A Categorification of Hall Algebras

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Many algebraic structures have been successfully categorified. However, quantum groups have been particularly hard to deal with. We are attempting to solve this by a new categorification process.

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Groupoidification is an attempt to reverse this process. As with any categorification process, this "reverse" direction is not systematic.

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- We use groupoids because they give us a way of categorifying vector spaces over real numbers.

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- We use "spans" because they give us a way to describe the matrix of a linear operator.

Definition

Given groupoids X and Y, a **span** from X to Y is defined as

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

where S is another groupoid and $p : S \to X$ and $q : S \to Y$ are functors

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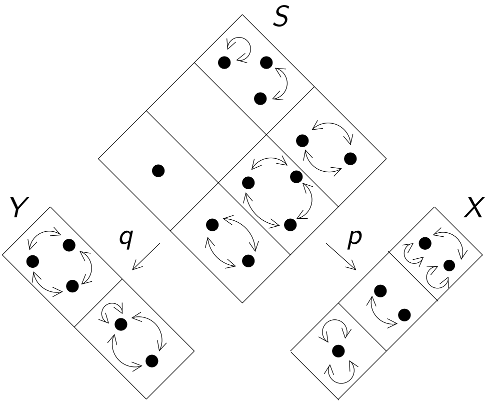
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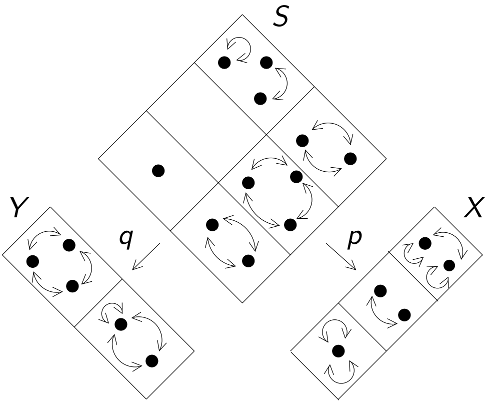

Formulaically we can describe the linear operator in terms of its matrix entries:

$$\underset{\sim}{S}_{[x][y]} = \sum_{[s]\in \underline{p^{-1}(x)} \bigcap \underline{q^{-1}(y)}} \frac{\#\mathrm{Aut}(y)}{\#\mathrm{Aut}(s)}$$

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When this sum converges, we call the groupoid **tame**.

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- The multiplication by $\#\mathrm{Aut}(x)$ is simply a choice of convention (specifically the one which is appropriate for Hall algebras).

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Given two functors $f\colon S \to Y$ and $g\colon T \to Y$, we define the **weak pullback** as:

$$\begin{tikzcd} & TS \arrow[dl, "\pi_T"'] \arrow[dr, "\pi_S"] & \\ T \arrow[dr, "g"'] & & S \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

where TS is the groupoid whose objects are triples (s, t, α) where $\alpha\colon f(s) \to g(t)$.

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With this we have another description of the linear operator obtained from a span.

Definition

Given a span of groupoids

$$\begin{tikzcd} & S \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y & & X \end{tikzcd}$$

the linear operator $\underset{\sim}{S} : \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{Y}]$ is given by $\underset{\sim}{S}\underset{\sim}{\Psi} = \underset{\sim}{S\Psi}$ where Ψ is a groupoid over X, $v \colon \Psi \to X$, and $S\Psi$ is the weak pullback:

$$\begin{tikzcd} & & S\Psi \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_\Psi"] & \\ & S \arrow[dl, "q"'] \arrow[dr, "p"] & & \Psi \arrow[dl, "v"'] \\ Y & & X & \end{tikzcd}$$

Considering the definition of weak pullback, we get the previously mentioned formula for the matrix entries of $\underset{\sim}{S}$.

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This can be rewritten using groupoid cardinality as follows:

$$\underset{\sim}{S}_{[x][y]} = \#\mathrm{Aut}(y)\ |(p \times q)^{-1}(x, y)|.$$

This version will be useful later with our example.

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- 1990 - Claude Ringel formalized the construction for certain abelian categories, and described the isomorphism of this algebra with (a piece of) a quantum group.
- The application of groupoidification to Hall algebras is very natural, since Hall algebras are constructed out of isomorphism classes of objects.

The Algebraic Construction

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$$V_1 \xrightarrow{f} V_2 \xleftarrow{g} V_3$$

- Define $Rep(Q)$ to be the category of finite dimensional representations of Q over a finite field $\mathbb{F}_q$.

From the category $Rep(Q)$, we construct the Hall algebra $\mathcal{H}(Q)$ as follows:

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- For three representations M, N, and E, define the set:

$$\mathcal{P}^E_{MN} = \{(f, g) \mid 0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0 \text{ is exact}\}$$

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- We then define an associative multiplication on $\mathcal{H}(Q)$ by:

$$u_M \cdot u_N = \sum_E \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(M)\#\mathrm{Aut}(N)} u_E$$

Isomorphism with Quantum Groups.

For any quiver Q, there is an underlying diagram given by ignoring the orientation of the edges. There is a quantum group associated to this diagram, namely $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra associated to the diagram. We will take a simple example to illustrate:

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Example - Consider the A_2 quiver:

$$\bullet \longrightarrow \bullet$$

The Lie algebra associated to A_2 is $\mathfrak{sl}(3)$. We can describe the quantum group $U_q(\mathfrak{sl}(3))$ in terms of generators and relations.

Generators: $\{E_i, F_i, K_i, K_i^{-1}\}$ for $i = 1, 2$. Relations:

$$K_i K_i^{-1} = 1$$

$$K_i E_i K_i^{-1} = q^2 E_i \qquad K_i F_i K_i^{-1} = q^{-2} F_i$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$E_i^2 E_j - (q+1) E_i E_j E_i + E_j E_i^2 = 0 \qquad i \neq j$$

$$F_i^2 F_j - (q+1) F_i F_j F_i + F_j F_i^2 = 0 \qquad i \neq j$$

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We are interested in the subalgebra $U_q(n^+)$ generated by $\{E_1, E_2\}$ and the two Serre relations between them.

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By Gabriel's Theorem, the category $Rep(Q)$ has a finite number of indecomposable representations, namely:

$$S_1 = \mathbb{F}_q \longrightarrow 0$$

$$S_2 = 0 \longrightarrow \mathbb{F}_q$$

$$X = \mathbb{F}_q \xrightarrow{1} \mathbb{F}_q$$

Note: the first two are also irreducible.

We will define a map from $U_q(n^+)$ to $\mathcal{H}(Q)$ by:

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We would then like to show that the Serre relations hold in $\mathcal{H}(Q)$.

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  - Since each vertex is a vector space, then dimension must be equal to the sum of the dimensions from the corresponding vertex of S_1 and S_2. this means E will be of the form $\mathbb{F}_q \xrightarrow{\alpha} \mathbb{F}_q$, where $\alpha \in \mathbb{F}_q$.

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    - We then check what values of α will make this an extension of S_1 by S_2.
- Next, we will calculate $\mathcal{P}^E_{S_1 S_2}$ for each E.

a few other results:

$$[S_1] \cdot [S_2] = [S_1 \oplus S_2] + [X]$$
$$[S_2] \cdot [S_1] = [S_1 \oplus S_2]$$
$$[S_i] \cdot [S_i] = (q+1)[S_i \oplus S_i]$$
$$[S_1] \cdot [X] = q[S_1 \oplus X]$$
$$[S_2] \cdot [X] = [S_2 \oplus X]$$
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Using these, we can produce the triple products.

$[S_1]^2 \cdot [S_2] = (q+1)[S_1 \oplus S_1 \oplus S_2] + (q+1)[S_1 \oplus X]$

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$[S_2] \cdot [S_1]^2 = (q+1)[S_1 \oplus S_1 \oplus S_2]$
Finally, we verify the Serre relation:

$$[S_1]^2 \cdot [S_2] - (q+1)[S_1] \cdot [S_2] \cdot [S_1] + [S_2] \cdot [S_1]^2 = 0$$

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- Next we construct a multiplication span. We start by defining a new groupoid $SES(Rep(Q))$:
  - Objects - Short exact sequences of objects in $Rep(Q)$.
  - Morphisms - Isomorphisms of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "\alpha"] & E \arrow[r, "g"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "f'"] & E' \arrow[r, "g'"] & M' \arrow[r] & 0 \end{tikzcd}$$

We define the multiplication span:

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\begin{tikzcd}
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This makes sense, because given a pair of representations (M, N) on the right, the span associates to it every short exact sequence with N as the subrep and M as the quotient. This is then projected down to representations E which appear as extensions of M by N.

We can then apply the degroupoidification to this span. Doing this, we get an operator:

$$m\colon \mathbb{R}[\underline{Rep(Q)_\circ}] \otimes \mathbb{R}[\underline{Rep(Q)_\circ}] \to \mathbb{R}[\underline{Rep(Q)_\circ}]$$

with

$$m(u_M \otimes u_N) = \sum_{E \in \mathcal{P}^E_{MN}} \#\mathrm{Aut}(E)\left|(p \times q)^{-1}(M,N,E)\right| u_E.$$

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We wish to show this matches the Hall algebra product $u_M \cdot u_N$.

For this, we must make a few observations.

- First, we note that the group $\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M)$ acts on the set $\mathcal{P}^E_{MN}$. This action is not necessarily free, but this is just the sort of situation groupoid cardinality is designed to handle.

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- First, we note that the group $\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M)$ acts on the set $\mathcal{P}^E_{MN}$. This action is not necessarily free, but this is just the sort of situation groupoid cardinality is designed to handle.
- Taking the weak quotient $\mathcal{P}^E_{MN}//(\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))$, we obtain a groupoid equivalent to one whose objects are short exact sequences of the form $0 \to N \to E \to M \to 0$ and morphisms are isomorphisms of short exact sequences (i.e. the subgroupoid $(p \times q)^{-1}(M, N, E)$ of $SES(Rep(Q))$).

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\begin{aligned}
|(p \times q)^{-1}(M, N, E)| &= |\mathcal{P}^{E}_{MN} // (\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))| \\
&= \frac{\#\mathcal{P}^{E}_{MN}}{\#\mathrm{Aut}(N)\,\#\mathrm{Aut}(E)\,\#\mathrm{Aut}(M)}
\end{aligned}
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We can then calculate.

$$\begin{aligned} |(p \times q)^{-1}(M, N, E)| &= |\mathcal{P}^E_{MN} /\!/ (\mathrm{Aut}(N) \times \mathrm{Aut}(E) \times \mathrm{Aut}(M))| \\ &= \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(N)\,\#\mathrm{Aut}(E)\,\#\mathrm{Aut}(M)} \end{aligned}$$

So, we obtain

$$m(u_M \otimes u_N) = \sum_{E \in \mathcal{P}^E_{MN}} \frac{\#\mathcal{P}^E_{MN}}{\#\mathrm{Aut}(M)\,\#\mathrm{Aut}(N)}\; u_E.$$

which is precisely the Hall algebra product $u_M \cdot u_N$.

A similar process can be applied to the adjoint span:

$$\begin{tikzcd} & SES(Rep(Q)) \arrow[dl, "\pi_M \times \pi_N"'] \arrow[dr, "\pi_E"] & \\ Rep(Q)_\circ \times Rep(Q)_\circ & & Rep(Q)_\circ \end{tikzcd}$$

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Unfortunately, these are not compatible, in the sense that they do not form a bialgebra. Instead, they give a twisted bialgebra. Algebraically we can describe this as a bialgebra in a braided monoidal category, and we have the beginning of an idea of how to describe this in terms of groupoids and spans.

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The End