Lie 2-algebras and Higher Gauge Theory

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Higher Gauge Theory

Ordinary gauge theory describes how point particles change as they move along paths in spacetime.

In a mathematical formulation this involves the study of **principal bundles** with some structure **group** G, and **connections** on these.

Higher gauge theory is a categorification of this:

- groups are promoted to **2-groups**,
- bundles are promoted to **2-bundles**,
- connections are promoted to **2-connections**

This talk will be about trying to understand 2-connections from a more algebraic point of view.

2-groups

A (strict) **2-group** $\mathfrak{G}$ is a category internal to the category **Grp** of groups. So $\mathfrak{G}$ consists of

- a group of objects G_0, and
- a group of morphisms G_1

such that all operations are homomorphisms of groups.

If G is a group then the associated groupoid $G[1]$ is a 2-group if and only if G is *abelian*. Higher gauge theory for the 2-group $U(1)[1]$ has been extensively studied in the subject of $U(1)$-**gerbes**. This is a mathematical formulation of 'higher dimensional electromagnetism'.

2-groups are the same thing as **crossed modules**. These are homomorphisms $t\colon H \to G$ together with an action of G on H satisfying some identities.

To understand these, its useful to think of the 2-group $\mathfrak{G}$ as a 2-category with one object.

There is a subgroup H consisting of all 2-morphisms of the form

$$\begin{tikzcd} \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd}$$

The target homomorphism restricts to a homomorphism t from H to the group $G = G_0$ of objects of the 2-group $\mathfrak{G}$ which sends each $h \in H$ to its target:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd}$$

The action of G on H can be described as follows:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1"] & \bullet \arrow[r, bend left, "1"] \arrow[r, bend right, "t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left, "g^{-1}"] \arrow[r, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow 1"] & \bullet \end{tikzcd}$$

Transition functions

Ordinary principal bundles have transition functions: these are maps

$$g_{ij} \colon U_i \cap U_j \to G$$

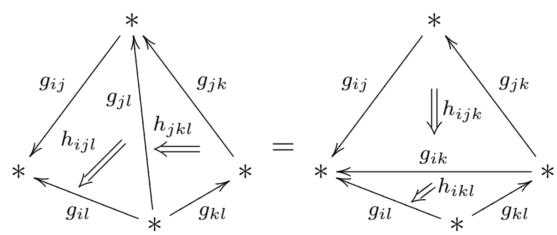
satisfying the cocycle condition $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$. A good way to think of this is as a diagram in the groupoid $G[1]$:

$$\begin{tikzcd}
 & * \arrow[dl, "g_{ij}(x)"'] & \\
* & & * \arrow[ul, "g_{jk}(x)"'] \arrow[ll, "g_{ik}(x)"]
\end{tikzcd}$$

In a categorification of the notion of a principal bundle, these diagrams should be weakened:

$$\begin{tikzcd}
 & * \arrow[dl, "g_{ij}(x)"'] \arrow[d, phantom, "\Downarrow h_{ijk}"] & \\
* & {} & * \arrow[ul, "g_{jk}(x)"'] \arrow[ll, "g_{ik}(x)"]
\end{tikzcd}$$

The morphism $h_{ijk}(x)$ should satisfy a coherence law of its own — namely the following tetrahedron should commute:



This amounts to a pair of maps

$$\begin{aligned} g_{ij} &\colon U_i \cap U_j \to G \\ h_{ijk} &\colon U_i \cap U_j \cap U_k \to H \end{aligned}$$

which satisfy the pair of equations

$$\begin{aligned} & g_{ij}\, g_{jk} = g_{ik}\, t(h_{ijk}) \\ & h_{ijl}\, \alpha(g_{ij})(h_{jkl}) = h_{ikl}\, h_{ijk} \end{aligned}$$

What kinds of objects have transition functions like these?

Gerbes and 2-bundles

A principal G bundle on a manifold M consists (in part) of another manifold P together with an action of G on P:

$$P \times G \to P$$

such that M is the orbit space of this G-action.

If $\mathfrak{G}$ is a 2-group we'd like to define a notion of a 'principal $\mathfrak{G}$ bundle'.

Categorify the definition of principal bundle:

manifolds $\to$ smooth groupoids
smooth maps $\to$ smooth functors

We would then want to have a smooth functor

$$P \times \mathfrak{G} \to P$$

between smooth groupoids satisfying some conditions.

There are several ways to do this categorification; one way is described in the thesis of Toby Bartels. Whichever way we choose, we want to find lots of examples!!

Connections

Classically, a **connection** on a principal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

is a preferred way of lifting paths — this is the notion of **parallel transport**.

Another point of view is that a connection on P is an invariant choice of **horizontal subspace** $H_u \subset T_uP$ for all $u \in P$.

This horizontal subspace defines a splitting of the short exact sequence

$$0 \to V_u \to T_uP \xrightarrow{d\pi} T_{\pi(u)}M \to 0$$

where the **vertical subspace** V_u is the kernel of $d\pi$.

The Atiyah Sequence

If $P \to M$ is a principal G-bundle then there is a canonical exact sequence of Lie algebras of vector fields:

$$\left\{ \begin{array}{c} \text{inv. vert.} \\ \text{vector fields} \\ \text{on } P \end{array} \right\} \to \left\{ \begin{array}{c} \text{inv.} \\ \text{vector} \\ \text{fields on } P \end{array} \right\} \to \left\{ \begin{array}{c} \text{vec. fields} \\ \text{on } M \end{array} \right\}$$

A **connection** on P is a splitting A of this short exact sequence.

Is A a *homomorphism* of Lie algebras??

In general the answer is no. The **curvature** F_A of the connection measures the failure of A to be a homomorphism of Lie algebras:

$$F_A(X,Y) = [A(X), A(Y)] - A[X,Y]$$

F_A satisfies an equation involving three vector fields X, Y and Z — the Bianchi identity.

Can we think of F_A as a *Lie algebra 2-cocycle*?

Classification of Lie algebra extensions

Consider an abstract extension of Lie algebras:

$$0 \to J \xrightarrow{i} K \xrightarrow{p} L \to 0$$

Let $A\colon L \to K$ be a linear splitting of p. We can measure the 'curvature' of A through the skew linear map $F_A\colon L \otimes L \to J$ defined by

$$F_A(x, y) = [A(x), A(y)] - A([x, y])$$

We can also define a linear map

$$\nabla_A\colon L \to \mathrm{Der}(J)$$

by

$$\nabla_A(x)(\xi) = [A(x), \xi] \text{ for } \xi \in J$$

∇_A is not a Lie algebra homomorphism; instead we have

$$\nabla_A([x, y]) = [\nabla_A(x), \nabla_A(y)] - \mathrm{ad}(F_A(x, y))$$

F_A satisfies an analogue of the 'Bianchi identity'.

Interlude: Lie 2-algebras

Lie 2-algebras were introduced by Baez and Crans in HDA6. A (**strict**) **Lie 2-algebra** is a category L internal to **LieAlg**. Thus L consists of

- a Lie algebra of objects L_0, and
- a Lie algebra of morphisms L_1

such that all structure maps are Lie algebra homomorphisms.

Baez and Crans go further and consider the notion of a **semi-strict Lie 2-algebra**. This is a 2-vector space L equipped with a skew bilinear functor

$$[\,,\,]\colon L \times L \to L$$

together an isomorphism (the **Jacobiator**)

$$J_{x,y,z}\colon [x,[y,z]] \to [[x,y],z] + [y,[x,z]]$$

natural in x, y and z. $J_{x,y,z}$ is required to satisfy the Jacobiator identity.

A good example of a Lie 2-algebra is the following. Let J be a Lie algebra. Recall that we have the adjoint homomorphism

$$\mathrm{ad}\colon J \longrightarrow \mathrm{Der}(J)$$

which we can think of as a chain complex. Define a Lie 2-algebra $\mathrm{DER}(J)$ with

- objects $= \mathrm{Der}(J)$, the Lie algebra of derivations of J, and
- morphisms $= \mathrm{Der}(J) \ltimes J$, the semi-direct product Lie algebra with bracket defined as usual by

$$[(f,x),(g,y)] = ([f,g],[x,y] + f(y) - g(x))$$

The source and target homomorphisms are defined by

$$s(f,x) = f,\ t(f,x) = f + \mathrm{ad}(x)$$

We can think of the pair (∇_A, F_A) as giving a **homomorphism** of Lie 2-algebras

$$F\colon L \to \mathrm{DER}(J),$$

where L is thought of as a *discrete* Lie 2-algebra.

For an object x of L, we set

$$F(x) = \nabla_A(x) \in \mathrm{Der}(J).$$

The 'curvature' $F_A(x,y)$ can be interpreted as a morphism

$$F([x,y]) \xrightarrow{F_2(x,y)} [F(x), F(y)]$$

in $\mathrm{DER}(J)$. The 'Bianchi identity' for $F_A(x,y)$ is the statement that the following diagram commutes:

$$\begin{tikzcd}
F[x,[y,z]] \arrow[r, "1"] \arrow[d, "F_2(x,[y,z])"'] & F[[x,y],z] + F[y,[x,z]] \arrow[d, "F_2([x,y],z)+F_2(y,[x,z])"] \\
{[F(x), F[y,z]]} \arrow[d, "{[1_{F(x)},F_2(y,z)]}"'] & {[F[x,y],F(z)] + [F(y),F[x,z]]} \arrow[d, "{[F_2(x,y),1_{F(z)}]+[1_{F(y)},F_2(x,z)]}"] \\
{[F(x),[F(y),F(z)]]} \arrow[r, "1"'] & {[[F(x),F(y)],F(z)] + [F(y),[F(x),F(z)]]}
\end{tikzcd}$$

We should think of the homomorphism F as a '*nonabelian Lie algebra cocycle*'.

Here is a classification theorem for extensions of Lie algebras. It must be well known.

Theorem. *Let L and J be Lie algebras. Then there is a bijection*

$$\mathrm{Ext}(L, J) \approx [L, \mathrm{DER}(J)]$$

between the set of equivalence classes of extensions of L by J and the set of homotopy classes of homomorphisms $L \to \mathrm{DER}(J)$.

It reduces to the usual description of $\mathrm{Ext}(L, J)$ when J is central in L.

Connections on gerbes

Several authors (Breen-Messing, Baez-Schreiber, and Jurčo et al) have developed a theory of **connections** on gerbes/2-bundles.

In particular Baez and Schreiber have shown how under certain conditions a connection on a gerbe can be used to define a notion of parallel transport over surfaces.

Connections on gerbes are difficult to understand; whereas a connection on a principal bundle can be described in terms of local gauge potentials

$$A_i,$$

a connection on a gerbe involves several gauge fields

$$A_i, \alpha_{ij}, B_i, \nu_i$$

satisfying lots of complicated equations.

We would like to have a description of connections on gerbes analogous to the description of connections on principal bundles in terms of Lie 2-algebras.

Derivations of L_∞ algebras

We need to understand derivations of Lie 2-algebras. To do this it is easiest to work with chain complexes.

Recall that in HDA6 Baez and Crans define the notion of a **2-term L_∞ algebra**. This is a chain complex L, concentrated in degrees 0: and 1:

$$L_1 \xrightarrow{d} L_0$$

together with skew linear maps

$$\ell_2 \colon L^{\otimes 2} \to L, \deg(\ell_2) = 0$$
$$\ell_3 \colon L^{\otimes 3} \to L, \deg(\ell_3) = 1$$

satisfying some conditions.

They prove the following result:

Theorem (Baez-Crans)**.** *There is a 2-equivalence of 2-categories*

$$\mathbf{2Term\ L_\infty} \simeq \mathbf{Lie2Alg}$$

We define instead a notion of derivation of a 2-term L_∞ algebra.

Let L be any L_∞ algebra. Then we can prove the following results.

Theorem. *There is a differential graded (DG) Lie algebra* $\mathrm{Der}(L)$*.*

One should think of these as derivations up to chain homotopy.

Theorem. *There is an L_∞ morphism* $\mathrm{ad}\colon L \to \mathrm{Der}(L)$*.*

This is a generalisation of the usual adjoint homomorphism $\mathrm{ad}\colon L \to \mathrm{Der}(L)$ for Lie algebras L.

Theorem. *There is a DG Lie algebra* $\mathrm{DER}(L)$ *whose underlying graded vector space is*

$$\mathrm{Der}(L) \oplus sL \ (s = \text{suspension}).$$

If L is a 2-term L_∞ algebra then $\mathrm{DER}(L)$ is concentrated in degrees 0, 1 and 2:

$$\mathrm{Der}_0(L) \leftarrow \mathrm{Der}_1(L) \oplus L_0 \leftarrow L_1$$

The underlying idea behind $\mathrm{DER}(L)$ is that it is supposed to be an infinitesimal version of the automorphism 3-group $\mathrm{AUT}(G)$ associated to a (weak) 2-group G.

Extensions of Lie 2-algebras

Let $f\colon L \to L'$ be an L_∞ morphism between 2-term L_∞ algebras. What do we mean by the **kernel** of f?

The correct notion is the **homotopy fibre** $H(f)$ of f; this has a natural structure of a 2-term L_∞ algebra such that the natural map $H(f) \to L$ extends to an L_∞ morphism.

Let L and J be 2-term L_∞ algebras. An **extension** of L by J consists of a 2-term L_∞ algebra K and an L_∞ morphism

$$p\colon K \to L$$

such that the underlying chain map is surjective, together with an L_∞ quasi-isomorphism

$$J \xrightarrow{\simeq} H(p).$$

The analogous notion for 2-groups was first considered by Breen in "*Théorie de Schreier Supérieure*".

Denote by $\mathrm{Ext}(L, J)$ the set of equivalence classes of extensions of L by J.

Classification Theorem

Theorem. *Let L and J be 2-term L_∞ algebras. Then there is a bijection*

$$\mathrm{Ext}(L, J) \approx [L, \mathrm{DER}(J)]$$

To any gerbe P on M, one can associate an analogue of the Atiyah short exact sequence.

$$0 \to \Gamma(\mathrm{ad}\, P) \to \Gamma(TP/G) \to \mathrm{Vect}(M) \to 0$$

This is now an exact sequence of Lie 2-algebras, where the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M is considered as a discrete Lie 2-algebra.

We can think of this as an extension of $\mathrm{Vect}(M)$ by $\Gamma(\mathrm{ad}\, P)$. By the theorem, this extension is classified by a '3-cocycle'

$$F\colon \mathrm{Vect}(M) \to \mathrm{DER}(\Gamma(\mathrm{ad}(P))).$$

The condition that this map is a homomorphism of L_∞ algebras neatly encodes the complicated equations for a connection on the gerbe.