Example 1**

The two standard classes of examples for strict 2-groups and crossed modules are the following:

- Let $G$ be any group, regarded as a groupoid with a single object. Then the automorphism functor 2-category $\mathrm{Aut}_{Cat}(G)$ is a 2-group. It corresponds to the crossed module

$$G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G) \xrightarrow{\mathrm{Id}} \mathrm{Aut}(G) \ .$$

- Every central extension

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1$$

  with the usual action of $G$ on $H$ defines a crossed module.

## 2.3 Transition of 2-functors and 2-anafunctors

We say a 2-functor $\mathrm{tra} : \mathcal{P}_2(X) \to T$ is $p$-locally $i$-trivializable if there is an equivalence

$$\begin{tikzcd}
\mathcal{P}_2(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\sim\swarrow t"] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \\
T' \arrow[r, "i"'] & T
\end{tikzcd} \ .$$

**Proposition 5** *The resulting* **transitions**

$$\begin{tikzcd}
& p_2^*\mathrm{tra}_U \arrow[dr, "p_{23}^*g"] \arrow[d, phantom, "f\Downarrow"] & \\
p_1^*\mathrm{tra}_U \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & {} & p_3^*\mathrm{tra}_U
\end{tikzcd}
\equiv
\begin{tikzcd}
& p_2^*\mathrm{tra}_U \arrow[d, bend left, "p_2^*\bar{t}"] \arrow[ddr, "p_{23}^*g"] & \\
& p^*\mathrm{tra} \arrow[u, bend left, "p_2^*t"] \arrow[dr, "p_3^*t"] & \\
p_1^*\mathrm{tra}_U \arrow[uur, "p_{12}^*g"] \arrow[ur, "p_1^*\bar{t}"] \arrow[rr, "p_{13}^*g"'] & & p_3^*\mathrm{tra}_U
\end{tikzcd}$$

| **1-anafunctors** | | **2-anafunctors** |
|---|---|---|
| $\begin{tikzcd} \mathcal{P}_1(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\stackrel{\sim}{\Swarrow}{}_t"] & \mathcal{P}_1(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd}$ | as $p$-local $i$-trivialization | $\begin{tikzcd} \mathcal{P}_2(U) \arrow[r, "p"] \arrow[d, "\mathrm{tra}_U"'] \arrow[dr, phantom, "\stackrel{\sim}{\Swarrow}{}_t"] & \mathcal{P}_2(X) \arrow[d, "\mathrm{tra}"] \\ T' \arrow[r, "i"'] & T \end{tikzcd}$ |
| $\begin{tikzcd} & p_2^*\mathrm{tra}_U \arrow[dr, "p_{23}^*g"] & \\ p_1^*\mathrm{tra}_U \arrow[ur, "p_{12}^*g"] \arrow[rr, "p_{13}^*g"'] & & p_3^*\mathrm{tra}_U \end{tikzcd}$ | as descent data/ transition data | $\begin{tikzcd} p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] \arrow[dr, phantom, "\Searrow p_{123}^*f", near start] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] \arrow[dl, phantom, "\Downarrow p_{134}^*f", near end] \\ p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[ur, "p_{13}^*g" description] \arrow[r, "p_{14}^*g"'] & p_4^*\mathrm{tra}_U \end{tikzcd} = \begin{tikzcd} p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] \arrow[dr, "p_{24}^*g" description] \arrow[dr, phantom, "p_{234}^*f \Swarrow", near start, shift left=6] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] \\ p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[r, "p_{14}^*g"'] \arrow[ur, phantom, "p_{124}^*f \Downarrow", near start] & p_4^*\mathrm{tra}_U \end{tikzcd}$ |
| $\begin{tikzcd} \mathcal{P}_1(U^\bullet) \arrow[r, "(\mathrm{tra}_U, g)"] \arrow[d, "p"'] & T' \\ \mathcal{P}_1(X) \end{tikzcd}$ | as spans | $\begin{tikzcd} \mathcal{P}_2(U^\bullet) \arrow[r, "(\mathrm{tra}_U, g, f)"] \arrow[d, "p"'] & T' \\ \mathcal{P}_2(X) \end{tikzcd}$ |

Table 4: We generalize 1-anafunctors to **2-anafunctors** by regarding an anafunctor as an instance of **descent data** or **transition data**.

*make a tetrahedron 2-commute*

$$\begin{tikzcd} p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] \arrow[dr, phantom, "p_{123}^*f \Searrow", near start] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] \arrow[dl, phantom, "\Downarrow p_{134}^*f", near end] \\ p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[ur, "p_{13}^*g" description] \arrow[r, "p_{14}^*g"'] & p_4^*\mathrm{tra}_U \end{tikzcd} \quad = \quad \begin{tikzcd} p_2^*\mathrm{tra}_U \arrow[r, "p_{23}^*g"] \arrow[dr, "p_{24}^*g" description] \arrow[dr, phantom, "p_{234}^*f \Swarrow", near start, shift left=6] & p_3^*\mathrm{tra}_U \arrow[d, "p_{34}^*g"] \\ p_1^*\mathrm{tra}_U \arrow[u, "p_{12}^*g"] \arrow[r, "p_{14}^*g"'] \arrow[ur, phantom, "p_{124}^*f \Downarrow", near start] & p_4^*\mathrm{tra}_U \end{tikzcd} .$$

We say tra is equipped with a smooth structure, if it is equipped with a fixed $p$-local $i$-trivialization such that all the transition data is smooth.

We may address these transitions as **differential 2-cocycles**, or, taking them as a categorification of anafunctors, as **2-anafunctors**.

For different choices of $i$, we find various structures invented by various authors:

**Proposition 6** *Principal and line* bundle gerbes *with connection, as well as the differential 2-cocycles characterizing them, are p-local i-transition data for 2-*

| **morphism of parallel transport codomain** | | | **name of corresponding 2-anafunctor** |
|---|---|---|---|
| $i : T'$ | $\to$ | $T$ | $p$-local $i$-transition |
| $\Sigma(\Sigma(U(1)))$ | $\stackrel{\mathrm{Id}}{\to}$ | $\Sigma(\Sigma(U(1)))$ | Deligne 2-cocycle |
| $\Sigma(\Sigma(U(1)))$ | $\to$ | $\Sigma(1d\mathrm{Vect}_{\mathbb{C}})$ | line bundle gerbe with connection (Murray) |
| | | | |
| $\Sigma(\mathrm{AUT}(H))$ | $\stackrel{\mathrm{Id}}{\to}$ | $\Sigma(\mathrm{AUT}(H))$ | fake-flat Breen-Messing 2-cocycle |
| $\Sigma(\mathrm{AUT}(H))$ | $\to$ | $\Sigma(H\mathrm{BiTor})$ | fake-flat Aschieri-Jurčo bibundle gerbe with connection |

Table 5: **Whatever the entries of this table mean** – an explanation of which does not fit in here – the message is that **various higher structures** that people have studied **are secretly examples of 2-anafunctors**.

*functors, with p a given surjective submersion and i as indicated in the following table.*

The above table says in particular that smooth 2-functors

$$\mathrm{tra} : \mathcal{P}_2(X) \to \Sigma(\mathrm{AUT}(H)$$

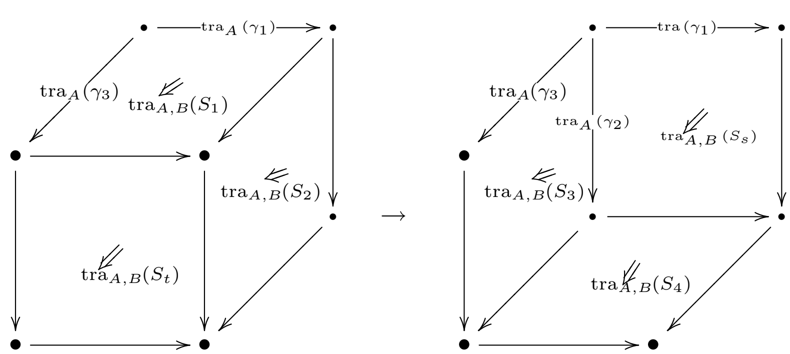
are in bijection with pairs consisting a $\mathrm{Lie}(\mathrm{AUT}(H))$-valued 1-form $A$ and a $\mathrm{Lie}(H)$-valued 2-form $B$

$$\mathrm{tra}\ :\ \begin{tikzcd} 0 \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\Swarrow"] & x \arrow[d, "\gamma_2"] \\ y \arrow[r, "\gamma_4"'] & x+y \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\Swarrow\ 1+B(\gamma_1,\gamma_3)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\ \bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet \end{tikzcd}\ ,$$

such that

$$\beta = F_A + \mathrm{ad}(B)$$

vanishes. This $\beta$ is the **2-form curvature** or **fake curvature**. The true curvature



is

$$H = d_A B\,.$$

People did consider connection data on gerbes that is not fake flat. By the above, this does not integrate to a parallel transport 2-functor with values in a 2-group.

But it does integrate to a pseudo 2-functor with values in a 3-group.

# 3 Parallel 3-Transport: Chern-Simons

As Danny Stevenson explains, we should expect general parallel transport with respect to a $n$-group $G_n$ to involve the $(n+1)$-group of **inner automorphisms**

$$\begin{array}{cccc}
\mathrm{Ad}_q \quad : & G_2 & \to & G_2 \\
 & \begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd} & \mapsto & \begin{tikzcd} \bullet \arrow[r, "q^{-1}"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, "q"] & \bullet \end{tikzcd}
\end{array}$$

of $G_n$.

For $G$ a 1-group, $G$-1-transport is the same as $(\mathrm{INN}(G) = (G \to G))$-2-transport:

$$\mathrm{curv}_A \;:\; \begin{tikzcd} x_s \arrow[r, "\gamma_1"] \arrow[d, "\gamma_3"'] \arrow[dr, phantom, "\swarrow S"] & x_1 \arrow[d, "\gamma_2"] \\ x_2 \arrow[r, "\gamma_4"'] & x_t \end{tikzcd} \;\mapsto\; \begin{tikzcd} \bullet \arrow[r, "1+A(\gamma_1)+\cdots"] \arrow[d, "1+A(\gamma_3)+\cdots"'] \arrow[dr, phantom, "\swarrow 1+F_A(\gamma_3,\gamma_1)+\cdots"] & \bullet \arrow[d, "1+A(\gamma_2)+\cdots"] \\ \bullet \arrow[r, "1+A(\gamma_4)+\cdots"'] & \bullet \end{tikzcd}\,.$$

Surfaces are sent to the integrated curvature of the parallel 1-transport.

But for $G_2$ a 2-group, $\mathrm{INN}(G_2)$-3-transport is inherently richer than $G_2$-2-transport.

$\mathrm{INN}(G_2)$ is no longer strict, but all nontrivial structure morphisms are unique. We can consider pseudo-2-transport

$$\mathcal{P}_2(X) \to \Sigma(\mathrm{INN}(G_2))$$

that strictly respects horizontal and vertical composition by itself.

**Proposition 7** *$p$-local $\mathrm{Id}_{\Sigma(\mathrm{INN}(G_2))}$-trnasitions yield the full Breen-Messing cocycle data, not restricted to vanishing fake curvature.*

We would like to understand $\mathrm{INN}(G_2)$ for the case where $G_2 = \mathrm{String}_G = (\hat{\Omega}_k G \to PG)$, the strict version of the string 2-group.

This is easiest at the level of Lie 3-algebras.

**Proposition 8** *For any semisimple Lie algebra $g$, and any level $k \in \mathbb{N}$, there is a Lie-3-algebra*

$$\mathrm{cs}(g)$$

*such that a 3-connection with values in that Lie 3-algebra is, locally, a $g$-valued 1-form $A$, a 2-form $B$ and the Chern-Simons 3-form*

$$H = k\mathrm{CS}(A) + dB\,.$$

**Proposition 9** *The Chern-Simons Lie 3-algebra sits inside the Lie 3-algebra of the inner automorphisms of the String 2-group*

$$\mathrm{cs}(g) \xrightarrow{\subset} \mathrm{Lie}(\mathrm{INN}(\mathrm{String}_G))\ .$$

This indicates that $\mathrm{INN}(\mathrm{String}_G)$-3-transport is in fact Chern-Simons 3-transport.

I expect that the above inclusion is in fact an equivalence, but this I could not prove yet.

If we consider Chern-Simons 2-gerbes *without* connection, the situation simplifies, since it is known that Chern-Simons 2-gerbes are characxterized by having WZW transition 1-gerbes:

noticing that the maximal strict sub-3-group in $\mathrm{INN}(\mathrm{String}_G)$ is $(U(1)ø\hat{\Omega}_k G \to PG)$ we have

**Proposition 10** *Transition gerbes for $(U(1) \to \hat{\Omega}_k G \to PG)$-2-gerbes (without connection) are WZW gerbes.*