# A categorification of Hecke algebras

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Joint work with J. Baez and J. Dolan

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- Given a finite group $G$, for any pair of finite $G$-sets $A$ and $B$, we have a groupoid

$$\hom(A, B) = (A \times B)//G$$

- Degroupoidifying this, we get the vector space of intertwiners from the permutation representation corresponding to $A$ to the permutation representation corresponding to $B$.
- When A = B, the vector space of intertwiners is an algebra.
- The multiplication in this algebra can be groupoidified. I.e., there's a span of groupoids which acts like a "multiplication".
- We use this to groupoidify Hecke algebras. Hecke algebras are $q$-deformed versions of the group algebras of symmetric groups.

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# The action groupoid

Given a $G$-set $S$, i.e. a set with an action of $G$, we can form the **action groupoid** $S//G$ with:

- Objects: elements $s \in S$;
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# Hecke algebras

Let $D$ be a Dynkin diagram. Denote the set of vertices of $D$ by $S$ and an edge between $s$ and $t$ in $S$ by $st$. We denote the label on $st$ by $m_{st}$.

**Definition**

Let $D$ be a Dynkin diagram and $q$ a nonzero complex number. The **Hecke algebra** corresponding to this data is the associative $\mathbb{Z}[q, q^{-1}]$-algebra with generators $\sigma_s$, for each $s \in S$, and relations:

$$\sigma_s \sigma_t \sigma_s \ldots = \sigma_t \sigma_s \sigma_t \ldots$$

where each side has $m_{st}$ factors, and

$$\sigma_s^2 = (q-1)\sigma_s + q$$

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# Example: The groupoidified Hecke algebra

Let $D$ be a Dynkin diagram and $q$ a prime power.
Then there is a corresponding algebraic group $G$ over $\mathbb{F}_q$.
$G$ has Borel subgroup $B \subset G$ and we can form a finite $G$-set

$$X = G/B,$$

a flag variety.
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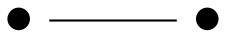
**Theorem**

*Degroupoidifying $H(D, q)$ yields the Hecke algebra associated to the Dynkin diagram $D$ with parameter $q$.*

# Example of the example

Consider the Dynkin diagram $A_2$:



We fix a prime power $q$. We have $G = SL(3, \mathbb{F}_q)$ and $B$ is the upper triangular matrices.
$X = G/B$ is the set of complete flags in $\mathbb{F}_q^3$, i.e.,

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$$dim V_i = i.$$

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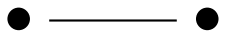

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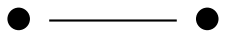

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# Projective perspective

In the projective space $\mathbb{F}_q P^2$, the flags are just a chosen point lying on a chosen line.

The vertices of our Dynkin diagram represent "figures" and the edges represent "incidence relations".



*point* —— *line*

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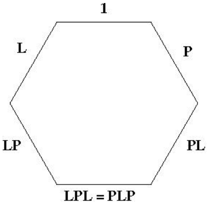
# The Coxeter group

The Coxeter group of $A_2$ has two generators:




The elements of this group correspond to the possible incidence relations between pairs of flags.
The multiplication in our groupoidified Hecke algebra will be a deformed version of this multiplication.

$$P^2 = (q-1)P + q1 \qquad L^2 = (q-1)L + q1$$

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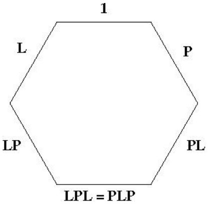


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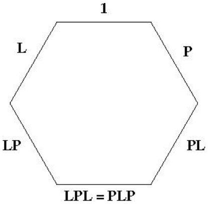


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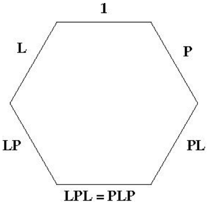


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where

$$P = \{((p,l),(p',l)) \mid p \neq p'\}$$

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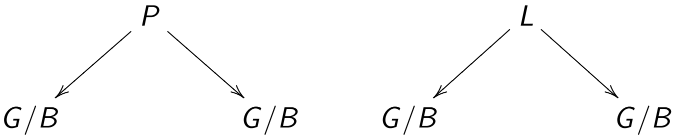


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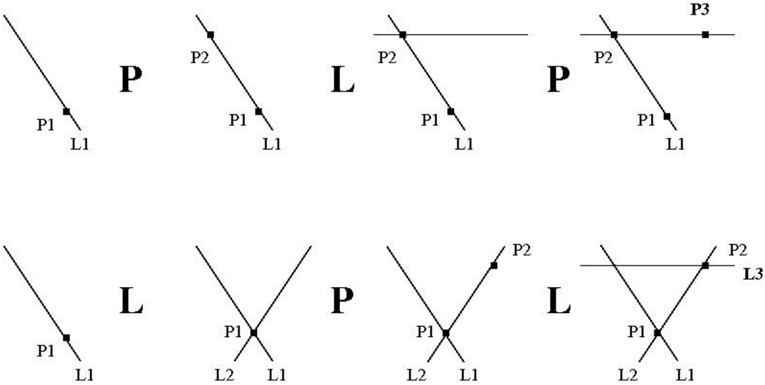
P1
L1
P
P2
P1
L1
L
P2
P1
L1
P
P3
P2
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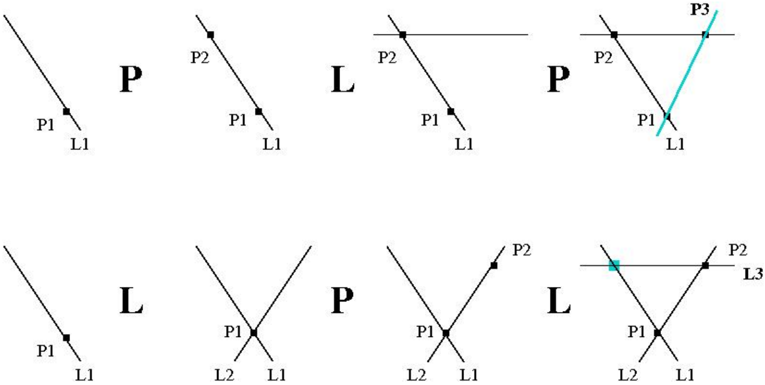
P1
L1
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# Groupoidified Hecke relations

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