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The Global Goursat Problem on $R \times S^1$

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The Goursat problem for the nonlinear wave equation $(\partial_\tau^2 - \partial_\rho^2)\,\phi + g(\phi) = 0$ on $R \times S^1$ is treated for Goursat data $\phi|_C$ in the Sobolev space $\mathbf{H}(C) = L_{2,1}(C)$, where the lightcone $C = \{\tau = |\rho|\}$ is identified with S^1 by means of the coordinate $\rho \in [-\pi, \pi]$. If $g \in C^1(R)$, there is a unique local solution given arbitrary Goursat data in $\mathbf{H}(C)$. In this case we describe conditions including nonlocal, nonlinear constraints on the Goursat data that are necessary and sufficient for the local solution to be C^2. If g is the derivative of a function G which is bounded from below, or if g satisfies $|g(y_1) - g(y_2)| \leqslant c_1 |y_1 - y_2| + c_2$ for some constants c_1 and c_2, there is a global solution given arbitrary data in $\mathbf{H}(C)$, and a global C^2 solution given data in $\mathbf{H}(C)$ satisfying the aforementioned conditions. We show that the map from Goursat data in $\mathbf{H}(C)$ to Cauchy data in $L_{2,1}(S^1) \oplus L_2(S^1)$ is continuous, with continuous inverse.

1. INTRODUCTION

The Goursat problem for a nonlinear wave equation, in which data are given on a lightcone, differs in a variety of ways from the more extensively studied Cauchy problem. Some of these appear in the context of wave equations on R^{n+1} [2]. For a through treatment of the global Goursat problem, however, it is preferable to work in $R \times S^n$, in part because the time evolution of Goursat data for a nonlinear wave equation on R^{n+1} is less conveniently described as a continuous one-parameter group on a Banach space. Moreover, conformally invariant wave equations on R^{n+1} can be rewritten as wave equations on $R \times S^n$, and there is a strong connection between the scattering theory of such equations and the global Goursat problem on $R \times S^n$ [1, 4].

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Consider the equation

$$\Box\phi + g(\phi) = 0,$$

where $\Box$ denotes the conformal wave operator on $R \times S^n$ and $g\colon R \to R$ is a sufficiently regular function. Then in the Goursat problem ϕ is determined by its restriction to the lightcone. The time evolution of this Goursat datum for ϕ lacks the domain of dependence properties associated with the Cauchy problem, and the infinitesimal generator of the evolution is nonlocal. Moreover, if ϕ is a C^2 solution, its Goursat datum satisfies certain nonlinear nonlocal constraints. These constraints make the regularity theory of the Goursat problem more subtle than that of the Cauchy problem.

In the present paper we treat regularity for the Goursat problem in the case $n = 1$. The constraints are tractable in this case, but one must take account of a certain special feature, namely that only for $n = 1$ does the generator of time evolution for the conformal wave equation on $R \times S^n$ have zero as an eigenvalue. This fact requires a slight strengthening of the energy norm successfully used to treat existence and uniqueness in the case $n > 1$ [1]. The space of Goursat data for which this strengthened energy norm is finite is denoted by $\mathbf{H}(C)$.

We show that if g is continuously differentiable, the Goursat problem has a unique solution locally in time given arbitrary Goursat data in $\mathbf{H}(C)$. In this case we describe conditions on Goursat data in $\mathbf{H}(C)$ that are necessary and sufficient for the local solution to be C^2. In particular, this implies that the conditions are preserved by the time evolution during the interval of existence. If g is the derivative of a function G which is bounded from below, or if g satisfies $|g(y_1) - g(y_2)| \leqslant c_1 |y_1 - y_2| + c_2$ for some constants c_1 and c_2, there is a global solution given arbitrary data in $\mathbf{H}(C)$, and a global C^2 solution given data in $\mathbf{H}(C)$ satisfying the constraints. These global solutions define a map from Goursat data to Cauchy data which is continuous from $\mathbf{H}(C)$ to the space of Cauchy data

$$\{\mathbf{\Phi}(0) \oplus \mathbf{\Phi}'(0) \mid \mathbf{\Phi}(0) \in L_{2,1}(S^1),\ \mathbf{\Phi}'(0) \in L_2(S^1)\}.$$

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2. GEOMETRY AND NOTATIONS

We let $S^1 = R/2\pi$, and specify points on S^1 by $\rho \in [-\pi, \pi]$. We give the manifold $R \times S^1$ the coordinates (τ, ρ), where $\tau \in R$, and $\rho \in [-\pi, \pi]$. We denote by $\tilde{M}$ the manifold $R \times S^1$ with the Lorentzian metric $d\tau^2 - d\rho^2$.

To treat the Goursat problem, we make use of the cones

$$C_t = \{(\tau, \rho) \mid \tau = t + |\rho|, |\rho| \leqslant \pi\}.$$

When the value of t is immaterial we call any of these cones C. We give C the coordinate ρ induced from the coordinate (τ, ρ) on $\tilde{M}$, thus identifying C with S^1. This identification is a homeomorphism, and yields (for each t) a map from S^1 to $\tilde{M}$ that is smooth except at $\{\rho = 0, \pi\}$. We use this identification to transfer to C the Riemannian metric $d\rho^2$ on S^1.

For arbitrary $t \in R$, S_t will denote the spacelike surface defined by the equation $\tau = t$. The subscript t will be suppressed when its value is immaterial. The S_t $(t \in R)$ are smooth compact submanifolds of $\tilde{M}$, each of which is diffeomorphic to S^1; the points of S will be specified by the function ρ indicated above.

We will use the notation $L_{p,q}(X)$ to denote the space of all real distributions f on a compact Riemannian manifold X that are in $L_p(X)$ together with their first q derivatives, and denote the norm in this space as $\|\cdot\|_{p,q}$. For $X = C$ we use the identification of C with S^1 to define $L_{p,q}(C)$.

3. The Cauchy and Goursat Problems

The Laplace–Beltrami operator on $\tilde{M}$ is

$$\square = \partial_\tau^2 - \partial_\rho^2.$$

Solutions of the wave equation and its inhomogeneous and nonlinear variants will be denoted by the lower-case Greek letters ϕ and ψ. We will use capital Greek letters to denote Cauchy data, e.g., $\Phi(\tau) = \phi|_{S_\tau}$ and $\Phi'(\tau) = \partial_\tau \Phi(\tau)$.

For any spacelike surface S, we define the Hilbert space of Cauchy data

$$\mathbf{H}(S) = L_{2,1}(S) \oplus L_2(S).$$

More specifically, the inner product in this space is defined by the equation

$$\langle \Phi_1 \oplus \Psi_1, \Phi_2 \oplus \Psi_2 \rangle = \langle \partial_\rho \Phi_1, \partial_\rho \Phi_2 \rangle + \langle \Phi_1, \Phi_2 \rangle + \langle \Psi_1, \Psi_2 \rangle, \quad (1)$$

where on the right side $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L_2(S)$.

We will use boldface capital Greek letters to denote Goursat data. Suppose ϕ is a function defined on $\tilde{M}$. To determine Goursat data at a particular point on C_t we introduce functions $\mathbf{\Phi}_+$, $\mathbf{\Phi}_-$, $\mathbf{\Phi}$, $c_0(t)$, and $c_\pi(t)$ as follows:

$$\mathbf{\Phi}_+(t, \rho) = \phi(t + \rho, \rho), \qquad 0 \leqslant \rho \leqslant \pi;$$
$$\mathbf{\Phi}_-(t, \rho) = \phi(t + \rho, -\rho), \qquad 0 \leqslant \rho \leqslant \pi.$$
$$\mathbf{\Phi}(t) = (\mathbf{\Phi}_+(t), \mathbf{\Phi}_-(t))$$
$$c_0(t) = \phi(t, 0), \qquad c_\pi(t) = \phi(t + \pi, \pi).$$

For any specific cone C, we define the Banach space $\mathbf{H}(C)$ in which the norm is given by

$$\|\mathbf{\Phi}(t)\|_{\mathbf{H}(C)} = \left[\int_0^\pi (\partial_\rho \mathbf{\Phi}_+(t, \rho))^2 + (\partial_\rho \mathbf{\Phi}_-(t, \rho))^2)\, d\rho\right]^{1/2} + |c_0(0)|. \quad (2)$$

This norm is equivalent to the norm in $L_{2,1}(C)$. The main part of the norm of $\mathbf{\Phi}(t)$ is the free energy

$$E(t) = \left[\int_0^\pi (|\partial_\rho \mathbf{\Phi}_+|^2 + |\partial_\rho \mathbf{\Phi}_-|^2)\, d\rho\right]^{1/2} \quad (3)$$

which is independent of t for every C^2 solution of the free wave equation (see Lemma 2 below).

We first examine temporal evolution in the Goursat format for the inhomogeneous wave equation.

Lemma 1. *If ϕ is a C^2 solution of the equation $\square\phi = h(\tau, \rho)$ on a neighborhood U of C_t, where h is a continuous function on this neighborhood, then*

$$\begin{cases} \partial_t \partial_\rho \mathbf{\Phi}_+(t, \rho) = \frac{1}{2}[\partial_\rho^2 \mathbf{\Phi}_+(t, \rho) + \mathbf{h}_+(t, \rho))] \\ \partial_t \partial_\rho \mathbf{\Phi}_-(t, \rho) = \frac{1}{2}[\partial_\rho^2 \mathbf{\Phi}_-(t, \rho) + \mathbf{h}_-(t, \rho))], \end{cases} \quad (4)$$

where $\mathbf{h}_+(t, \rho) = h(t + \rho, \rho)$, $\mathbf{h}_-(t, \rho) = h(t + \rho, -\rho)$, $0 \leqslant \rho \leqslant \pi$.

Proof. Noting that $\partial_t \mathbf{\Phi}_+ = \partial_\tau \phi$, $\partial_\rho \mathbf{\Phi}_+ = (\partial_\rho + \partial_\tau)\phi$ and $\partial_t \mathbf{\Phi}_- = \partial_\tau \phi$, $\partial_\rho \mathbf{\Phi}_- = (-\partial_\rho + \partial_\tau)\phi$, Eq. (4) follows from the direct calculation. ∎

Given the hypothesis in Lemma 1 with the inhomogeneous term f replaced by a nonlinear function of ϕ, Eq. (4) implies certain nonlinear nonlocal constraints satisfied by the Goursat data. We consider this matter further when we study the regularity problem.

Next we prove a statement of energy conservation for the nonlinear equation. We first compute the differential of a one-form ε; physically, ε

may be interpreted as a component of the energy-momentum tensor associated with ϕ. Let ϕ be a C^2 function on an open set $U \subset \tilde{M}$ and let $G: R \to R$ be a C^1 function. Set

$$\varepsilon = \tfrac{1}{2}[(\partial_\rho \phi)^2 + (\partial_\tau \phi)^2]\, d\rho + G(\phi)\, d\rho + \partial_\rho \phi \partial_\tau \phi\, d\tau.$$

Then on U we have

$$d\varepsilon = \partial_\tau \phi [\Box \phi + G'(\phi)]\, d\tau \wedge d\rho. \tag{5}$$

Now we are ready to state the following

LEMMA 2. *Let ϕ be a C^2 function on the contractible open set $\acute{U} \subset \tilde{M}$. Let ε be the one-form given above. Then for all $S_\tau \subset U$ and $C_t \subset U$,*

$$\int_{S_\tau} \varepsilon = \int_{-\pi}^{\pi} [\tfrac{1}{2}((\partial_\rho \phi(\tau, \rho))^2 + (\partial_\tau \phi(\tau, \rho))^2) + G(\phi(\tau, \rho))]\, d\rho$$

$$\int_{C_t} \varepsilon = \tfrac{1}{2}[E(t)]^2 + \int_0^{\pi} [G(\mathbf{\Phi}_-(t, \rho)) + G(\mathbf{\Phi}_+(t, \rho))]\, d\rho,$$

where $E(t)$ is the free energy associated with $\mathbf{\Phi}(t)$ as defined in (3). If in addition ϕ satisfies the equation $\Box\phi + G'(\phi) = 0$ in U, then there is a constant E such that

$$\int_{S_\tau} \varepsilon = \int_{C_t} \varepsilon = E$$

for any S_τ or C_t lying in U.

From this we conclude that $E(t)$ is independent of t if ϕ is a solution of the free wave equation.

Proof. One easily calculates that

$$\int_{S_\tau} \varepsilon = \int_{-\pi}^{\pi} [\tfrac{1}{2}((\partial_\rho \phi(\tau, \rho))^2 + (\partial_\tau \phi(\tau, \rho))^2) + G(\phi)]\, d\rho.$$

On C_t we use the relation $\tau = t + |\rho|$ and the definition of $\mathbf{\Phi}_-$ and $\mathbf{\Phi}_+$ to check that

$$\int_{C_t} \varepsilon = \int_0^{\pi} [\tfrac{1}{2}((\partial_\rho \mathbf{\Phi}_+(t, \rho))^2 + (\partial_\rho \mathbf{\Phi}_-(t, \rho))^2) + G(\mathbf{\Phi}_-(t, \rho)) + G(\mathbf{\Phi}_+(t, \rho))]\, d\rho.$$

If ϕ is a solution to the equation $\Box\phi + G'(\phi) = 0$, then ε is a closed form by

Eq. (5). Therefore the integrating ε over any S_τ or C_t lying in U gives the same constant E, as these 1-submanifolds of $\tilde{M}$ are all homotopic. ∎

4. LOCAL EXISTENCE OF SOLUTIONS

In this section we study the local existence and uniqueness of solutions to the Goursat problem for the nonlinear wave equation in $R \times S^1$

$$\Box\phi + g(\phi) = 0, \tag{6}$$

and the continuity of the map thus defined from Goursat data in $\mathbf{H}(C_t)$ to Goursat data in $\mathbf{H}(C_{t'})$. First we rewrite the equation in an abstract Cauchy form,

$$\mathbf{\Phi}(t) = T_t \mathbf{\Phi}(0) + k_t(g(\mathbf{\Phi})),$$

where $\mathbf{\Phi}(t)$ has values in the Banach space $\mathbf{H}(C)$, and estimate both terms on the right-hand side in terms of the norm of $\mathbf{\Phi}(0)$.

To begin with, consider the Goursat problem for the free wave equation in the form

$$\begin{aligned} &\Box\phi = 0 \\ &\phi(\rho, \rho) = \mathbf{\Phi}_+(0, \rho), \qquad 0 \leqslant \rho \leqslant \pi \\ &\phi(\rho, -\rho) = \mathbf{\Phi}_-(0, \rho), \qquad 0 \leqslant \rho \leqslant \pi \\ &\mathbf{\Phi}_+(0, 0) = \mathbf{\Phi}_-(0, 0), \qquad \mathbf{\Phi}_+(0, \pi) = \mathbf{\Phi}_-(0, \pi). \end{aligned}$$

Suppose that ϕ is a C^2 solution and $0 \leqslant t \leqslant \pi$. Then

$$c_0(t) = \mathbf{\Phi}_+\left(0, \frac{t}{2}\right) + \mathbf{\Phi}_-\left(0, \frac{t}{2}\right) - c_0(0)$$

$$c_\pi(t) = \mathbf{\Phi}_+\left(0, \frac{t}{2}\right) + \mathbf{\Phi}_-\left(0, \frac{t}{2}\right) - 2c_0(0) + c_\pi(0),$$

and for $\rho \leqslant \pi - t/2$, we have

$$\mathbf{\Phi}_\pm(t, \rho) = \mathbf{\Phi}_\pm\left(0, \frac{t + 2\rho}{2}\right) + \mathbf{\Phi}_\mp\left(0, \frac{t}{2}\right) - c_0(0).$$

For $\pi - t/2 \leqslant \rho \leqslant \pi$, using the periodicity of ϕ as function of ρ, we have

$$\mathbf{\Phi}_\pm(t, \rho) = \mathbf{\Phi}_\pm\left(0, \frac{t + 2\rho}{2} - \pi\right) + \mathbf{\Phi}_\mp\left(0, \frac{t}{2}\right) - 2c_0(0) + c_\pi(0).$$

It is easy to see that

$$\lim_{\rho\to\sigma}\Phi_+(t,\rho)=\lim_{\rho\to 0}\Phi_-(t,\rho)=c_0(t)$$
$$\lim_{\rho\to\pi}\Phi_+(t,\rho)=\lim_{\rho\to\pi}\Phi_-(t,\rho)=c_\pi(t).$$

Thus if we define $T_t\Phi(0)$ to equal $\Phi(t)$ as given by the formulas above, we see that

$$\|T_t\Phi(0)\|_{\mathbf{H}(C)}\leqslant(1+c\sqrt{t})\|\Phi(0)\|_{\mathbf{H}(C)}. \tag{7}$$

Now consider the inhomogeneous problem. Suppose $\Box\phi=h$, where ϕ is C^2 and h is a continuous function. Let

$$\Phi(t)=T_t\Phi(0)+k_th,$$

where T_t is defined as above, and k_t is a linear operator from $C(\tilde{M})$ to Goursat data on C_t. Let us compute k_th, supposing that $0\leqslant t\leqslant\pi$. It is easy to see that for $\rho\leqslant\pi-t/2$ we have

$$\begin{aligned}\Phi_\pm(t,\rho)&=\phi(t+\rho,\pm\rho)\\&=\phi\left(\frac{t+2\rho}{2},\pm\frac{t+2\rho}{2}\right)+\phi\left(\frac{t}{2},\mp\frac{t}{2}\right)-\phi(0,0)\\&\quad+\int_0^{(t+2\rho)/2}\int_0^{t/2}h(p+q,\pm(p-q))\,dq\,dp\\&=(T_t\Phi(0))_\pm+\int_0^{(t+2\rho)/2}\int_0^{t/2}h(p+q,\pm(p-q))\,dq\,dp.\end{aligned}$$

Similarly by using the periodicity of ϕ in the variable ρ, for $\pi-t/2\leqslant\rho\leqslant\pi$ we have

$$\begin{aligned}\Phi_\pm(t,\rho)&=(T_t\Phi(0))_\pm+\int_0^{(t+2\rho)/2-\pi}\int_\pi^{p+\pi}h(p+q,\pm(p-q))\,dq\,dp\\&\quad+\int_0^\pi\int_0^{t/2}h(p+q,\pm(p-q))\,dq\,dp\\&\quad+\int_\pi^{(t+2\rho)/2}\int_{p-\pi}^{t/2}h(p+q,\pm(p-q))\,dq\,dp.\end{aligned}$$

Define k_th by the above formulas, namely

$$(k_th)_\pm(\rho)=\int_0^{(t+2\rho)/2}\int_0^{t/2}h(p+q,\pm(p-q))\,dq\,dp,\qquad \rho\leqslant\pi-\frac{t}{2};$$

$$\begin{aligned}(k_th)_\pm(\rho)&=\int_0^{(t+2\rho)/2-\pi}\int_\pi^{\rho+\pi}h(p+q,\pm(p-q))\,dq\,dp\\&\quad+\int_0^\pi\int_0^{t/2}h(p+q,\pm(p-q))\,dq\,dp\\&\quad+\int_\pi^{(t+2\rho)/2}\int_{p-\pi}^{t/2}h(p+q,\pm(p-q))\,dq\,dp,\qquad \pi-\frac{t}{2}\leqslant\rho\leqslant\pi.\end{aligned}$$

Let D_t be the union of the C_τ for $0\leqslant\tau\leqslant t$. Then it is easy to see that if $h\in C(\tilde{M})$, then

$$\|k_th\|_{\mathbf{H}(C)}\leqslant c\,\|h\|_{L_2(D_t)},$$

thus allowing us to define k_t as a bounded operator from $L_2(D_t)$ to $\mathbf{H}(C)$. From this estimate and (7), we conclude that

$$\|\Phi(t)\|_{\mathbf{H}(C)}\leqslant[1+c\sqrt{t}]\|\Phi(0)\|_{\mathbf{H}(C)}+c\,\|h\|_{L_2(D_t)},\qquad t\in[0,\pi]. \tag{8}$$

It is easy to see that if $h\in L_2(D_t)$, the function Φ given above satisfies the inhomogeneous wave equation in the distributional sense. Hence we have the following.

THEOREM 1. *Let h be a function in $L_2(D_t)$, $t\leqslant\pi$, and let $\Phi(0)\in\mathbf{H}(C)$. Then there is a unique continuous map $\Phi\colon[0,t]\to\mathbf{H}(C)$ such that the associated function ϕ satisfies*

$$\Box\phi=h,\qquad \text{in}\quad D_t;$$
$$\phi|_{C_0}=\Phi(0)$$

in the distributional sense. Moreover the estimate (8) holds.

Proof. We only need to prove the uniqueness. Suppose that Φ_1 and Φ_2 are two solutions. Then we have

$$\Box(\phi_1-\phi_2)=0$$
$$(\phi_1-\phi_2)|_{C_0}=0.$$

Therefore $\phi_1(\tau,\rho)-\phi_2(\tau,\rho)=f_1(\tau+\rho)+f_2(\tau-\rho)$. From

$$f_1(2\tau)+f_2(0)=0,\qquad f_2(2\tau)+f_1(0)=0,$$

we conclude that

$$\phi_1(\tau, \rho) - \phi_2(\tau, \rho) = -f_1(0) - f_2(0) = \phi_2(0, 0) - \phi_1(0, 0) = 0. \quad \blacksquare$$

Turning now to the nonlinear case, suppose that h is replaced by a given function of the unknown field, i.e., $h = g(\phi)$. The estimate (8) yields the following local existence theorem and continuity of the map from $\mathbf{\Phi}(0)$ to $\mathbf{\Phi}(t)$.

THEOREM 2. *If g is a continuously differentiable function on R, $\mathbf{\Phi}(0) \in \mathbf{H}(C)$, then there is a $t_0 > 0$ and a unique continuous $\mathbf{\Phi}: [0, t_0] \to \mathbf{H}(C)$ such that*

$$\mathbf{\Phi}(t) = T_t \mathbf{\Phi}(0) + N_t(\mathbf{\Phi}),$$

where

$$N_t(\mathbf{\Phi}) = -k_t(g(\mathbf{\Phi})).$$

Proof of Theorem 2. Noting that $\|\mathbf{\Phi}(t)\|_{L_\infty} \leqslant c\,\|\mathbf{\Phi}(t)\|_{\mathbf{H}(C)}$ and that $g \in C^1$ implies that the mapping: $\mathbf{\Phi} = (\mathbf{\Phi}_+, \mathbf{\Phi}_-) \to (g(\mathbf{\Phi}_+), g(\mathbf{\Phi}_-))$ is boundedly Lipschitzian from $\mathbf{H}(C)$ to $L_2(C)$. Then we can use the estimate (8) and an iteration procedure to prove the theorem, as in [1].

From this it follows that the equation

$$\Box \phi = P(\phi),$$

P any polynomial, has a solution on some $D(t)$ given Goursat data $\mathbf{\Phi}(0) \in \mathbf{H}(C)$.

THEOREM 3. *Suppose that $\mathbf{\Phi}_1(t)$, $\mathbf{\Phi}_2(t) \in \mathbf{H}(C)$, $0 \leqslant t \leqslant t_0$, are two solutions in $\mathbf{H}(C)$ to the Goursat problem of Eq.* (6). *Then we have the estimate*

$$\|\mathbf{\Phi}_1(t) - \mathbf{\Phi}_2(t)\|_{\mathbf{H}(C)} \leqslant 2(1 + c\sqrt{t})\,\|\mathbf{\Phi}_1(0) - \mathbf{\Phi}_2(0)\|_{\mathbf{H}(C)}\, e^{(cA(t))^2 t}, \qquad (9)$$

where c is a positive constant independent of $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$, and

$$A(t) = \max\{|g'(y)| \mid |y| \leqslant M(t)\},$$

and $M(t)$ is defined by

$$M(t) = \max\{\|\mathbf{\Phi}_1(\tilde{t})\|_{L_\infty(C)}, \|\mathbf{\Phi}_2(\tilde{t})\|_{L_\infty(C)} \mid 0 \leqslant \tilde{t} \leqslant t\}.$$

It should be remarked that if g satisfies conditions (18) or (17) in Section 6, then we have

$$\|\mathbf{\Phi}_1(t)\|_{\mathbf{H}(C)} \leqslant f(t, \|\mathbf{\Phi}_1(0)\|_{\mathbf{H}(C)})$$

and

$$\|\mathbf{\Phi}_2(t)\|_{\mathbf{H}(C)} \leqslant f(t, \|\mathbf{\Phi}_2(0)\|_{\mathbf{H}(C)}),$$

where $f(t, y): R^2 \to R$ maps bounded sets in R^2 to bounded sets in R (for details see Section 6). Therefore in this case we have

$$\|\mathbf{\Phi}_1(t) - \mathbf{\Phi}_2(t)\|_{\mathbf{H}(C)} \leqslant F(t, \|\mathbf{\Phi}_1(0)\|_{\mathbf{H}(C)}, \|\mathbf{\Phi}_2(0)\|_{\mathbf{H}(C)})\,\|\mathbf{\Phi}_1(0) - \mathbf{\Phi}_2(0)\|_{\mathbf{H}(C)},$$

where $F(t, y_1, y_2): R^3 \to R$ maps bounded sets in R^3 to bounded sets in R, i.e., for any fixed t, the map from $\mathbf{\Phi}(0)$ to $\mathbf{\Phi}(t)$ is boundedly Lipschitzian.

Proof of Theorem 3. Let $\eta(t) = \mathbf{\Phi}_1(t) - \mathbf{\Phi}_2(t)$. Then $\eta(t)$ satisfies the equation

$$\begin{aligned} \eta(t) &= T_t \eta(0) + k_t(g(\mathbf{\Phi}_1) - g(\mathbf{\Phi}_2)) \\ &= T_t \eta(0) + k_t(v\eta). \end{aligned}$$

For any $0 \leqslant t_1 \leqslant t_0$ and $t \in [0, t_1]$, using (8),

$$\begin{aligned} \|\eta(t)\|_{\mathbf{H}(C)} &\leqslant (1 + c\sqrt{t_1})\,\|\eta(0)\|_{\mathbf{H}(C)} + c\,\|v\eta\|_{L_2(D_t)} \\ &\leqslant (1 + c\sqrt{t_1})\,\|\eta(0)\|_{\mathbf{H}(C)} + cA(t_1)\left[\int_0^t \|\eta(\tau)\|^2_{\mathbf{H}(C)}\, d\tau\right]^{1/2}. \end{aligned}$$

Let $B(t) = \|\eta(t)\|^2_{\mathbf{H}(C)}$, $a = (1 + c\sqrt{t_1})\,\|\eta(0)\|_{\mathbf{H}(C)}$. Therefore

$$\begin{aligned} B(t) &\leqslant \left(a + cA(t)\left[\int_0^t B(\tau)\, d\tau\right]^{1/2}\right)^2 \\ &\leqslant 2a^2 + 2(cA(t))^2 \int_0^t B(\tau)\, d\tau, \end{aligned}$$

using the Gronwall inequality we conclude that

$$B(t) \leqslant 2a^2 e^{2(cA(t_1))^2 t},$$

i.e.,

$$\|\eta(t_1)\|_{\mathbf{H}(C)} = \sqrt{B(t_1)} \leqslant 2a e^{(cA(t_1))^2 t_1},$$

which is exactly the estimate (9). $\blacksquare$

We have proved an estimate from Goursat data to Goursat data. Given certain conditions on g there are similar estimates from Goursat data to Cauchy data, and from Cauchy data to Goursat data. We will study this matter in Section 7.

5. Regularity of Solutions

Regularity of solutions and sufficient condition on the Goursat data of the nonlinear wave equation on C_0 under which the solution given in the previous section to the Goursat problem is actually in C^2 and solves the wave equation classically. We first find a necessary condition, and then prove that it is actually sufficient. To be precise, we prove the following.

THEOREM 4. *Suppose that g is differentiable on R, and $\mathbf{\Phi}(0) \in H(C)$. Let $\mathbf{\Phi}: [0, t_0] \to H(C)$ be the solution found in the previous section to the Goursat problem for the wave equation (6) with Goursat data $\mathbf{\Phi}(0)$. Then the associated function ϕ is in $C^2(D_{t_0})$ and solves Eq. (6) classically if and only if $\mathbf{\Phi}_\pm(0) \in C^2[0, \pi]$ and satisfies the following nonlinear nonlocal constraints*

$$\partial_\rho \mathbf{\Phi}_\pm(0, \pi) - \partial_\rho \mathbf{\Phi}_\pm(0, 0) + \int_0^\pi g(\mathbf{\Phi}_\mp(0, \rho))\, d\rho = 0, \tag{10}$$

and

$$\partial_\rho^2 \mathbf{\Phi}_\pm(0, \pi) - \partial_\rho^2 \mathbf{\Phi}_\pm(0, 0)$$
$$= g(\mathbf{\Phi}_\pm(0, \pi)) - g(\mathbf{\Phi}_\pm(0, 0)) - 2 \int_0^\pi g'(\mathbf{\Phi}_\mp(0, \rho))\, \partial_t \mathbf{\Phi}_\mp(0, \rho)\, d\rho, \tag{11}$$

where $\partial_t \mathbf{\Phi}_\pm(0, \rho)$ is given in terms of $\mathbf{\Phi}(0)$ by

$$\partial_t \mathbf{\Phi}_\pm(0, \rho) - \partial_t \mathbf{\Phi}_\pm(0, 0)$$
$$= \tfrac{1}{2}\left[\partial_\rho \mathbf{\Phi}_\pm(0, \rho) - \partial_\rho \mathbf{\Phi}_\pm(0, 0) - \int_0^\rho g(\mathbf{\Phi}_\pm(0, \rho'))\, d\rho'\right],$$

with $\partial_t \mathbf{\Phi}_\pm(0, 0) = \tfrac{1}{2}(\partial_\rho \mathbf{\Phi}_+(0, 0) + \partial_\rho \mathbf{\Phi}_-(0, 0))$.

Proof. Suppose that ϕ is a classical solution of the wave equation (6) in the domain D_{t_0}, the union of the C_t for $0 < t < t_0$. Then by Lemma 1, in D_{t_0} we can rewrite the wave equation (6) as

$$\partial_t \partial_\rho \mathbf{\Phi}_\pm(t, \rho) = \tfrac{1}{2}[\partial_\rho^2 \mathbf{\Phi}_\pm(t, \rho) - g(\mathbf{\Phi}_\pm(t, \rho))]. \tag{12}$$

Integrating with respect to ρ gives

$$\partial_t \mathbf{\Phi}_\pm(t, \rho) - \partial_t \mathbf{\Phi}_\pm(t, 0)$$
$$= \tfrac{1}{2}\left[\partial_\rho \mathbf{\Phi}_\pm(t, \rho) - \partial_\rho \mathbf{\Phi}_\pm(t, 0) - \int_0^\rho g(\mathbf{\Phi}_\pm(t, \rho'))\, d\rho'\right]. \tag{13}$$

In particular we have

$$\partial_t \mathbf{\Phi}_\pm(t, \pi) - \partial_t \mathbf{\Phi}_\pm(t, 0) = \frac{1}{2}\left[\partial_\rho \mathbf{\Phi}_\pm(t, \pi) - \partial_\rho \mathbf{\Phi}_\pm(t, 0) - \int_0^\pi g(\mathbf{\Phi}_\pm(t, \rho))\, d\rho\right]. \tag{14}$$

Noting that

$$\partial_\rho \mathbf{\Phi}_+(t, 0) + \partial_\rho \mathbf{\Phi}_-(t, 0) = 2\partial_t \mathbf{\Phi}_\pm(t, 0),$$
$$\partial_\rho \mathbf{\Phi}_+(t, \pi) + \partial_\rho \mathbf{\Phi}_-(t, \pi) = 2\partial_t \mathbf{\Phi}_\pm(t, \pi),$$

we see that (14) yields the first constraint on the Goursat data

$$\partial_\rho \mathbf{\Phi}_\pm(t, \pi) - \partial_\rho \mathbf{\Phi}_\pm(t, 0) + \int_0^\pi g(\mathbf{\Phi}_\mp(t, \rho))\, d\rho = 0. \tag{15}$$

Now let us differentiate (15) with respect to t. Using (12), it is easy to see that

$$\partial_\rho^2 \mathbf{\Phi}_\pm(t, \pi) - \partial_\rho^2 \mathbf{\Phi}_\pm(t, 0)$$
$$= g(\mathbf{\Phi}_\pm(t, \pi)) - g(\mathbf{\Phi}_\pm(t, 0)) - 2 \int_0^\pi g(\mathbf{\Phi}_\mp(t, \rho))\, \partial_t \mathbf{\Phi}_\mp(t, \rho)\, d\rho, \tag{16}$$

which is the second constraint on the Goursat data, where $\partial_t \mathbf{\Phi}_\pm$ is given by the formula (13). Setting $t = 0$ in (15) and (16), we conclude that (10) and (11) are necessary conditions for ϕ to be a C^2 solution.

Next we prove that if $\mathbf{\Phi}(0) \in \mathbf{H}(C)$ has $\mathbf{\Phi}_\pm \in C^2[0, \pi]$ and satisfies conditions (10) and (11), there is a C^2 solution of wave equation (6) in D_{t_0} with $\mathbf{\Phi}(0)$ as its Goursat data. It can be easily checked by direct differentiation that the solution given by formulas in Section 4 is a C^2 solution in D_{t_0} except possibly on the characteristics

$$\{(\tau, \rho) \mid \tau + |\rho| = 2\pi\}.$$

Therefore we only need to check that $\mathbf{\Phi}_\pm(t, \rho)$ has continuous first and second order derivatives across the line $t/2 + \rho = \pi$. Set $h(\tau, \rho) = -g(\phi(\tau, \rho))$. First we can calculate $\partial_\rho \mathbf{\Phi}_\pm$ and $\partial_t \mathbf{\Phi}_\pm$ as follows: For $0 \leqslant t/2 + \rho < \pi$,

$$\partial_\rho \Phi_\pm(t,\rho) = \partial_\rho \Phi_\pm\left(0, \frac{t}{2}+\rho\right) + \int_0^{t/2} h\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq$$

$$\partial_t \Phi_\pm(t,\rho) = \frac{1}{2}\left[\partial_\rho \Phi_\pm\left(0, \frac{t}{2}+\rho\right) + \partial_\rho \Phi_\mp\left(0, \frac{t}{2}\right)\right] + \frac{1}{2}\left[\int_0^{t/2} h\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq + \int_0^{t/2+\rho} h\left(\rho+\frac{t}{2}, \pm\left(\rho-\frac{t}{2}\right)\right) d\rho\right].$$

For $t/2+\rho>\pi$, $\rho \leqslant \pi$, we have

$$\partial_\rho \Phi_\pm(t,\rho) = \partial_\rho \Phi_\pm\left(0, \frac{t}{2}+\rho-\pi\right) + \int_{t/2+\rho-\pi}^{t/2} h\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq + \int_\pi^{t/2+\rho} h\left(\frac{t}{2}+\rho-\pi+q, \pm\left(\frac{t}{2}+\rho-\pi-q\right)\right) dq$$

$$\partial_t \Phi_\pm(t,\rho) = \frac{1}{2}\left[\partial_\rho \Phi_\pm\left(0, \frac{t}{2}+\rho-\pi\right) + \partial_\rho \Phi_-\left(0, \frac{t}{2}\right) + \int_0^{t/2+\rho} h\left(p+\frac{t}{2}, \pm\left(p-\frac{t}{2}\right)\right) d\rho\right] + \frac{1}{2}\left[\int_{t/2+\rho-\pi}^{t/2} h\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq + \int_\pi^{t/2+\rho} h\left(\frac{t}{2}+\rho-\pi+q, \pm\left(\frac{t}{2}+\rho-\pi-q\right)\right) dq\right].$$

From the expression above we can see immediately that the condition (10) is enough to guarantee the continuity of the first order derivatives of the solution across the line $t/2+\rho=\pi$. By similar calculation the condition given in (11) will guarantee the continuity of all second order derivatives. Calculating all second order derivatives is a lengthy process, so here we just give formulas for $\partial_\rho^2 \Phi_\pm$ and leave the rest to readers. For $t/2+\rho<\pi$,

$$\partial_\rho^2 \Phi_\pm(t,\rho) = \partial_\rho^2 \Phi_\pm\left(0, \frac{t}{2}+\rho\right) + \int_0^{t/2}\left[h_\tau\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) + h_\rho\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right)\right] dq.$$

For $t/2+\rho>\pi$, $\rho\leqslant\pi$, we have

$$\begin{aligned}\partial_\rho^2 \Phi_\pm(t,\rho) = {} & \partial_\rho^2 \Phi_\pm\left(0, \frac{t}{2}+\rho-\pi\right) \\ & + \int_{t/2+\rho-\pi}^{t/2} h_\tau\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq \\ & + \int_{t/2+\rho-\pi}^{t/2} h_\rho\left(\frac{t}{2}+\rho+q, \pm\left(\frac{t}{2}+\rho-q\right)\right) dq \\ & + h(t+2\rho-\pi, \mp\pi) - h(t+2\rho-\pi, \pm\pi) \\ & + \int_\pi^{t/2+\rho} h_\tau\left(\frac{t}{2}+\rho-\pi+q, \pm\left(\frac{t}{2}+\rho-\pi-q\right)\right) dq \\ & + \int_\pi^{t/2+\rho} h_\rho\left(\frac{t}{2}+\rho+q-\pi, \pm\left(\frac{t}{2}+\rho-q-\pi\right)\right) dq.\end{aligned}$$

In order for $\partial_\rho^2 \Phi_\pm$ to be continuous across $t/2+\rho=\pi$, we only need that

$$\partial_\rho^2 \Phi_\pm(0,\pi) = \partial_\rho^2 \Phi_\pm(0,0) + \int_0^\pi [h_\tau(q,\mp q) + h_\rho(q,\mp q)]\, dq.$$

Note that

$$\begin{aligned}-\int_0^\pi [h_\tau(q,\mp q)+h_\rho(q,\mp q)]\, dq &= \int_0^\pi [2\partial_t g(\Phi_\mp(t,\rho)) - \partial_\rho g(\Phi_\mp(t,\rho))]\, d\rho\,|_{t=0} \\ &= g(\Phi_\mp(0,0)) - g(\Phi_\mp(0,\pi)) + 2\int_0^\pi g'(\Phi_\mp)\,\partial_t \Phi_\mp(t,\rho)\, d\rho\,|_{t=0}.\end{aligned}$$

The formulas above are exactly the condition we obtained in (11). ∎

From the above calculation we know that if the constraints are satisfied, there is a local C^2 solution. Since any Goursat data for a local C^2 solution satisfies the nonlinear constraints (10), and (11), therefore the constraints are preserved by the time evolution locally in time.

6. Global Existence of the Solutions

In this section we will study the global existence of the solutions to the Goursat problem of the nonlinear wave equation (6) in the space $\mathbf{H}(C)$.

THEOREM 5. *There is a global solution $\Phi(t)\in \mathbf{H}(C)$, $0\leqslant t<\infty$, to the Goursat problem of the wave equation* (6), *if g is continuously differentiable and*

$$|g(y_1)-g(y_2)|\leqslant c_1+c_2|y_1-y_2| \qquad \forall y_1, y_2\in R \tag{17}$$

or

$$G(y)\geqslant -c_2 \qquad \forall y\in R, \tag{18}$$

where c_1 and c_2 are positive constants, $G(y)=\int_0^y g(z)\,dz$

It is easy to see that $g(y)=y^r+P_{r-1}(y)$, where r is an odd integer, and P_{r-1} is a polynomial of degree less than or equal to $r-1$, satisfies the condition (18). Any bounded C^1 function or any function with bounded derivative satisfies the condition (17). In these cases we have global existence in $\mathbf{H}(C)$. In order to prove this theorem, we need to establish the following conservation of energy:

THEOREM 6. *If $\Phi(t)$ is a solution to the Goursat problem of the wave equation* (6) *in the space* $\mathbf{H}(C)$ *defined by* (2), *then*

$$I(\Phi(t))=\tfrac{1}{2}[E(t)]^2+\int_0^\pi G(\Phi(t,\rho))\,d\rho$$

is independent of t, where $E(t)$ is the free energy of $\Phi(t)$, and

$$\int_0^\pi G(\Phi(t,\rho))\,d\rho=\int_0^\pi [G(\Phi_+(t,\rho))+G(\Phi_-(t,\rho))]\,d\rho.$$

In fact, to prove Theorem 4, we only need to prove that $\|\Phi(t)\|_{\mathbf{H}(C)}$ will not blow up in a finite time, i.e., for any fixed $0<t_0<\infty$, $\|\Phi(t)\|_{\mathbf{H}(C)}$ cannot approach ∞ as t tends to t_0.

Case 1. If g satisfies the condition (17), we have, by using (8),

$$\begin{aligned}\|\Phi(t)\|_{\mathbf{H}(C)}&\leqslant(1+c\sqrt{t})\|\Phi(0)\|_{\mathbf{H}(C)}+c\,\|g(\Phi)\|_{L_2(D_t)}\\&\leqslant(1+c\sqrt{t})\|\Phi(0)\|_{\mathbf{H}(C)}+c+c\left[\int_0^t\|\Phi(\tau)\|^2_{\mathbf{H}(C)}\,d\tau\right]^{1/2}\end{aligned}$$

for $0\leqslant t\leqslant\pi$, which implies that $\|\Phi(t)\|_{\mathbf{H}(C)}$ cannot blow up in a finite time.

Case 2. If g satisfies the condition (18), we have, by using Theorem 6,

$$[E(t)]^2=2I(\Phi(0))-2\int_0^\pi G(\Phi(t,\rho))\,d\rho\leqslant 4c_2\pi+2I(\Phi(0)).$$

Noting that $\int_{-t}^0|\partial_\rho\phi(t,s)|^2\,ds\leqslant 2I(\Phi(0))+4c_2\pi$ by using the 1-form ε defined earlier, we have

$$\begin{aligned}c_0(t)=\phi(t,0)&=\int_{-t}^0\partial_\rho\phi(t,s)\,ds-\phi(t,-t)\\&\leqslant\sqrt{t}\left[\int_{-t}^0|\partial_\rho\phi(t,s)|^2\,ds\right]^{1/2}+c\,\|\Phi(0)\|_{\mathbf{H}(C)}\\&\leqslant 2\pi[2I(\Phi(0))+4c_2\pi]^{1/2}+c\,\|\Phi(0)\|_{\mathbf{H}(C)}\end{aligned}$$

for all $t\in[0,\pi]$. Combining two inequalities above gives

$$\|\Phi(t)\|_{\mathbf{H}(C)}\leqslant c(\|\Phi(0)\|_{\mathbf{H}(C)}+1)$$

for $0\leqslant t\leqslant\pi$, where c is a constant independent of $\|\Phi(0)\|_{\mathbf{H}(C)}$, which implies that $\|\Phi(t)\|_{\mathbf{H}(C)}$ cannot blow up in a finite time.

So the only thing left to do is prove Theorem 5. First let us recall that if $\Phi(0)\in\mathbf{H}(C)$ and $\Phi_\pm(0)$ is in $C^2[0,\pi]$ and satisfies nonlinear constraints (10) and (11), we know by Lemma 2 that $I(\Phi(t))=I(\Phi(0))$ for every t. Now for any $\Phi(0)\in\mathbf{H}(C)$, we can approximate $\Phi(0)$ by a sequence of $\{\Phi_{n_\pm}(0)\}\in\mathbf{H}(C)$, where $\Phi_{n_\pm}(0)$ are in $C^2[0,\pi]$ and satisfies constraints (10) and (11), i.e.,

$$\|\Phi_n(0)-\Phi(0)\|_{\mathbf{H}(C)}\to 0$$

Let $\Phi_n(t)$ be the solution on the Goursat problem of the nonlinear wave equation (6) with initial Goursat data $\Phi_n(0)$. Then by Theorem 3, we know that

$$\|\Phi_n(t)-\Phi(t))\|_{\mathbf{H}(C)}\to 0 \qquad as \quad n\to\infty$$

so we have

$$I(\Phi(t))-I(\Phi(0))=\lim_{n\to\infty}[I(\Phi_n(t))-I(\Phi_n(0))]=0,$$

for every t. Therefore we have the conservation of energy. ∎

It should be pointed out that if g satisfies (18) or (17), and $\phi^s\in\mathbf{H}(S)$, then there is a unique solution ϕ to the Cauchy problem for the wave equation (6) such that $\Phi(0)\oplus\partial_\tau\Phi(0)=\phi^s$, and

$$\|\Phi(\tau)\oplus\Phi(\tau)\|_{\mathbf{H}(S)}\leqslant f(\tau,\|\phi^s\|_{\mathbf{H}(S)}),$$

where $f(t,y)\colon R^2\to R$ maps bounded sets in R^2 to bounded sets in R. Moreover if $\phi^s\in C^2(S^1)\oplus C^1(S^1)$, then $\phi\in C^2(R\times S^1)$. These facts can be proved by using standard arguments in partial differential equations. The details are omitted here.

7. The Map from Cauchy Data to Goursat Data

For nonlinear or inhomogeneous wave equations on $R \times S^1$ with global regular solutions to the Cauchy problem there are maps $\Omega_\pm$ from Cauchy data on S_0 to Goursat data on the cones

$$C_\pm = \{\pm\tau = \pi - |\rho|\}.$$

As explained in [1], these maps can be construed as the wave transforms for wave equations on R^2, and if they are invertible the map $\Sigma = \Omega_+ \Omega_-^{-1}$ can be construed as the scattering transform.

Here we prove the continuity of the maps $\Omega_\pm$ and their inverses with respect to the norms defined by (1), (2) for the nonlinear equation

$$\Box\phi + g(\phi) = 0,$$

where g is a C^1 function satisfying (18) or (17). For convenience we deal below only with Ω_-, because C_- is one of our standard cones C_t, namely $C_{-\pi}$. The cone C_+ is the image of C_- under the isometry of $\tilde{M}$ given by $(\tau, \rho) \to (-\tau, \rho)$, which fixes S_0. Thus the theorem below extends to the case Ω_+ by symmetry considerations.

Theorem 7. *Let $g: R \to R$ be a C^1 function satisfing (18) or (17). Let $\phi^s \in C^2(S^1) \oplus C^1(S^1)$, and let ϕ be the global C^2 solution of Eq. (6) with $\Phi(0) \oplus \Phi'(0) = \phi^s$. Let $\mathbf{\Phi} = \phi|_{C_-}$. Then the map $\Omega_0: C^2(S^1) \oplus C^1(S^1) \to \mathbf{H}(C)$ given by $\Omega_0(\phi^s) = \mathbf{\Phi}$ extends uniquely to a continuous map $\Omega_-: \mathbf{H}(S) \to \mathbf{H}(C)$. Moreover, Ω_- is boundedly Lipschitzian.*

Proof. Since $g \in C^1(R)$ and $\|\mathbf{\Phi}\|_{L_\infty} \leqslant c\,\|\mathbf{\Phi}\|_{\mathbf{H}(C)}$, $\|\Phi\|_{L_\infty(S)} \leqslant \|\Phi\|_{2,1(C)}$, we know that g is boundedly Lipschitzian from $L_{2,1}(S)$ to $L_2(S)$, and from $\mathbf{H}(C)$ to $L_2(C)$. Suppose that $\phi_i^s \in C^2(S) \oplus C^1(S)$, where $i = 1, 2$. Let ϕ_i be the corresponding C^2 solution of (6) on $\tilde{M}$. Let $\psi = \phi_1 - \phi_2$, $\psi^s = \phi_1^s - \phi_2^s$, and $\mathbf{\Psi} = (\phi_1 - \phi_2)|_{C_-}$. Then by the boundedly Lipschitzian character of the nonlinear time evolution in $\mathbf{H}(S)$, for all $\tau \in [-\pi, 0]$ we have

$$\|\mathbf{\Psi}(\tau)\|_{L_{2,1}(S)},\ \|\partial_\tau \mathbf{\Psi}(\tau)\|_{L_2(S)} \leqslant h(\|\phi_1^s\|_{\mathbf{H}(S)}, \|\phi_2^s\|_{\mathbf{H}(S)}) \|\psi_s\|_{\mathbf{H}(S)}, \tag{19}$$

where $h: R^2 \to R$ is bounded on bounded sets.

Define the 1-form ε on $\tilde{M}$ by

$$\varepsilon = \tfrac{1}{2}[(\partial_\rho \phi)^2 + \partial_\tau \phi)^2]\, d\rho + \partial\rho\, \phi \partial_\tau \phi\, d\tau.$$

As in the proof of Lemma 2,

$$\int_{S_\tau} \varepsilon = \tfrac{1}{2}[\|\psi^s\|^2_{\mathbf{H}(S)} - \|\mathbf{\Psi}\|^2_{L_2(S)}]$$

and

$$\int_C \varepsilon = \tfrac{1}{2}[\|\mathbf{\Psi}\|_{\mathbf{H}(C)} - |\psi(-\pi, 0)|]^2.$$

It is easy to check that

$$d\varepsilon = \partial_\tau \psi\, \Box\psi\, d\tau \wedge d\rho = \partial_\tau \psi [g(\phi_2) - g(\phi_1)]\, d\tau \wedge d\rho,$$

and if R is a region of $\tilde{M}$ bounded by $\{\tau = 0\}$ and C_-, we have

$$2\int_R d\varepsilon = [\|\psi^s\|^2_{\mathbf{H}(S)} - \|\mathbf{\Psi}\|^2_{L_2(S)}] - [\|\mathbf{\Psi}\|_{\mathbf{H}(C)} - |\psi(-\pi, 0)|]^2$$

$$= 2\int_R \partial_\tau \psi (g(\phi_2) - g(\phi_1))\, d\tau \wedge d\rho.$$

Thus

$$[\|\mathbf{\Psi}\|_{\mathbf{H}(C)} - |\psi(-\pi, 0)|]^2$$

$$\leqslant \|\psi^s\|^2_{\mathbf{H}(S)} + 2\int_{[-\pi, 0] \times S^1} |\partial_\tau \psi (g(\phi_2) - g(\phi_1))|\, d\tau \wedge d\rho$$

and by (19), the latter is less than or equal to

$$2\,\|\partial_\tau \psi\|_{L_2([-\pi,0] \times S^1)} \|g(\phi_2) - g(\phi_1)\|_{L_2([-\pi,0] \times S^1)} \leqslant k(\|\phi_1^s\|_{\mathbf{H}(S)}, \|\phi_2^s\|_{\mathbf{H}(S)}) \|\psi^s\|^2_{\mathbf{H}(S)}$$

for some function $k: R^2 \to R$ that is bounded on bounded sets. Thus we have

$$\|\mathbf{\Psi}\|_{\mathbf{H}(C)} = [\|\mathbf{\Psi}\|_{\mathbf{H}(C)} - |\psi(-\pi, 0)|] + |\psi(-\pi, 0)|$$
$$\leqslant (k(\|\phi_1^s\|_{\mathbf{H}(S)}, \|\phi_2^s\|_{\mathbf{H}(S)}) + 1)^{1/2} \|\psi^s\|_{\mathbf{H}(S)} + c\,\|\Psi(-\pi)\|_{L_{2,1}(S)}$$
$$\leqslant [(ch + k)(\|\phi_1^s\|_{\mathbf{H}(S)}, \|\phi_2^s\|_{\mathbf{H}(S)}) + 1]^{1/2} \|\psi^s\|_{\mathbf{H}(S)}.$$

Hence the map Ω_0 is boundedly Lipschitzian with respect to norms defined by (1) and (2). Since $C^2(S^1) \oplus C^1(S^1)$ is dense in $\mathbf{H}(S)$, this implies that Ω_0 extends uniquely to a continuous map $\Omega_-: \mathbf{H}(S) \to \mathbf{H}(C)$, which is also bounded Lipschitzian. ∎

If g satisfies (18) or (17), we proved global existence for the Goursat problem in the previous section. It is easy to prove, using the estimate in (8), that

$$\Omega_-^{-1}: \mathbf{H}(C) \to \mathbf{H}(S)$$

exists and is boundedly Lipschitzian. We omit the details.

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