Lectures on Higher Gauge Theory – III

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Higher Gauge Theory

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- topological groups and principal bundles.

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Now let's do the theory of connections!

For this it helps to work in a convenient category of smooth spaces. We'll use Souriau's 'diffeological spaces' — but we'll call them 'smooth spaces'.

Smooth Spaces

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Defn. A **smooth space** is a set X with, for each open set C, a collection of functions $\phi\colon C \to X$ called **plots** such that:

- If $\phi\colon C \to X$ is a plot and $f\colon C' \to C$ is a smooth map between open sets, then $\phi \circ f\colon C' \to X$ is a plot.
- If $i_\alpha\colon C_\alpha \to C$ is an open cover of an open set C by open subsets C_α, and $\phi\colon C \to X$ has the property that $\phi \circ i_\alpha$ is a plot for all α, then ϕ is a plot.
- Every map from a point to X is a plot.

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- Every map from a point to X is a plot.

Defn. Given smooth spaces X, Y, a map $f\colon X \to Y$ is **smooth** if $\phi \circ f\colon C \to Y$ is a plot whenever $\phi\colon C \to X$ is a plot.

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Let M be a smooth space (e.g. a manifold).

Let G be a **smooth group**: a smooth space with smooth group operations (e.g. a Lie group).

Let $\mathfrak{g}$ be the Lie algebra of G.

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We would like to compute this holonomy from a $\mathfrak{g}$-valued 1-form A on M, as follows. Solve this differential equation:

$$\frac{d}{dt} g(t) = A(\gamma'(t))\, g(t)$$

with initial value $g(t_0) = 1$. Then let:

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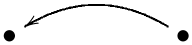
We say G is **exponentiable** if the above differential equation always has a smooth solution. Any Lie group is exponentiable. Henceforth assume all our smooth groups are exponentiable!

Connections as Functors

The holonomy along a path doesn't depend on its parametrization. When we compose paths, their holonomies multiply:

When we reverse a path, we get a path with the inverse holonomy:



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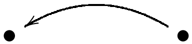
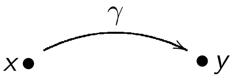


So, let $\mathcal{P}_1(M)$ be the **path groupoid** of M:

- objects are points $x \in M$: $\bullet\, x$
- morphisms are thin homotopy classes of smooth paths $\gamma\colon [0,1] \to M$ such that $\gamma(t)$ is constant near $t = 0, 1$:



Thm. $\mathcal{P}_1(M)$ is a **smooth groupoid**: it has a smooth space of objects, a smooth space of morphisms, and all the groupoid operations are smooth.

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Thm. Suppose M is a smooth space and G is a smooth group. There is a 1-1 correspondence between smooth functors

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G$$

and $\mathfrak{g}$-valued 1-forms A on M.

Internalization

Now let's categorify everything in sight and get a theory of holonomies for paths *and surfaces!*

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The crucial trick is 'internalization'. Given a familiar gadget x and a category K, we define an 'x in K' by writing the definition of x using commutative diagrams and interpreting these in K.

We need examples where $K = C^\infty$ is the category of smooth spaces:

- A **smooth group** is a group in C^∞.
- A **smooth groupoid** is a groupoid in C^∞.
- A **smooth strict 2-group** is a strict 2-group in C^∞.
- A **smooth strict 2-groupoid** is a strict 2-groupoid in C^∞.

2-Connections as 2-Functors

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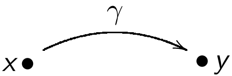

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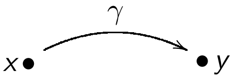
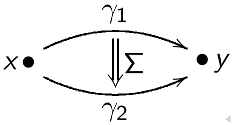


- as 2-morphisms, thin homotopy classes of smooth maps $\Sigma\colon [0,1]^2 \to M$ such that $\Sigma(s,t)$ is independent of s in a neighborhood of $s = 0$ and $s = 1$, and constant in a neighborhood of $t = 0$ and $t = 1$:



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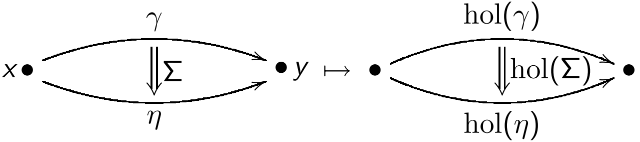
This suggests:

Defn. If $\mathcal{G}$ is a smooth strict 2-group, a **2-connection** on the trivial $\mathcal{G}$-2-bundle over a smooth space M is a smooth 2-functor

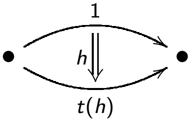
$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}.$$



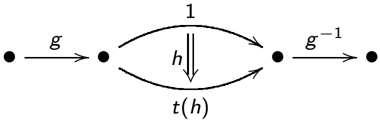
Crossed Modules

A strict 2-group $\mathcal{G}$ is determined by (G, H, t, ρ), where:

- the group G consists of all morphisms of $\mathcal{G}$,
- the group H consists of all 2-morphisms of $\mathcal{G}$ with source 1,
- the homomorphism $t\colon H \to G$ sends each 2-morphism in H to its target:



- ρ is the action of G on H given by:



This data (G, H, t, ρ) satisfies some equations making it a **crossed module**. Any crossed module determines a unique strict 2-group.

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Differentiating everything in a smooth crossed module, we get an **infinitesimal crossed module** $(\mathfrak{g}, \mathfrak{h}, dt, d\rho)$. This is just another way of repackaging a **strict Lie 2-algebra**: a Lie 2-algebra with trivial Jacobiator.

2-Connections on Trivial 2-Bundles

Thm. Suppose M is a smooth space. Suppose $\mathcal{G}$ is a smooth strict 2-group, let (G, H, t, ρ) be its smooth crossed module, and $(\mathfrak{g}, \mathfrak{h}, dt, d\rho)$ its infinitesimal crossed module.

There is a 1-1 correspondence between 2-connections on the trivial $\mathcal{G}$-2-bundle over M:

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G}$$

and pairs (A, B) consisting of a $\mathfrak{g}$-valued 1-form A and an $\mathfrak{h}$-valued 2-form B on M with vanishing **fake curvature**:

$$dA + A \wedge A + dt(B) = 0.$$

2-Connections on Locally Trivial 2-Bundles

Just as a 2-connection on a trivial 2-bundle is a smooth 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathcal{G},$$

a 2-connection on a *locally* trivial 2-bundle is a smooth 2-functor

$$\mathcal{P}_2(M) \xleftarrow{\;\sim\;} \mathcal{P}_2(M,\mathcal{U}) \xrightarrow{\;\mathrm{hol}\;} \mathcal{G}$$

where $\mathcal{U} = \{U_i\}$ is an open cover of M, and $\mathcal{P}_2(M,\mathcal{U})$ is a smooth 2-groupoid 'weakly equivalent' to $\mathcal{P}_2(M)$.

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So, a 2-connection is like a 'Morita morphism' or 'Hilsum–Skandalis map' from $\mathcal{P}_2(M)$ to $\mathcal{G}$.

Abusing notation a bit, let $\mathcal{U}$ be the disjoint union of the sets U_i, and

$$p\colon \mathcal{U} \to M$$

the map sending $x \in U_i$ to $x \in M$. Form the pullback

$$\begin{array}{ccc} \mathcal{U} \times_M \mathcal{U} & \xrightarrow{p_1} & \mathcal{U} \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{U} & \xrightarrow[p]{} & M \end{array}$$

This gives a diagram of smooth 2-groupoids

$$\begin{array}{ccc} \mathcal{P}_2(\mathcal{U} \times_M \mathcal{U}) & \xrightarrow{p_{1*}} & \mathcal{P}_2(\mathcal{U}) \\ {\scriptstyle p_{2*}}\downarrow & & \\ \mathcal{P}_2(\mathcal{U}) & & \end{array}$$

Next, define $\mathcal{P}_2(M,\mathcal{U})$ to be the weak pushout

$$\begin{array}{ccc} \mathcal{P}_2(\mathcal{U} \times_M \mathcal{U}) & \xrightarrow{p_{1*}} & \mathcal{P}_2(\mathcal{U}) \\ {\scriptstyle p_{2*}}\downarrow & & \downarrow \\ \mathcal{P}_2(\mathcal{U}) & \longrightarrow & \mathcal{P}_2(M,\mathcal{U}) \end{array}$$

in the semistrict 3-category of:

- smooth strict 2-groupoids,
- smooth 2-functors,
- smooth pseudonatural transformations,
- smooth modifications.

Defn. Let M be a smooth space with open cover $\mathcal{U} = \{U_i\}$. Let $\mathcal{G}$ be a smooth strict 2-group. Then a **2-connection** on a $\mathcal{G}$-2-bundle locally trivialized over the sets U_i is a 2-functor

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What does this amount to, more explicitly?

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1. For each i a smooth 2-functor:

$$\mathrm{hol}_i : \quad \mathcal{P}_2(U_i) \longrightarrow \mathcal{G}$$

$$x \underset{\eta}{\overset{\gamma}{\Downarrow \Sigma}} y \quad \longmapsto \quad \bullet \underset{\mathrm{hol}_i(\eta)}{\overset{\mathrm{hol}_i(\gamma)}{\Downarrow \mathrm{hol}_i(\Sigma)}} \bullet$$

2. For each i, j a smooth pseudonatural isomorphism:

$$g_{ij} : \mathrm{hol}_i|_{\mathcal{P}(U_i \cap U_j)} \to \mathrm{hol}_j|_{\mathcal{P}(U_i \cap U_j)}$$

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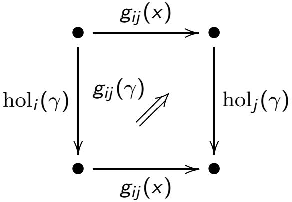
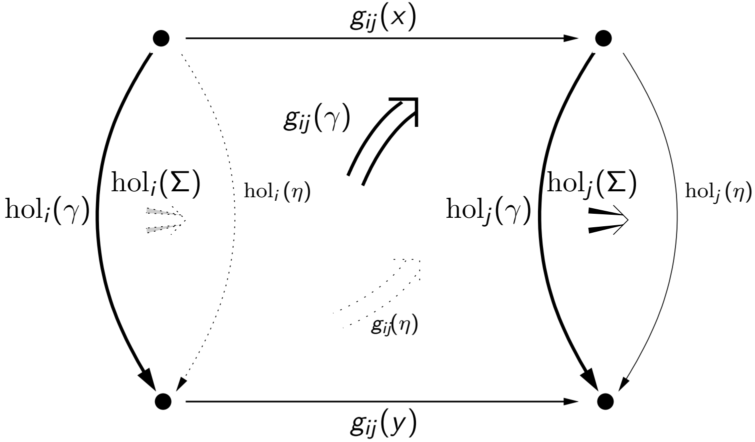
So: for each point $x \in U_i \cap U_j$, a morphism $g_{ij}(x) \colon \bullet \to \bullet$ in $\mathcal{G}$ depending smoothly on x.

For each path $\gamma \colon x \to y$ in $U_i \cap U_j$, a 2-morphism in $\mathcal{G}$:

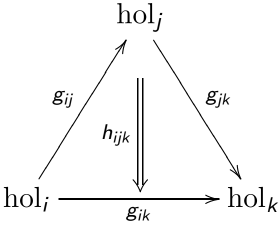


depending smoothly on γ...

... and making this diagram commute for any surface $\Sigma\colon \gamma \Rightarrow \eta$ in $U_i \cap U_j$:



3. For each i, j, k a smooth modification:



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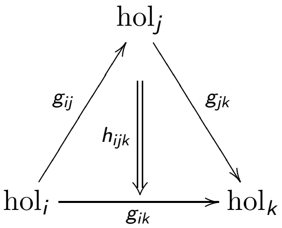

So: for each point $x \in U_i \cap U_j \cap U_k$, a 2-morphism

$$h_{ijk}(x) \colon g_{ij}(x) g_{jk}(x) \Rightarrow g_{ik}(x)$$

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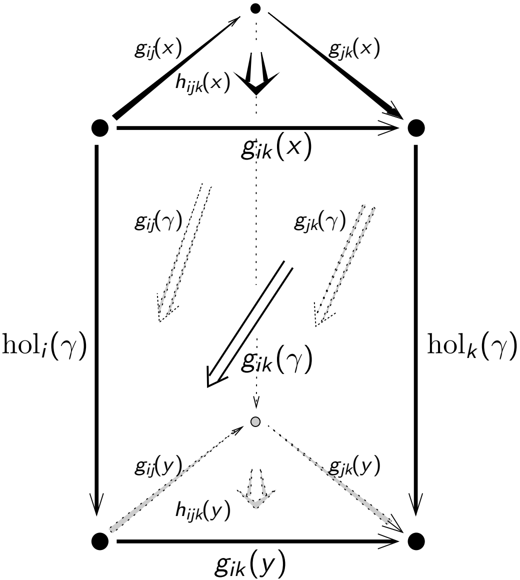

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Thm. Suppose M is a smooth space with open cover $\{U_i\}$. Suppose $\mathcal{G}$ is a smooth strict 2-group, let (G, H, t, ρ) be its smooth crossed module, and $(\mathfrak{g}, \mathfrak{h}, dt, d\rho)$ its infinitesimal crossed module.

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1. The holonomy 2-functor hol_i is specified by a $\mathfrak{g}$-valued 1-form A_i and an $\mathfrak{h}$-valued 2-form B_i on U_i, satisfying the fake flatness condition:

$$dA_i + A_i \wedge A_i + dt(B_i) = 0$$

2. The pseudonatural isomorphism $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$ is specified by the transition function

$$g_{ij} \colon U_i \cap U_j \to G$$

together with an $\mathfrak{h}$-valued 1-form a_{ij} on $U_i \cap U_j$, satisfying the equations:

$$A_i = g_{ij} A_j g_{ij}^{-1} + g_{ij} d g_{ij}^{-1} - dt(a_{ij})$$

$$B_i = \rho(g_{ij})(B_j) + da_{ij} + a_{ij} \wedge a_{ij} + d\rho(A_i) \wedge a_{ij}$$

3. For $g_{ij} \circ g_{jk} \xrightarrow{h_{ijk}} g_{ik}$ to be a modification, the functions h_{ijk} must satisfy the equations:

$$g_{ij}\, g_{jk}\, t(h_{ijk}) = g_{ik}$$

$$h_{ijk}\, h_{ikl} = \alpha(g_{ij})(h_{jkl})\, h_{ijl}$$

and

$$a_{ij} + \rho(g_{ij}) a_{jk} = h_{ijk}\, a_{ik}\, h_{ijk}^{-1} + h_{ijk}\, d\rho(A_i)\, h_{ijk}^{-1} + h_{ijk}\, dh_{ijk}^{-1}$$

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Punchline. *Except for fake flatness, these weird-looking equations show up already in Breen and Messing's definition of a connection on a nonabelian gerbe! A special case appears in the work of Martins and Picken. Other special cases are also known.*

So, these structures are really intrinsic to higher gauge theory.