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THE UBIQUITY OF COXETER-DYNKIN DIAGRAMS
(AN INTRODUCTION TO THE A-D-E PROBLEM)

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1. PREFACE AND APOLOGY

The problem of the ubiquity of the Dynkin-diagrams A_k, D_k, E_k was formulated by V.I. ARNOLD as problem VIII in [52] as follows.

The A-D-E *classifications*. The Coxeter-Dynkin graphs A_k, D_k, E_k appear in many independent classification theorems. For instance

(a) classification of the platonic solids (or finite orthogonal groups in euclidean 3-space),

(b) classification of the categories of linear spaces and maps (representations of quivers),

(c) classification of the singularities of algebraic hypersurfaces, with a definite intersection form of the neighboring smooth fibre,

(d) classification of the critical points of functions having no moduli,

(e) classification of the Coxeter groups generated by reflections, or, of Weyl groups with roots of equal length.

The problem is to find the common origin of all the A-D-E classification theorems and to substitute a priori proofs to a posteriori verifications of the parallelism of the classifications.

During the 13th Dutch Mathematical congress on April 6 and 7, 1977 in Rotterdam we organized a series of lectures designed to acquaint the participants with the problem mentioned above. More specifically we aimed to indicate how one obtains Coxeter-Dynkin diagrams in some of the various areas of mathematics listed in the problem. The text below is essentially a printed version of the talks given in this series of lectures with but little editing, and with only a few extra comments, mainly of a bibliographical nature. Thus the text below is an introduction to the problem stated above; it is far too incomplete to constitute a survey of the field and it does not contain new results. The oral lectures corresponding to sections 2, 3, 4 were given by F.D. Veldkamp, the material of section 5 was presented by W. Hesselink, that of section 6 by M. Hazewinkel and that of section 7 by D. Siersma. The final redaction of this text was done by M. Hazewinkel.

2. COXETER DIAGRAMS AND GROUPS OF REFLECTIONS

2.1. Coxeter diagrams

A Coxeter diagram is a graph will all its edges labelled by an element of $\{3,4,5,\ldots\} \cup \{\infty\}$. As a rule the label 3 is suppressed. Thus one has for example the Coxeter diagrams

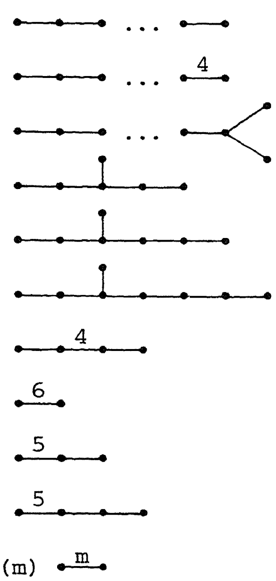
(a) (b) (c) , n-1 vertices (d)

2.2. Group associated to a Coxeter diagram

Let Γ be a Coxeter diagram. Let S be its set of vertices. For all $s,s' \in S$, $s \neq s'$, define $m(s,s') = 2$ if there is no edge connecting s and s', and $m(s,s')$ = label of edge connecting s and s', otherwise. We now associate to Γ the group $W(\Gamma)$ generated by the symbols $s \in S$ subject to the relations $(ss')^{m(s,s')} = 1$, $s^2 = 1$ for all $s,s' \in S$, $s \neq s'$. If Γ is the disconnected union of two subgraphs Γ_1 and Γ_2, then $W(\Gamma)$ is the direct product $W(\Gamma_1) \times W(\Gamma_2)$, because in this case $s_1s_2 = s_2s_1$ for all $s_1 \in \Gamma_1$, $s_2 \in \Gamma_2$.

2.3. EXAMPLES. If Γ is the graph (a) of 2.1 above then $W(\Gamma) = \mathbb{Z}/(2) \times \mathbb{Z}(2)$, the Klein fourgroup. If Γ is the graph (b) of 2.1 then $W(\Gamma)$ is the semidirect product $\mathbb{Z}/(2) \times_s \mathbb{Z}$, where $\mathbb{Z}/(2)$ acts on $\mathbb{Z}$ as $\sigma x = -x$, where σ is the generator of $\mathbb{Z}/(2)$; the isomorphism is induced by $s_1 \mapsto (\sigma,0)$, $s_2 \mapsto (\sigma,1)$. Similarly $W(\Gamma)$ is the dihedral group $\mathbb{Z}_2 \times_s \mathbb{Z}/(m)$ if Γ is the diagram (d) of 2.1. Finally if Γ is diagram (c) of 2.1 then $W(\Gamma) = S_n$, the permutation group on n letters. Here the isomorphism is induced by mapping the i-th vertex of Γ to the transposition $(i,i+1) \in S_n$. (Cf. [8], Ch.4, §1, exercise 4 or §2.4, example, for a proof.)

2.4. THEOREM. *Let Γ be a connected Coxeter diagram. Then $W(\Gamma)$ is finite if and only if Γ is one of the following Coxeter diagrams:*



A_ℓ *($\ell \geq 1$ vertices)*

B_ℓ *($\ell \geq 2$ vertices)*

D_ℓ *($\ell \geq 4$ vertices)*

E_6

E_7

E_8

F_4

G_2

H_3

H_4

$I_2(m)$ *($m = 5$ or $7 \leq m < \infty$)*.

2.5. Bilinear form associated to Γ

Let Γ be a Coxeter diagram with vertex set S. For each $s,s' \in S$, let $b_{s,s'}$ be the real number $b_{s,s'} = -\cos(m(s,s')^{-1}\pi)$, where we take $m(s,s') = 1$ if $s = s'$. Let E be the direct sum vector space $E = \mathbb{R}^{(S)}$ and let B_Γ be the symmetric bilinear form on E defined by the matrix $(b_{s,s'})$.

2.6. THEOREM. *The group* $W(\Gamma)$ *is finite if and only if* B_Γ *is positive nondegenerate.*

For a proof cf. [8], Ch.V, §4.8. Given this theorem (whose proof uses the realization of $W(\Gamma)$ as a group of reflections which will be discussed below), theorem 2.4 follows readily (cf. [8], Ch.VI, §4, théorème 1). E.g. $B_{\underset{n}{\bullet\!-\!\bullet\!-\!\bullet}}$ is positive definite iff $n \leq 5$.

2.7. Realization of $W(\Gamma)$

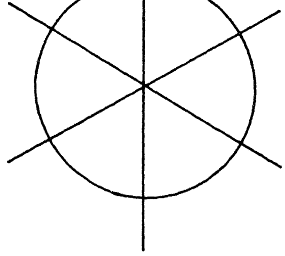
Let Γ, S, E be as in 2.5 above. Let $GL(E)$ be the group of real vector space automorphisms of E. To each $s \in S$ we associate the reflection

$$\sigma_s(x) = x - 2B_\Gamma(e_s, x)e_s,$$

where e_s is the canonical basis vector in $E = \mathbb{R}^{(S)}$ corresponding to $s \in S$.

This induces an injective embedding $W(\Gamma) \to GL(E)$, and, incidentally shows that the map $i\colon S \ni s \mapsto$ generator of $W(\Gamma)$ corresponding to s, is injective; the pair $(W(\Gamma), i(S))$ is a Coxeter system in the sense of [8], Ch.IV, §1. Cf. [8], Ch.V, §4 for all this.

Let Γ be one of the Coxeter diagrams listed in theorem 2.4. The reflecting hyperplanes of the σ_s then cut up $\mathbb{R}^\ell$ into connected pieces, the chambers. Taking the intersection of these with the unit sphere $S^{\ell-1} \subset \mathbb{R}$ we find a partition of $S^{\ell-1}$ into spherical simplices. In the case of dihedral group belonging to $I_2(3) = A_2$ the picture is ($\ell = 2$).



2.8. The crystallographic condition

Let $W(\Gamma)$ be realized as a group of reflections as in 2.7 above. Then the crystallographic condition says that there is a lattice $\mathbb{Z}^\ell \subset \mathbb{R}^\ell$ which is invariant under $W(\Gamma)$. The groups of type A,B,D,E, F,G of the list in theorem 2.4 satisfy this condition, but the groups of type H and $I_2(m)$, $m = 5$, or $m \geq 7$ do not satisfy this condition. This condition has, of course, to do with the crystallographic symmetry groups (BRAVAIS, MÖBIUS, HESSEL, 1830-1840; cf. [19], 9.3 and 4.2).

2.9. Notational remark

Instead of $\bullet\overset{4}{-}\bullet$ in a Coxeter diagram one also writes $\bullet\!\Rightarrow\!\bullet$ and instead of $\bullet\overset{6}{-}\bullet$ one also uses $\bullet\!\Rrightarrow\!\bullet$. Thus $\bullet-\bullet\Rightarrow\bullet-\bullet$ is an alternative version of F_4.

3. LIE GROUPS, LIE ALGEBRAS AND DYNKIN-DIAGRAMS

3.1. Lie algebras

Let k be a field, e.g. $k = \mathbb{R}, \mathbb{C}$. A finite dimensional Lie algebra over k is a finite dimensional vector space L over k equipped with a bilinear multiplication $L \times L \to L$, $(x,y) \longmapsto [x,y]$, such that $[x,x] = 0$ and $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x,y,z \in L$. Then, of course, also $[x,y] = -[y,x]$ for all $x,y \in L$. An *ideal* $\underline{a} \subset L$ is a subvectorspace such that $[x,y] \in \underline{a}$ for all $x \in L$, $y \in \underline{a}$; a *subalgebra* of L is a subvectorspace $\underline{h}$ such that $[x,y] \in \underline{h}$ for all $x,y \in \underline{h}$. A Lie algebra L is called *abelian* if $[x,y] = 0$ for all $x,y \in L$. (Then every subvectorspace is an ideal.)

A Lie algebra L is *simple* if it is not abelian and if L and $\{0\}$ are the only ideals of L. If $\underline{a}$ is an ideal in a Lie algebra L then $\underline{a}$ is also a Lie algebra and $L/\underline{a}$ inherits a Lie algebra structure from L. Thus the simple Lie algebras appear as the natural building blocks for all Lie algebras. Below we shall outline the classification of the simple Lie algebras over $\mathbb{C}$, cf. 3.3 for the result.

One of the main reasons for the importance of Lie algebras in mathematics and physics is their intimate connection with Lie groups, cf. 3.13 below. A basis of the Lie algebra L(G) of a Lie group G is, in physicists terms, a set of infinitesimal generators for G.

3.2. EXAMPLE. Let $g\ell_n(k)$ be the vector space of all n×n matrices over k. We define a bracket multiplication on $gl_n(k)$ by $[X,Y] = XY - YX$. This makes $g\ell_n(k)$ a Lie algebra. Let $s\ell_n(k)$ be the subvector space of all matrices $X \in g\ell_n(k)$ with trace $(X) = 0$. Then $s\ell_n(k)$ is an ideal in $g\ell_n(k)$. The quotient is the abelian Lie algebra of dimension 1. Let $\underline{h}$ be the subvectorspace of $s\ell_n(k)$ consisting of all diagonal matrices $\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 + \ldots + \lambda_n = 0$. Then $\underline{h}$ is an abelian subalgebra of $s\ell_n(k)$ of dimension $n-1$; $\underline{h}$ is not an ideal of $s\ell_n(k)$ if $n \geq 2$.

3.3. List of simple complex Lie algebras

There are four big families A_n, $n \geq 1$; B_n, $n \geq 2$; C_n, $n \geq 3$; D_n, $n \geq 4$ and 5 exceptional simple complex Lie algebras E_6, E_7, E_8, F_4, G_2. The A_n, B_n, C_n, D_n are easily defined, e.g. $A_n = s\ell_{n+1}(\mathbb{C})$; cf. [40], section 2.7, for the remaining ones.

As we shall see it is no coincidence that we here encounter similar labels as in theorem 2.4 above. For the Dynkin diagrams $A_n,\ldots,G_2$ cf. 3.12 below.

3.4. Real simple Lie algebras

Let L be a Lie algebra over $\mathbb{R}$. Then by extension of scalars one finds a natural complex Lie algebra structure on $L_{\mathbb{C}} = L \otimes_{\mathbb{R}} \mathbb{C}$. If now L is a complex Lie algebra then any real Lie algebra L_0 such that L is isomorphic over $\mathbb{C}$ to $L_0 \otimes \mathbb{C}$ is called a *real form* of L. Every simple complex Lie algebra has several nonisomorphic real forms (cf. [31], Ch.III, §6), and these real forms have been classified by E. CARTAN ([18]; cf. also e.g. [1]).

3.5. We now want to indicate how one associates a Dynkin diagram (a class of objects closely related to Coxeter diagrams) to a simple Lie algebra over $\mathbb{C}$. This association proceeds in two steps: (i) to a simple Lie algebra there corresponds a root system, (ii) a root system gives rise to a Dynkin diagram. We now first describe in sections 3.6 - 3.11 how root systems translate into Dynkin diagrams. Step (i) above is the subject of 3.12 below.

3.6. Abstract root systems

Let V be a finite dimensional vector space over a field k of characteristic zero. A *root system* $R \subset V$ is a subset R of V such that

(i) R is finite, $0 \notin R$, and R generates V as a vector space over k;

(ii) for every $\alpha \in R$, there exists an element $\alpha^* \in V^*$, the dual space of V, such that $\alpha^*(\alpha) = 2$ and such that the reflection $s_\alpha(x) = x - \alpha^*(x)\alpha$ maps R into R;

(iii) if $\alpha, \beta \in R$, then $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The reflection s_α, whose existence is required by condition (ii) is necessarily unique, thus (iii) makes sense (cf. [40], Ch.V, §1).

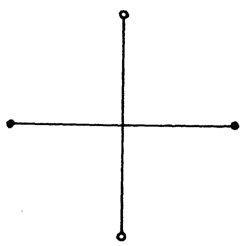
In the following we shall take $k = \mathbb{R}$ or $\mathbb{C}$. It does not matter much which we take. If $R \subset V$ is a complex root system, then

$$R \subset \sum_{\alpha \in R} \alpha\mathbb{R} \subset V$$

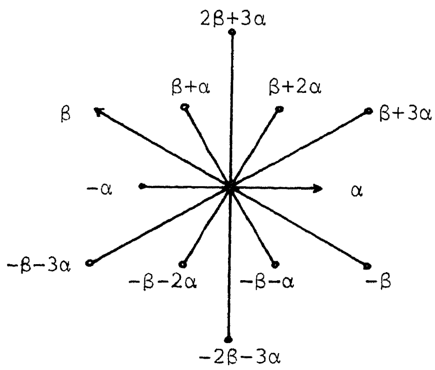
is a real root system in $\sum \alpha\mathbb{R}$ and this sets up bijective correspondence between real and complex root systems. Cf. also [40], Ch.VI, §1, prop.1.

The root system $R \subset V$ is called *reduced* if for all $\alpha \in R$ the only roots proportional to α are α and $-\alpha$. The *rank* of a root system $R \subset V$ is the dimension of V. Two root systems $R \subset V$ and $R' \subset V'$ are isomorphic if there exists an isomorphism $\phi: V \to V'$ of vector spaces such that $\phi(R) = R'$.

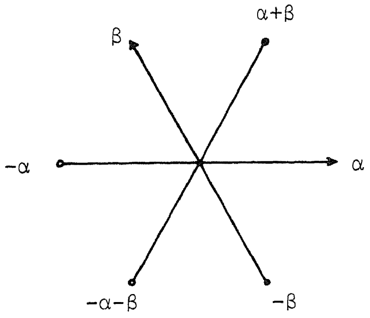
3.7. EXAMPLES. The reduced root systems of rank 2 are



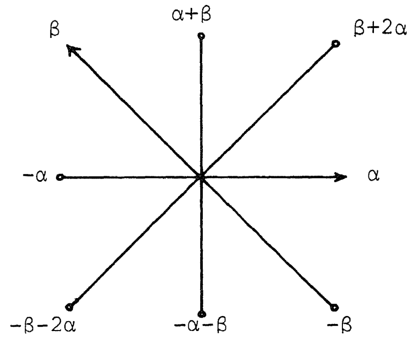
$(A_1 \times A_1)$



(G_2)



(A_2)



(B_2)

3.8. Weyl group and Coxeter system of a root system

Let $R \subset V$ be a (real) root system. The Weyl group $W(R)$ is then defined as the subgroup of $GL(V)$ generated by the reflections s_α, $\alpha \in R$. Because R generates V, s_α is uniquely determined by its action on R, and because R is finite this means that $W(R)$ is a finite group.

EXAMPLES. $W(A_1 \times A_1) = \mathbb{Z}/(2) \times \mathbb{Z}/(2)$, $W(A_2) = S_3$, the permutation group on 3 letters.

Let $R \subset V$ be a root system. A basis for R (or a simple set of roots) is a subset $S \subset R$ which is a basis for V and which is such that every $\alpha \in R$ can be uniquely written in the form $\alpha = \sum m_i \alpha_i$, $m_i \in \mathbb{Z}$, $\alpha_i \in S$ with either $m_i \geq 0$ for all i (positive roots) or $m_i \leq 0$ for all i (negative roots). It is now a theorem that every root system has a basis ([40], Ch.V, §8). Let S be a basis for R and let $S' \subset W(R)$ be the set of reflexions $\{s_\alpha \mid \alpha \in S\} \subset W(R)$. Then $(W(R), S')$ is a Coxeter system in the sense of 2.7 above ([8], Ch.VI, §1.5, théorème 2).

3.9. Invariant metric

Let $R \subset V$ be a real root system. There is a symmetric positive definite bilinear form $(\ ,\ )$ on V which is invariant with respect to $W(R)$. This follows simply from the fact that $W(R)$ is finite; indeed, if (,)' is any positive definite symmetric form on V then

$$(x,y) = \sum_{w \in W(R)} (wx,wy)'$$

works. In terms of (,) the coefficient $\alpha^*(x)$ appearing in the reflection s_α is equal to $\alpha^*(x) = (\alpha,\alpha)^{-1} s(\alpha,x)$.

With respect to this metric W(R) acts as a finite group of orthogonal transformations. The invariant bilinear form (,) is by no means unique. For each $\alpha,\beta \in R$, let $n(\alpha,\beta) = (\beta^*,\alpha) = 2(\beta,\beta)^{-1}(\beta,\alpha)$. If ϕ is the angle between α and β (with respect to the invariant metric discussed above) then $4\cos^2\phi = n(\beta,\alpha)n(\alpha,\beta)$. Now $n(\alpha,\beta)$ is an integer by condition (iii) of the definition of a root system. Hence $4\cos^2\phi = 0,1,2,3,4$ which severely limites the possible values for ϕ and $n(\alpha,\beta)$, $n(\beta,\alpha)$. In fact there are only seven possibilities (for α and β non-proportional, $|\alpha| \le |\beta|$). They are:

(i) $n(\alpha,\beta) = 0$, $n(\beta,\alpha) = 0$, $\phi = 2^{-1}\pi$,
(ii) $n(\alpha,\beta) = 1$, $n(\beta,\alpha) = 1$, $\phi = 3^{-1}\pi$, $|\alpha| = |\beta|$
(iii) $n(\alpha,\beta) = -1$, $n(\beta,\alpha) = -1$, $\phi = 3^{-1}2\pi$, $|\alpha| = |\beta|$
(iv) $n(\alpha,\beta) = 1$, $n(\beta,\alpha) = 2$, $\phi = 4^{-1}\pi$, $|\beta| = \sqrt{2}|\alpha|$
(v) $n(\alpha,\beta) = -1$, $n(\beta,\alpha) = -2$, $\phi = 4^{-1}3\pi$, $|\beta| = \sqrt{2}|\alpha|$
(vi) $n(\alpha,\beta) = 1$, $n(\beta,\alpha) = 3$, $\phi = 6^{-1}\pi$, $|\beta| = \sqrt{3}|\alpha|$
(vii) $n(\alpha,\beta) = -1$, $n(\beta,\alpha) = -3$, $\phi = 6^{-1}5\pi$, $|\beta| = \sqrt{3}|\alpha|$.

3.10. Cartan matrix and Dynkin diagram of a root system

Let $S \subset R$ be a basis for the reduced root system $R \subset V$. The *Cartan matrix* (with respect to S) is the matrix $(n(\alpha,\beta))_{\alpha,\beta \in S}$. One now has the proposition that a reduced root system is determined (up to isomorphism) by its Cartan matrix ([40], Ch.V, prop.8,8' or [8], Ch.VI, §1, Prop.15, Cor.). Also if both α,β are part of a basis of R only possibilities (i), (iii), (v), (vii) of the list in 3.8 above are possible; cf. [8], Ch.VI, §1, théorême 1.

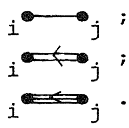
We now assign a Dynkin diagram to the root system $R \subset V$ as follows: the vertices correspond to the element of a basis $S \subset R$. Two vertices $i,j \in S$ are joined according to the following recipe:

(i) if $n(i,j) = n(j,i) = 0$ then i and j are not joined;

(ii) if $n(i,j) = n(j,i) = -1$

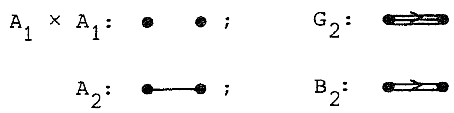
(iii) if $2n(i,j) = n(j,i) = -2$

(iv) if $3n(i,j) = n(j,i) = -3$



This exhausts all possibilities. And we also see that the Dynkin diagram of $R \subset V$ (relative to S) determines the Cartan matrix of $R \subset V$ (relative to S) and hence R itself according to the theorem quoted above.

3.11. EXAMPLES. The Dynkin diagrams of the reduced root systems of example 3.7 above are respectively



3.12. The root system of a simple Lie algebra over $\mathbb{C}$

We now proceed to indicate how one obtains the classification theorem 3.3, i.e., given 3.6 - 3.11 above, how one constructs a root system from a (semi) simple Lie algebra over $\mathbb{C}$. We shall outline the general theory and treat a specific example (viz. $s\ell_n(\mathbb{C})$) in two parallel columns. In the following L is some fixed simple Lie-algebra over $\mathbb{C}$, and in example of course $L = s\ell_n(\mathbb{C})$.

(i) *Cartan subalgebra*

Let $x \in L$, then ad x: $L \to L$, $y \longmapsto [x,y]$ is a linear endomorphism of L. We say that x is semisimple if ad x is diagonalizable. A *Cartan* subalgebra of L is maximal abelian subalgebra with the additional property that all its elements are semisimple in L. Cartan subalgebras $\underline{h}$ always exist.

The subalgebra $\underline{h}$ of $s\ell_n(\mathbb{C})$ consisting of all diagonal matrices (of trace zero) is a Cartan subalgebra of $s\ell_n(\mathbb{C})$. The dimension of $\underline{h}$ is $n - 1$.

(ii) *Roots and root vectors*

Let $\alpha \in \underline{h}^*$, the complex linear dual of $\underline{h}$. We define $L^\alpha = \{x \in L \mid [h,x] =$

Let $\omega_i(\mathrm{diag}(\lambda_1,\dots,\lambda_n)) = \lambda_i$. Then $\omega_i - \omega_j$: $\underline{h} \to \mathbb{C}$ is a root

$= \alpha(h)x$ for all $h \in \underline{h}\}$. Then α is called a root (of L with respect to $\underline{h}$) if $\alpha \neq 0$ and $L^{\alpha} \neq 0$. One then has that $\dim L_{\alpha} = 1$ for all roots α and if Σ is the set of all roots then $L = \underline{h} \otimes \oplus_{\alpha\in\Sigma} L\alpha$ as a vector space.

if $i \neq j$. A nonzero element of $L^{\omega_i - \omega_j}$ is E_{ij} the matrix with zero entries everywhere except a 1 at spot (i,j).

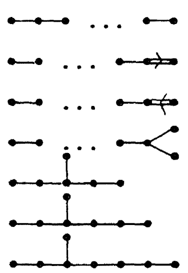
(iii) *Root system and basis*

Σ is a reduced root system in $\underline{h}^*$ ([8], Ch.VI, §1, théorème 2) and hence has a basis. Moreover Σ is *irreducible* which means that there is no nontrivial decomposition $R = R_1 \cup R_2$ with $R_1 \subset V_1$, $R_2 \subset V_2$ root systems, $V = V_1 \times V_2$. This root system determines L up to isomorphism ([31], Ch.III, §5, theorem 5.4; [8], Ch.VI, §5, théorème 8).

$V = \underline{h}^* = \{\sum_{i=1}^{n} \xi_i\omega_i \mid \sum \xi_i = 0\}$. The reflection s_α associated to $\omega_i - \omega_j$ interchanges ω_i and ω_j and leaves all other ω_k invariant. Hence $(\omega_i - \omega_j)^* \in \underline{h}$ is $\mathrm{diag}(0,\dots,0,1,0,\dots,0,-1,0,\dots,0)$ with the 1 in spot i and -1 in spot j. A basis (or set of simple roots) is e.g. $\alpha_1 = \omega_1 - \omega_2$, $\alpha_2 = \omega_2 - \omega_3, \dots, \alpha_{n-1} = \omega_{n-1} - \omega_n$.

(iv) *Dynkin diagram*

Now construct the Dynkin diagram of the root system Σ (cf. 3.10 above). This Dynkin diagram is connected because Σ is irreducible. The Dynkin diagrams which arise in this way are



A_n (n vertices, n≥1)
B_n (n vertices, n≥2)
C_n (n vertices, n≥3)
D_n (n vertices, n≥4)
E_6
E_7
E_8

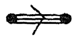
We find $\langle\alpha_i^*, \alpha_j\rangle = 0$ if $i < j-1$ or $i > j+1$, $\langle\alpha_i^*, \alpha_i\rangle = 2$, $\langle\alpha_{i-1}^*, \alpha_i\rangle = \langle\alpha_{i+1}^*, \alpha_i\rangle = -1$. It follows that the Dynkin diagram of $s\ell_n(\mathbb{C})$ is

A_{n-1}

((n-1) vertices)

The Weyl group of $s\ell_n(\mathbb{C})$ is S_n.

F_4 

G_2 

By removing the arrows one finds the Coxeter diagram of the Weyl group W(R) of L.

3.13. On the connections between Lie groups and Lie algebras

Some, first presumably largely superfluous, preliminaries on analytic manifolds. Let $k = \mathbb{R}$ or $\mathbb{C}$. An *analytic manifold* of dimension n over k is a Hausdorff topological space M together with an open covering $\mathcal{U} = \{U_i \mid i \in I\}$ and homeomorphisms $\phi_i\colon\ U_i \to \phi(U_i) \subset k^n$ onto some open subset of k^n, such that for all $i,j \in I$

$$\phi_j\phi_i^{-1}\colon\ \phi_i(U_i \cap U_j) \to U_i \cap U_j \to \phi_j(U_i \cap U_j)$$

is an analytic mapping. A function $f\colon U \to k$ in M is *analytic* if $f\phi_i^{-1}\colon\ \phi_i(U \cap U_i) \to U \cap U_i \to k$ is analytic for all i. Let $F_M(U)$ be the ring of analytic functions on U. A mapping $\phi\colon M \to N$ between analytic manifolds M and N is *analytic* if for all open $V \subset N$ and analytic functions $f \in F_N(V)$ the function $f\phi$ on $f^{-1}(V) \subset M$ is analytic.

Let $p \in M$. We define $F_M(p)$, the k-algebra of germs of analytic functions in p, as the set of equivalence classes of analytic functions $f\colon U \to k$ defined on some neighbourhood U of x, under the equivalence relation $f\colon U \to k \sim g\colon V \to k$ iff there is a neighbourhood $W \subset U \cap V$ of x on which f and g agree. A *tangent vector* to M at p is a k-linear mapping $t\colon F_M(p) \to k$ such that $t(fg) = (tf)g(p) + f(p)(tg)$. There is an obvious k-vector space structure on M_p, the set of tangent vectors to M at p, and $\dim(M_p) = n$. An *analytic (tangent) vector field* X on an open subset $Y \subset M$ is a collection of derivations $X_U\colon F_M(U) \to F_M(U)$, one for each open $U \subset Y$, such that $r_{U,V} \circ X_U = X_V \circ r_{U,V}$ for all open $V \subset U$. Here $r_{U,V}\colon F_M(U) \to F_M(V)$ is restriction. Given a vector field X on $U \subset M$ and a point $p \in U$ one defines a tangent vector $X_p \in M_p$ by $X_p(f) = (Xf)(p)$.

If $\phi\colon M \to N$ is analytic and $t \in M_p$ then $(d\phi)_p(t)(g) = t(g\phi)$ defines a tangent vector $(d\phi)_p(t) \in N_{\phi(p)}$, giving us a k-linear mapping $(d\phi)_p\colon M_p \to N_{\phi(p)}$.

A *Lie group* is now an analytic manifold G which is equipped with analytic mappings "product": $G \times G \to G$ and "inverse": $G \to G$ and an element $e \in G$ which make G a group. Example: $G = GL_n(\mathbb{C})$, the group of invertible n×n matrices over $\mathbb{C}$. (Here the covering $\mathcal{U}$ defining the analytic structure has just one element.) Other examples are the orthogonal groups, symplectic groups, unitary groups, special linear groups, projective linear groups,... .

Let G be a Lie group, let $y \in G$ then $\lambda_y: G \to G$, $x \mapsto yx$ is an analytic mapping. A vector field X on G is said to be *left invariant* if for all open $U \subset G$, $f \in F(U)$ we have $X_{y^{-1}U}(f\lambda_y) = X_U(f)\lambda_y$. Now let $t \in G_e$ be a tangent vector at the identity element. We define a left invariant vector field $X(t)$ on G by $(X(t)f)(x) = t(f\lambda_x)$. This sets up a bijection between G_e and left invariant vector fields on G. (Easy.) Now if X,Y are any two vector fields on G the so is $[X,Y] = XY - YX$, and $[X,Y]$ is left invariant if X and Y are left invariant. This defines a Lie algebra structure on the vector space of left invariant vector fields on G and hence a Lie algebra structure on the tangent space G_e. This is the Lie algebra L(G) associated to G. Locally the structure of G is determined by L(G). More precisely:

(i) for every $m \in L(G)$ there exists a unique analytic map $e_m: k \to G$, such that $e_m(s_1)e_m(s_2) = e_m(s_1+s_2)$ and such that $(de_m)_0(1) = m$ (where we have identified the tangent space at 0 to the analytic manifold "k" with k itself);

(ii) exp: $L(G) \to G$, $m \mapsto e_m(1)$ is a local analytic isomorphism of analytic manifolds;

(iii) locally near e the group structure of G is given by $\exp(m)\exp(m') = \exp(F(m,m'))$ where $F(m,m') = m + m' + \frac{1}{2}[m,m'] + \frac{1}{12}([m,[m,m']] + [m',[m',m]]) - \frac{1}{24}(\dots) + \dots$ is some well-defined universal expression (Campbell-Baker-Hausdorff formula);

(iv) connected Lie subgroups of G correspond biuniquely to Lie subalgebras of L(G);

(v) connected normal Lie subgroups correspond biuniquely to ideals in L(G);

(vi) G is quasi-simple ($\Longleftrightarrow$ G is connected and has only discrete proper normal subgroups) iff L(G) is simple.

EXAMPLE. $G = GL_n(\mathbb{C})$, $L(G) = M_n(\mathbb{C})$, the Lie algebra of all $n\times n$ matrices under $[A,B] = AB - BA$. The map exp: $M_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is given by $A \mapsto e^A = I + A + (2!)^{-1}A^2 + (3!)^{-1}A^3 + \ldots$ (whence the notation "exp" in general).

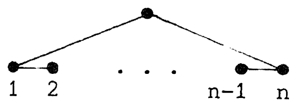
For the proofs of all these facts, cf. any of the standard books on Lie groups and Lie algebras, e.g. [33], [31] and, in a slightly different context [35].

3.14. Extracting information from Dynkin diagrams

(i) Let I be the set of vertices of a connected subgraph of a Dynkin diagram. Then $\sum_{i\in I} \alpha_i$ is a positive root. Every root $\sum_i m_i\alpha_i$ with $m_i = 0,1$ is obtainable in this way. In the case of A_n one thus obtains all positive roots.

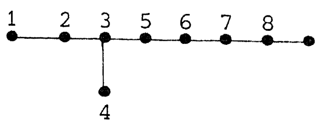
(ii) $\mathrm{Aut}(D) = \mathrm{Aut}_{\mathrm{Lie}}(G)/\mathrm{Int}(G)$. Here D is the Dynkin diagram of the simple Lie group G, Int(G) is the group of interior automorphisms of G and $\mathrm{Aut}_{\mathrm{Lie}}(G)$ is the group of automorphisms of G. One has $\mathrm{Aut}(A_n) = \mathbb{Z}/(2)$, $\mathrm{Aut}(D_4) = S_3$, $\mathrm{Aut}(D_n) = \mathbb{Z}/(2)$ for $n \geq 5$, $\mathrm{Aut}(E_6) = \mathbb{Z}/(2)$ and $\mathrm{Aut}(D) = \{1\}$ for all other Dynkin diagrams D.

(iii) The so-called *completed Dynkin diagrams* play an important role in the determining of all maximal compact subgroups of compact (real) simple Lie groups, cf. [54]. One adds a vertex corresponding to the largest root, cf. [8], Ch.VI, §4.3 for details. The completed Dynkin diagrams $\tilde{A}_n$, $\tilde{D}_n$ and $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ are

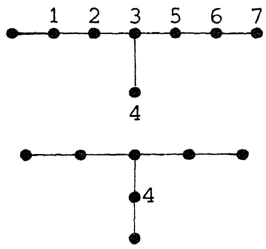


$\tilde{A}_n$

$\tilde{D}_n$



$\tilde{E}_8$



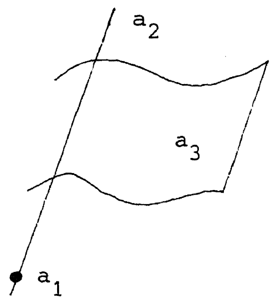
$\tilde{E}_7$

$\tilde{E}_6$

4. TITS GEOMETRIES

<u>4.1. EXAMPLE</u>. $\mathbb{P}^n(\mathbb{C})$ *as a Tits-geometry*. We start with an example. Let $\mathbb{P}^n(\mathbb{C})$ be a complex projective space of (complex) dimension n, and let $PGL_{n+1}(\mathbb{C}) = PSL_{n+1}(\mathbb{C})$ be its group of linear projective automorphisms. We show how the geometry of $\mathbb{P}^n(\mathbb{C})$, i.e. the sets of points, lines, planes,... of $\mathbb{P}^n(\mathbb{C})$ together with their incidence relations are recoverable from the Lie group $PGL_{n+1}(\mathbb{C})$.

Let F_j be the set of all (j-1)-dimensional linear subspaces of $\mathbb{P}^n(\mathbb{C})$, $j = 1,2,\ldots,n$. If $i \neq j$, $x \in F_i$, $y \in F_j$ we write $x|y$ if x and y are incident, i.e. if $x \subset y$ if $i < j$ or if $y \subset x$ if $i > j$. A *flag* is a sequence of elements $(a_1,\ldots,a_t)$, $a_i \in F_{i_j}$, $i_1 < \ldots < i_t$ such that $a_i | a_{i+1}$ for all $i = 1,\ldots,t-1$. If $t = n$ the flag is *maximal*. The terminology comes from the picture of a maximal flag in $\mathbb{P}^3$.



Choose a basis $e_1,e_2,\ldots,e_{n+1}$ of $\mathbb{C}^{n+1}$. Interpreting points of $\mathbb{P}^n(\mathbb{C})$ as lines through 0 in $\mathbb{C}^{n+1}$, lines in $\mathbb{P}^n(\mathbb{C})$ as planes through 0 in $\mathbb{C}^{n+1}$,... we find a maximal flag

$$E = (\langle e_1\rangle, \langle e_1,e_2\rangle, \ldots, \langle e_1,\ldots,e_n\rangle).$$

The stabilizer of E in $G = PGL_{n+1}(\mathbb{C})$ is then the subgroup B of all upper triangular matrices (with respect to the chosen basis).

$$B = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix} \right\}$$

The subgroups conjugate to B are all stabilizers of a maximal flag, and these are precisely the maximal solvable subgroups of G, that is the *Borel subgroups*.

A *parabolic* subgroup is a subgroup of G which contains a Borel subgroup. The parabolic subgroups containing B above are the groups

$$P = \left\{ \begin{pmatrix} \begin{matrix} * \\ \vdots \\ * \cdots * \end{matrix} & & * \\ & \begin{matrix} * \\ \vdots \\ * \cdots * \end{matrix} & \\ & \ddots & \begin{matrix} * \\ \vdots \\ * \cdots * \end{matrix} \end{pmatrix} \right\}$$

(different block sizes are allowed); i.e. they are the groups consisting of all blocks upper triangular matrices for a given sequence of block sizes $n_1, \ldots, n_s$, $n_1 + \ldots + n_s = n+1$. These groups are the stabilizers of flags contained in E, e.g. if n = 3, then the parabolic subgroups $\neq$ B,G containing B are

$$\left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\} \left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\} \left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\} \left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}\right\} \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\} \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\}$$

which are respectively the stabilizers of the flags $(\langle e_1, e_2\rangle, \langle e_1, e_2, e_3\rangle)$, $(\langle e_1\rangle, \langle e_1, e_2, e_3\rangle)$, $(\langle e_1\rangle, \langle e_1, e_2\rangle)$, $(\langle e_1\rangle)$, $(\langle e_1, e_2\rangle)$, $(\langle e_1, e_2, e_3\rangle)$.

Every parabolic subgroup of G is conjugate to precisely one parabolic subgroup containing B. In particular the subspaces of $\mathbb{P}^n(\mathbb{C})$, i.e. the elements of the F_j, j = 1,...,n, correspond to the maximal parabolic subgroups $\neq$ G. In case n = 3 the last three of the parabolic subgroups listed above are maximal.

Now let P' be any parabolic subgroup, then $P' = gPg^{-1}$ with $P \supset B$ where B is the standard Borel subgroup given above. Now the normalizer of a parabolic subgroup P is P itself and it follows that $\{h \mid hPh^{-1} = P'\} = gP$ so that $P' \mapsto gP$ sets up a bijective correspondence between parabolic subgroups conjugate to a given $P \supset B$ and cosets of P in G. Let $P_{(i)}$ be the stabilizer of $(\langle e_1,\ldots,e_i\rangle)$; then we see that

$$F_i = \{(i-1)-\dim \text{ subspaces}\} \xleftrightarrow{1-1} \{gP_{(i)}g^{-1} \mid g \in G\} \xleftrightarrow{1-1} G/P_{(i)}.$$

Furthermore we recover the incidence relations as follows:
$gP_{(i)} \mid g'P_{(j)} \iff gP_{(i)}$ and $g'P_{(j)}$ correspond to elements of the same maximal flag

$$\iff \exists g'' \text{ such that } gP_{(i)} \cap g'P_{(j)} \supset g''B$$

$$\iff gP_{(i)} \cap g'P_{(j)} \neq \phi.$$

4.2. The Tits geometry of a (quasi-)simple Lie group G

Let G be a quasi-simple Lie group and let $\underline{g}$ be its Lie-algebra. Let $\underline{h}$ be a Cartan subalgebra, let R be the root system of $\underline{g}$ with respect to $\underline{h}$ and let $S = \{\alpha_1,\ldots,\alpha_\ell\}$ be a set of simple roots. For each $\alpha = \sum m_i\alpha_i$ we set $\operatorname{supp}(\alpha) = \{\alpha_i \mid m_i \neq 0\}$. For each subset $I \subset S$ we set

$$\underline{p}_I = \underline{h} \oplus \sum_{\alpha>0} \mathbb{C} e_\alpha \oplus \sum_{\substack{\alpha<0 \\ \operatorname{supp}(\alpha)\subset I}} \mathbb{C} e_\alpha ,$$

where e_α is a nonzero element of $\underline{g}^\alpha$. In particular we have

$$\underline{p}_\phi = \underline{h}^\alpha \oplus \sum_{\alpha>0} \mathbb{C} e_\alpha.$$

Then $B = \langle\exp(\underline{p}_\phi)\rangle$ is a Borel subgroup and the $P_I = \langle\exp \underline{p}_I\rangle$ are the parabolic subgroups containing B. Every parabolic subgroup of G is conjugate with precisely one P_I.

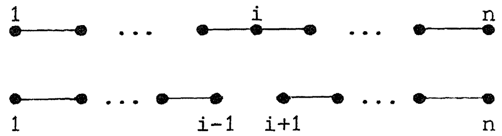
E.g. if $G = PGL_4(\mathbb{C})$ and $S = \{\alpha_1,\alpha_2,\alpha_3\}$ as in the example of 3.12 above, then the six parabolic subgroups listed in 4.1 above correspond respectively to the following subsets of S:

$$\{\alpha_1\},\ \{\alpha_2\},\ \{\alpha_3\},\ \{\alpha_2,\alpha_3\},\ \{\alpha_1,\alpha_3\},\ \{\alpha_1,\alpha_2\}.$$

For each $i \in \{1,2,\ldots,\ell\}$ let $P_{(i)}$ be the maximal parabolic subgroup $P_{(i)} = P_{I(i)}$, where $I(i) = S\backslash\{\alpha_i\}$. Now define sets of points, lines,... by $F_i = G/P_{(i)}$ and define the incidence relations by $xP_{(i)} | yP_{(j)} \Leftrightarrow xP_{(i)} \cap yP_{(j)} \neq \phi$. This is the Tits geometry (or Tits building) of G.

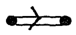
4.3. Reducing Tits geometries

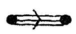
Let $\alpha_i \in S$ be a given vertex of the Dynkin diagram Take any $a \in F_i = G/P_i$. The geometry of all x which are incident with this given a corresponds to the diagram one obtains by removing α_i and all edges through α_i. Thus in the case of $\mathbb{P}^n(\mathbb{C})$ if $a \in F_i$, i.e. if a is an (i-1)-dimensional linear subspace we have



and the "residual geometry" of all $x|a$ consists of a $\mathbb{P}^{i-1}(\mathbb{C})$ (consisting of those $x|a$ with $\dim(x) < i-1$) and a $\mathbb{P}^{n-i}(\mathbb{C})$ (consisting of those $x|a$ with $\dim(x) > i-1$). Thus one can establish various properties of the Tits geometries by reduction to the geometries of rank 2:

A_2: , the projective plane $\mathbb{P}^2$

B_2:  , points and lines on a quadric in $\mathbb{P}^4$

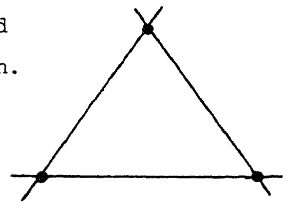
G_2:  , a geometry related to the Cayley numbers.

4.4. EXAMPLE 4.1 continued (The Skeleton geometry). Consider again the situation of 4.1 above. The subgroup of G which stabilizes all the $\langle e_{i_1}, e_{i_2}, \ldots, e_{i_p} \rangle$ is

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} = \exp(\underline{h}).$$

The Weyl group W acts as permutations on the coefficients of the matrices in T; it is the automorphism group of the skeleton $Sk = \{(\langle e_{i_1},\ldots,e_{i_p}\rangle)\}$. Let W_i be the stabilizer in W of $\langle e_1,\ldots,e_i\rangle$. Then $W_i = \{s_{\alpha_1},\ldots,s_{\alpha_{i-1}},s_{\alpha_{i+1}},\ldots,s_{\alpha_n}\}$. The (i-1)-dimensional subspaces of Sk are the wW_iw, $w \in W$, or, the cosets wW_i, $w \in W$. The geometry Sk is described by the W/W_i just as the geometry $\mathbb{P}^n$ is described by the G/P_i. The Skeleton geometry Sk is an "n-dimensional projective geometry over the field of one element".

In case n = 2 it consists of three points and three lines with incidence relations as shown. (An i-dimensional projective space over a finite field of q-elements has $1 + q + q^2 + \ldots + q^i$ points; so an i-dimensional projective space over the field of 1 element should have $i+1$ points.)



4.5. Bibliographical notes

The reference [45] is a good first introduction to the subject of Tits geometries; [47] and [48] are useful after one has read [45], and [50] describes a number of applications. The standard reference, containing all proofs, is [49], which also contains an extensive bibliography.

5. DYNKIN CURVES AND SINGULARITIES

5.1. Introduction

Here is, how, very roughly, the Dynkin diagram of a quasi-simple Lie group G arises as the fibre of a resolution of singularities of a certain variety associated to G. Let G be a quasi-simple algebraic complex Lie group. Let U(G) be the algebraic variety of its unipotent elements. This variety has singularities. Let $U_{sing}(G)$ be the subvariety of singular points. There is a more or less canonical desingularisation $\pi: V(G) \to U(G)$ and there is a single open and dense conjugacy class $C \subset U_{sing}(G)$ of so-called subregular unipotents.

For $x \in C$ the fibre $\pi^{-1}(x)$ is a connected one dimensional variety which is a union of projective lines. The intersection graph of this union of projective lines is the unfolded Dynkin diagram of G. In the following we shall try to precisize all this to some extent.

5.2. Algebraic varieties over $\mathbb{C}$ and singular points

For the purposes of this section an *affine algebraic variety* V is the set of solutions in $\mathbb{C}^r$ (for some r) of a collection of polynomials in r variables $X_1,\ldots,X_r$ and a *projective variety* is the set of solutions in $\mathbb{P}^r(\mathbb{C})$ (for some r) of a collection of homogeneous polynomials in r + 1 variables $X_0,X_1,\ldots,X_r$.

Let $V \subset \mathbb{C}^r$ be an affine algebraic variety, $x \in V$. Let $f_1(X),\ldots,f_n(X)$ be the polynomials defining V. Then we can write $f_i(X_1-x_1,\ldots,X_r-x_r) = L_i(X) + g_i(X)$ where $L_i(X)$ is homogeneous of degree 1 in X and all monomials in $g_i(X)$ have degree ≥ 2 in X. An r-vector $a \neq 0$ (starting in x) is now said to be a tangent vector to V at x if $L_i(a) = 0$, $i = 1,\ldots,n$. Let $T_x(V)$ be the linear space spanned by the tangent vectors to V at x. The point $x_0 \in V$ is called *smooth* if $\dim(T_x(V))$ is constant in a neighbourhood of x_0 in V; otherwise x_0 is called *singular*. The variety V is smooth if all its points are smooth. A projective variety $V \subset \mathbb{P}^r(\mathbb{C})$ can be seen as r + 1 affine varieties $V_i = V \cap U_i$ glued together where $U_i = \{x \in \mathbb{P}^r(\mathbb{C}) \mid x_i \neq 0\} = \mathbb{C}^r$, and a point $x \in V_i \subset V$ is smooth if it is smooth as a point of V_i. Cf. [41], Ch.II, §1 for more details.

<u>EXAMPLE</u>. Let $V \subset \mathbb{C}^2$ be the curve defined by $X_1^2 - X_2^3 = 0$. Then $(0,0) \in V$ and $\dim(T_{(0,0)}(V)) = 2$ and $\dim(T_x(V)) = 1$ for all $x \in V$, $x \neq (0,0)$. Hence (0,0) is singular and all other points of V are smooth.

5.3. Algebraic Lie groups over $\mathbb{C}$

An algebraic Lie group over $\mathbb{C}$ is (for the purpose of these lectures) a closed connected subgroup G of $GL_n(\mathbb{C})$, the group of complex invertible n×n matrices, such that the points of G are the solutions of a set of polynomials in the matrix coefficients. Now

$GL_n(\mathbb{C})$ can be identified with the variety in $\mathbb{C}^{n^2+1}$ defined by the polynomial $\det((X_{ij}))X_0 - 1$. Hence G is an affine algebraic variety in the sense of 5.2 above.

Examples of such Lie groups are:

$$GL_n(\mathbb{C}),\quad B_n(\mathbb{C}) = \left\{\begin{pmatrix} * & \cdots & \cdots & * \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}\right\}$$

$$U_n(\mathbb{C}) = \left\{\begin{pmatrix} 1 & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}\right\}$$

$$SL_n(\mathbb{C}) = \{x \in GL_n(\mathbb{C}) \mid \det(x) = 1\},$$

$$SO_n(\mathbb{C}) = \{x \in SL_n(\mathbb{C}) \mid x^t x = I\},$$

$$SB_n(\mathbb{C}) = \{x \in B_n(\mathbb{C}) \mid \det(x) = 1\},$$

where x^t is the transposed matrix of x and I is the $n \times n$ identity matrix. In the following we shall write $GL_n, \ldots, SB_n$ instead of $GL_n(\mathbb{C}), \ldots, SB_n(\mathbb{C})$.

5.4. <u>The variety of unipotent elements</u>

A matrix $x \in GL_n$ is said to be unipotent if all its eigenvalues are 1, or, equivalently, if $(x-I)^n = 0$. Let G be as in 5.3 above. Then $U(G) = \{g \in G \mid (g-I)^n = 0\}$ is called the unipotent variety of G. This is a closed subset of G defined by polynomial equations, hence it is an affine algebraic variety in the sense of 5.2 above.

<u>EXAMPLE A</u>. $G = SL_2$. Then $U(SL_2) = \{\begin{pmatrix} 1+x & y \\ z & 1-x \end{pmatrix} \mid x^2 + yz = 0\}$. This is isomorphic to the complex cone $\{(x,y,z) \in \mathbb{C}^3 \mid x^2 + yz = 0\}$ with top in (0,0,0). This top corresponds to $I \in SL_2$. The point $I \in U(SL_2)$ is singular, all other points are smooth.

EXAMPLE B. $G = SB_n$. Then $U(G) = U_n$, which is a smooth variety.

EXAMPLE C. $G = SL_n$. Then $U(G) = \{gxg^{-1} \mid g \in SL_n, x \in U_n\} = \bigcup_{g \in SL_n} gU_ng^{-1}$.

Thus we have written U(G) as a union of smooth varieties in this case. This is a general phenomenon, cf. below in 5.5.

5.5. The variety $\mathbb{B}(G)$ of Borel subgroups

A Lie subgroup $G \subset GL_n$ is *solvable* if it is conjugate in GL_n to a subgroup of B_n. If G is solvable then U(G) is smooth (as in example B), cf. [35], 19.1. A maximal solvable Lie subgroup of G is called a *Borel subgroup* (cf. also section 4 above). Every two Borel subgroups are conjugate ([35], 21.3) and it follows that the set of Borel subgroups is the homogeneous variety G/B because the normalizer of B in G is B itself ([35], 29.3). In fact G/B is a projective variety ([35], 21.3).

THEOREM ([35], 23.4).

(i) $\mathbb{B}(G)$ *is a non-empty smooth connected compact variety on which* G *acts transitively (by* $(g,B) \mapsto gBg^{-1}$; *i.e. all Borel subgroups are conjugate);*

(ii) $G = \bigcup_{B \in \mathbb{B}(G)} B$;

(iii) $U(G) = \bigcup_{B \in \mathbb{B}(G)} U(B)$, *and all the* U(B) *are smooth and connected.*

In case $G = GL_n$, part (ii) is proved by the fact that every $x \in GL_n$ is triangulizable.

EXAMPLE A (continued). $\mathbb{B}(SL_2) = SL_2/SB_2 \simeq \mathbb{P}^1(\mathbb{C})$ as is easily checked by hand.

EXAMPLE C (continued): $\mathbb{B}(SL_n)$ consists of SB_n and its conjugates.

5.6. Reductive Lie groups

The intersection $\bigcap_{B \in \mathbb{B}(G)} U(B)$ is a normal subgroup of G and one can take the quotient of G by this subgroup without changing the singularities of U(G). We shall therefore from now on suppose that this normal subgroup is trivial, i.e. that G is *reductive*. The groups GL_n, SL_n, SO_n are reductive but B_n and U_n are not reductive if $n \geq 2$.

5.7. Conjugacy classes

Let $x \in G$. Then $C(x) = \{gxg^{-1} \mid g \in G\}$, the conjugacy class of x_1 is a connected homogeneous and smooth subvariety of G.

THEOREM (RICHARDSON-LUSZTIG [55],[44]). *The variety* U(G) *is a disjoint union of a finite number of conjugacy classes.*

EXAMPLE C (continued). In the case of $G = SL_n$ this follows from the theory of the Jordan normal form.

5.8. Regular unipotents

THEOREM (STEINBERG [43], pp. 108, 110).

(i) *There is precisely one conjugacy class* $C_{reg} \subset U(G)$ *which is open and dense in* U(G);

(ii) *the variety* U(G) *is smooth in the points of* C_{reg};

(iii) *for every* $x \in C_{reg}$ *there is precisely one* $B \in \mathbb{B}(G)$ *such that* $x \in U(B)$;

(iv) *for every* $x \in U(G) \setminus C_{reg}$ *there are infinitely many* $B \in \mathbb{B}(G)$ *such that* $x \in U(B)$.

The elements of C_{reg} are called the *regular unipotents*. They can be characterized in various ways (cf. [43], 3.7).

EXAMPLE A (continued). The cone $U(SL_2)$ is the union $\{I\} \cup C_{reg}$ where C_{reg} is the conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Through every $x \in C_{reg}$ there passes precisely one line U(B) on the cone. All these lines pass through I.

EXAMPLE C (continued). In case $G = SL_n$, C_{reg} is the conjugacy class of the "one Jordan block" matrix with eigenvalue 1. E.g. if n = 4 C_{reg} is the conjugacy class of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5.9. The Springer desingularisation

Let $V(G) = \{(B,x) \mid x \in U(B)\} \subset \mathbb{B}(G) \times G$, and $\pi: V(G) \to U(G)$ be defined by $\pi(B,x) = x$. Then $V(G)$ is a closed subvariety of $\mathbb{B}(G) \times G$. The algebraic morphism π is a desingularisation in that the following theorem holds.

THEOREM ([42],[15],[44],[43], 3.9).

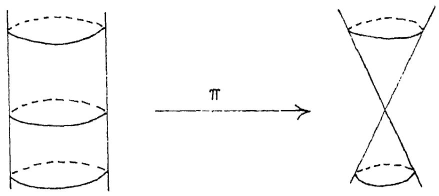
(i) $V(G)$ *is smooth and connected;*

(ii) π *is surjective and proper (that is* $\pi^{-1}(Y)$ *is compact if* Y *is compact);*

(iii) $\pi: \pi^{-1}(C_{reg}) \to C_{reg}$ *is an isomorphism and* $\pi^{-1}(C_{reg})$ *is open and dense in* $V(G)$ *(i.e.* π *is a birational morphism).*

The fibre $\pi^{-1}(x)$ for $x \in U(G)$ is the set of all Borel subgroups of G containing x, i.e. it is the set of fixed points of $x \in G$ acting on $\mathbb{B}(G) \simeq G/B$ as in the theorem of section 5.5 above. It follows that $\pi^{-1}(x)$ is a projective variety. This variety is also connected ([43], 3.9, prop.1).

EXAMPLE A (continued). The desingularisation of the cone $U(SL_2)$ looks as follows:



(where we have, of course, only drawn the real points of the 2-dimensional complex surfaces involved).

5.10. The parabolic lines of $\mathbb{B}(G)$

For simplicity we assume that G is quasi-simple. We have seen in section 4.2 above how to associate a parablic subgroup P to every subset I of the set of simple roots S. For each $\alpha_i \in S$ let P_i be the parabolic subgroup corresponding to $I = \{\alpha_i\}$. These are the minimal parabolic subgroups $\neq B$ in G. (Do not confuse them with the $P_{(i)}$, the maximal parabolic subgroups used in 4.2.) Of the six parabolic subgroups of 4.1 above the first three are minimal. They are also called *simple parabolic subgroups*, as is every parabolic subgroup conjugate to one of these.

For each P_i, $\mathbb{B}(P_i) = P_i/B$, cf. [49], 3.2.3, is isomorphic to $\mathbb{P}^1(\mathbb{C})$. We shall call $\mathbb{B}(P) \subset \mathbb{B}(G)$ a parabolic line of type i if P is conjugate to P_i.

THEOREM ([43], p.146).

(i) *Every point* $B \in \mathbb{B}(G)$ *lies on* ℓ *parabolic lines, one of each type (here* ℓ *is the number of vertices of the Dynkin diagram);*

(ii) *two parabolic lines of different type intersect each other in at most one point.*

EXAMPLE C (contined). The Dynkin diagram of SL_n is $\underset{1}{\bullet}\!-\!\underset{2}{\bullet} \cdots \bullet\!-\!\underset{n-1}{\bullet}$.
The Borel subgroup SB_n lies on the parabolic lines $\mathbb{B}(P_1),\ldots,\mathbb{B}(P_{n-1})$.
If n = 4 then P_1,P_2,P_3 are respectively equal to

$$P_1 = \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\}, \quad P_2 = \left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}\right\}, \quad P_3 = \left\{\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}\right\} .$$

In this case one easily checks by hand that $\mathbb{B}(P_i) \simeq P_i/B \simeq \mathbb{P}^1(\mathbb{C})$.

5.11. Subregular unipotents

As in 5.10 above let G be quasi-simple, so that the Dynkin diagram of G is connected.

THEOREM (STEINBERG-TITS [43], p.145,153).

(i) *There is precisely one conjugacy class* C_{sub} *which is open and dense in* $U(G) \setminus C_{reg}$.

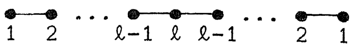
(ii) *For* $x \in U(G)$ *we have* $x \in C_{sub} \Leftrightarrow \dim(\pi^{-1}(x)) = 1$.

(iii) *If* $x \in C_{sub}$, *then the fibre* $\pi^{-1}(x) = \{B \in \mathbb{B}(G) \mid x \in U(B)\}$ *is a connected one dimensional projective variety. It is a finite union of projective lines whose intersection diagram is the unfolded Dynkin diagram of G.*

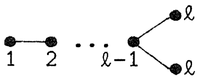
Here the unfolded versions of $A_n,\ldots,G_2$ *are defined as follows:*

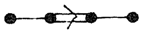
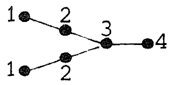
(a) A_n, D_n, E_n *remain the same*

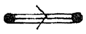
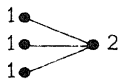
(b) 

becomes 

(c) 

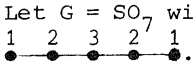
becomes 

(d)  *becomes* 

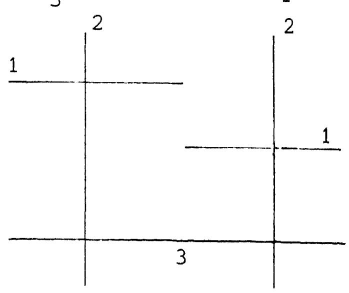
(e)  *becomes* 

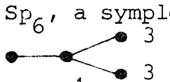
Notice that, apart from the numbering of the vertices, all unfolded Dynkin diagrams are of the types A_n, D_n, E_6, E_7, E_8.

Thus if G has Dynkin diagram B_ℓ, then part (iii) of the theorem above says that $\pi^{-1}(x)$ consists of a union of 2 lines each of types $1,2,\ldots,\ell-1$ and one line of type ℓ, which intersect as indicated by the diagram. (Two lines intersect iff the corresponding vertices are joined.)

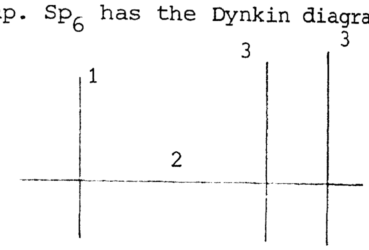
<u>EXAMPLE D</u>. Let $G = SO_7$ with Dynkin diagram B_3. The unfolded Dynkin diagram is 

. Thus the Dynkin curve $\pi^{-1}(x)$ for $x \in C_{sub}$ consists of 5 projective lines, two of type 1, two of type 2, and one of type 3, which intersect as indicated in the picture on the right.

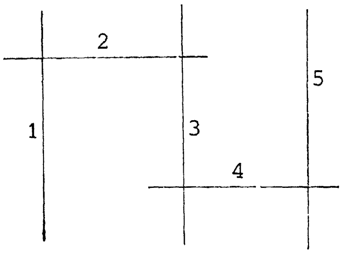


<u>EXAMPLE E</u>. Let $G = Sp_6$, a symplectic group. Sp_6 has the Dynkin diagram C_3 with unifolding 

. Thus the Dynkin curve $\pi^{-1}(x)$ consists of two lines of type 3 and one each of type 1 and 2, which intersect each other as in the picture on the right.



EXAMPLE C (continued). $G = SL_n$ has Dynkin diagram A_{n-1} with unfolding $\underset{1}{\bullet}\!-\!\underset{2}{\bullet}\!-\!\bullet \ \cdots \ \bullet\!-\!\underset{n-1}{\bullet}$. Thus $\pi^{-1}(x)$ consists of (n-1) lines, one each of type 1,2,...,n-1 which intersect each other as indicated in the diagram on the right for the case n = 6.



5.12. Local description of singularities with a Dynkin curve as exceptional fibre in a resolution

THEOREM (BRIESKORN [15]). *In a neighbourhood of a subregular element* x U(G) *is isomorphic with a neighbourhood of the origin in* $X_\ell \times \mathbb{C}^r$ *where* X_ℓ *is a surface in* $\mathbb{C}^3$ *with rational singularity in* (0,0,0). *This means that* X_ℓ *is one of the following surfaces with isolated singularity*

$$A_\ell: \{(x,y,z) \in \mathbb{C}^3 \mid x^{\ell}+1 + yz = 0\} \quad , \quad \ell \geq 1,$$

$$D_\ell: \{(x,y,z) \in \mathbb{C}^3 \mid x^{\ell+1} + xy^2 + z^2 = 0\}, \quad \ell \geq 3,$$

$$E_6: \{(x,y,z) \in \mathbb{C}^3 \mid x^4 + y^3 + z^2 = 0\} \quad ,$$

$$E_7: \{(x,y,z) \in \mathbb{C}^3 \mid x^3y + y^3 + z^2 = 0\} \quad ,$$

$$E_8: \{(x,y,z) \in \mathbb{C}^3 \mid x^5 + y^3 + z^2 = 0\} \quad .$$

(There is a nonlinear coordinate transformation which takes D_3 into A_3.)

5.13. Transversal sections

A different more concrete method for getting at the structure of the singularities at $x \in C_{sub}$ is as follows. Construct a smooth subvariety S of G through x such that $T_x(S) + T_x(C_{sub}) = T_x(G)$. By the implicit function theorem a neighbourhood of x in U(G) is isomorphic with a neighbourhood of (x,0) in $(S \cap U(G)) \times \mathbb{C}^r$ for a certain r. By choosing S cleverly one finds that $S \cap U(G) \simeq X_\ell$. Cf. [4], [15], [32].

EXAMPLE G. Let $G = GL_3$. Take n = 3. The matrix x_0 is then a subregular unipotent.

$$x_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 1+v & 1 & 0 \\ w & 1 & z \\ y & 0 & 1+t \end{pmatrix}.$$

The variety of matrices x with $\det(x) \neq 0$ is a transversal section, and $S \cap U(GL_3)$ consists of the matrices $x \in S$ which satisfy $\mathrm{trace}(x) = 3$, $\det(x) = 1$ and

$$\det\begin{pmatrix} 1+v & 1 \\ w & 1 \end{pmatrix} + \det\begin{pmatrix} 1+v & 0 \\ y & 1+t \end{pmatrix} + \det\begin{pmatrix} 1 & z \\ 0 & 1+t \end{pmatrix} = 3.$$

This gives $v = -t$, $w = -t^2$, $t^3 + yz = 0$. Hence $S \cap U(GL_3)$ is the singularity A_2, and one verifies that the Dynkin curve consists of two intersecting lines. (Remark: $U(GL_3) = U(SL_3)$, so whether one considers GL_3 or SL_3 does not matter much.)

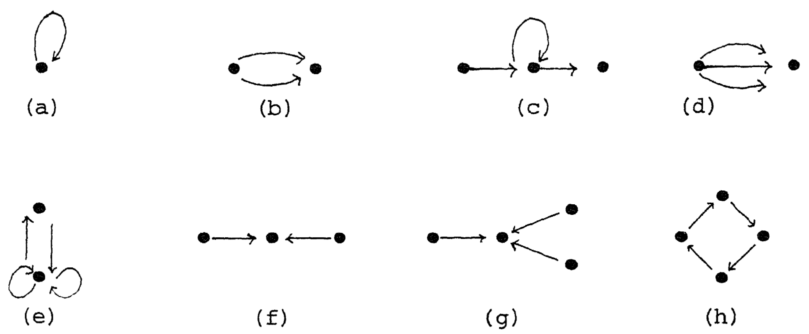
6. QUIVERS AND THEIR REPRESENTATIONS

6.1. Introduction

A *quiver* Q is a finite connected directed graph. A *representation* over a field K assigns to each vertex of the graph a vector space over K and to each arrow a homomorphism of vector spaces. It now turns out that a quiver Q has (up to isomorphism) only finitely many indecomposable representations if and only if the underlying undirected graph of Q is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

6.2. Quivers and representations

A *quiver* is a finite connected directed graph. Thus it consists of a finite set P_Q of vertices and a finite set A_Q of arrows between elements of P_Q. Let $s,r\colon A_Q \to P_Q$ be the two maps which assign to an arrow $a \in A_Q$ its initial vertex $s(a)$ and its end vertex $r(a)$. Some examples of quivers are



Let K be a field. A *representation* V of a quiver Q assigns to each $p \in P_Q$ a vector space V(p) over k (finite dimensional) and to each arrow $a \in A_Q$ a homomorphism of vector spaces $V(a): V(s(a)) \to V(r(a))$. The *zero representation* assigns to each $p \in P_Q$ the zero vector space (and to each $a \in A_Q$ the zero mapping). Given two representations V_1, V_2 their direct sum is the representation $(V_1 \oplus V_2)(p) = V_1(p) \oplus V_2(p)$, $(V_1 \oplus V_2)(a) = V_1(a) \oplus V_2(a)$. A representation V is called *indecomposable* if it cannot be written as a direct sum $V = V_1 \oplus V_2$ with both V_1 and $V_2 \neq 0$.

Finally two representations V_1 and V_2 are said to be *isomorphic* if there exists for each $p \in P_Q$ an isomorphism $\phi(p): V_1(p) \to V_2(p)$ such that for all $a \in A_Q$, $\phi(r(a)) \circ V_1(a) = V_2(a) \circ \phi(s(a))$.

<u>EXAMPLE (a)</u>. A representation of quiver (a) above consists of a vector space and an endomorphism; i.e. after choosing a basis a representation is given by a square matrix M. Two representations M, M' are isomorphic iff there is an invertible matrix S such that $M' = SMS^{-1}$. A representation M over an algebraically closed field k is indecomposable iff its Jordan canonical form consists of one Jordan block, and the indecomposables over k are classified by their sizes and the eigenvalue appearing.

<u>EXAMPLE (b)</u>. Here a representation is given by two (not necessarily square) matrices M, N and two representations (M,N), (M',N') are

isomorphic if and only if there exist invertible matrices S and T such that SM = M'T, SN = N'T. Thus the theory of the representations of quiver (b) is the theory of Kronecker pencils of matrices. Cf. [25] for the results of this theory.

6.3. Gabriel's theorem

A quiver Q is said to be of *finite type* if, up to isomorphism, there are only finitely many indecomposable representations of Q; the quiver Q is said to be *tame* if there are infinitely many isomorphism classes of indecomposable representations but these can be parametrized by a finite set of integers together with a polynomial irreducible over k; the quiver Q is said to be *wild* if for every finite dimensional algebra E over k there are infinitely many pairwise nonisomorphic representations of Q which have E as their endomorphism algebra. These three classes of quivers are clearly exclusive; they are, as it turns out, also exhaustive.

THEOREM (GABRIEL [23]). *A quiver Q is of finite type if and only if its underlying undirected graph is one of the Dynkin diagrams* A_n, D_n, E_6, E_7, E_8.

EXAMPLES. The quivers (f) and (g) of the examples of 6.2 above are of finite type.

Let Q be a quiver. We chose a fixed ordering of P_Q. For each representation V of Q we now define n(V), the dimension vector of V, as the vector $n(V) = (\dim(V(p_1)),\dots,\dim(v(p_\ell)))$.

THEOREM (GABRIEL, cf. also [7]). *Let Q be a quiver of finite type. The map* $V \mapsto n(V)$ *sets up a bijective correspondence between the indecomposable representations of Q and the set of positive roots of the underlying Dynkin diagram of V.*

6.4. Nazarova's extension of Gabriel's theorem

THEOREM ([38]). *The quivers of tame type are precisely the quivers whose underlying undirected graph is one of the extended Dynkin diagrams* $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. (Cf. section 3.14 above for a discription of these Dynkin diagrams.)

EXAMPLES. The quivers (a), (b), (h) of the examples of 6.2 above are tame. The quivers (c), (d), (e) are wild.

6.5. Quadratic form of a quiver

Let Q be a quiver with ℓ vertices. We associate to Q a quadratic form in ℓ-variables as follows:

$$B_Q(x_1,\ldots,x_\ell) = \sum_{i=1}^{\ell} x_i^2 - \sum_{a\in A_Q} x_{s(a)}x_{r(a)}.$$

EXAMPLES. The quadratic forms of the quivers (a), (b), (c), (d), (f), (g) of 6.2 above are respectively 0, $x_1^2+x_2^2-2x_1x_2$, $x_1^2+x_3^2-x_1x_2-x_2x_3$, $x_1^2+x_2^2-3x_1x_2$, $x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3$, $x_1^2+x_2^2+x_3^2+x_4^2-x_1x_2-x_2x_3-x_2x_4$.

THEOREM ([7]). *A quiver Q is of finite type (resp. tame) iff B_Q is positive definite (resp. semipositive definite).*

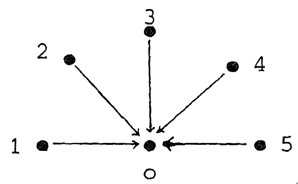
6.6. Proof of "Q is of finite type" ⇒ B_Q is positive definite (Tits)

Let Q be a quiver of finite type and let $n = (n_1,\ldots,n_\ell)$ be a fixed dimension vector. Because Q is of finite type there are only finitely many isomorphism classes of representations V such that $n(V) = n$. Now giving a representation with $n(V) = n$ is the same as specifying an $n_{r(a)}\times n_{s(a)}$ matrix for each $a \in Q_a$. This gives us a $\sum_{a\in Q_A} n_{s(a)}n_{r(a)}$ dimensional space of representations. The group $G = GL_{n_1}(k)\times\ldots\times GL_{n_\ell}(k)$ acts on this space of representations by $(M_a)_{a\in A_Q} \to (T_{r(a)}M_aT_{s(a)}^{-1})_{a\in A_Q}$ and the isomorphism classes of representations V with $n(V) = n$ are precisely the orbits X/G. The subgroup $H = \{(sI_n,\ldots,sI_{n_\ell}) \mid s\in k\}$ of G acts trivially. Because X/G is finite it follows (if we are working over an infinite field) that dim G-1 ≥ dim(X). Hence $n_1^2+\ldots+n_\ell^2-1 \geq \sum_a n_{s(a)}n_{r(a)}$; i.e. $B_Q(n_1,\ldots,n_\ell) \geq 1$. This holds for all sequences of positive integers $n = (n_1,\ldots,n_\ell)$ and hence, because clearly $B_Q(x_1,\ldots,x_\ell) \geq$ $\geq B_Q(|x_1|,|x_2|),\ldots,|x_\ell|)$, it follows that B_Q is positive definite.

6.7. EXAMPLE. Let Q be a quiver with underlying Dynkin diagram A_ℓ. For all $r,s \in \mathbb{N}$ with $1 \leq r < s \leq n$. Let $V_{r,s}(i) = k$ for $r \leq i \leq s$

and $V_{r,s}(j) = 0$ for $j < r$ or $j > s$. For $a \in Q_A$ we set $V_{r,s}(a) = \mathrm{id}$ if a joints two points in $\{i \mid r \le i \le s\}$ and $V_{r,s}(a) = 0$ otherwise. Then $V_{r,s}$ is an indecomposable representation of Q and all indecomposable representations of Q are isomorphic to one of these.

6.8. <u>EXAMPLE</u> ([24],[26]). Consider the quiver Q_5:



with the vertices numbered as indicated. This quiver is wild. We show that every finite dimensional algebra A arises as an endomorphism algebra of Q_5. To this end consider first Q_4, the quiver obtained from Q_5 by removing the vertex 5 and the arrow incident with it. We now first construct a representation U of Q_4 over a field k with $\dim(U) = 2n+1$, $n = 1,2,\ldots$ such that the endomorphism algebra of U is k. To this end let E be an $n+1$-dimensional vector space over k with basis $e_1,\ldots,e_{n+1}$ and F an n-dimensional vector space with basis $f_1,\ldots,f_n$. We set $U(0) = E \oplus F$, $U(1) = E \oplus 0$, $U(2) = 0 \oplus F$, $U(3) = \{(\lambda(f),f) \mid f \in F\}$, $U(4) = \{(\delta(f),f) \mid f \in F\}$. Where $\lambda,\delta\colon F \to E$ are defined by $\lambda(f_i) = e_i$, $\delta(f_i) = e_{i+1}$. The maps associated to the arrows are the natural inclusions. An endomorphism of U is then given by an endomorphism α of $U(0) = E \oplus F$, which preserves the subspaces $U(1),\ldots,U(4)$. One easily checks that this means α is multiplication with an element of k, i.e. one finds $\mathrm{End}(U) = k$. Now let A be any finite dimensional algebra over k and let $a_1,\ldots,a_m$ be a set of generators of A (as a k-module). Let $a_0 = 1$ and see to it that m is even, $m \ge 2$. Let U be the representation of Q_4 constructed above with $\dim(U) = m+1$. We now define a representation V of Q_5 by $V(0) = A \otimes U(0)$, $V(i) = A \otimes U(i)$, $i = 1,\ldots,4$, $V(5) = \{\sum_{i=0}^{m} aa_i \otimes e_i \mid a \in A\} \subset A \otimes U(0)$, where $e_0,\ldots,e_m$ is a basis for $U(0)$. An endomorphism of V is an endomorphism of $V(0)$ which preserves the five subspaces $V(j)$, $j = 1,\ldots,5$. Because $\mathrm{End}(U) = k$

the endomorphisms of $V(0)$ which preserve $V(1),\dots,V(4)$ are necessarily of the form $\phi \otimes 1$ where ϕ is a k-vector space endomorphism of A. Now $(\phi \otimes 1)(\sum_{i=0}^{m} aa_i \otimes e_i) = \sum_{i=0}^{m} \phi(aa_i) \otimes e_i$ and it follows that if $\phi \otimes 1$ also preserves $V(5)$, there must be, for all $a \in A$, a $b(a)$ such that $\sum_{i=0}^{m} \phi(aa_i) \otimes e_i = \sum_{i=0}^{m} b(a)a_i \otimes e_i$. Now $1 \otimes e_0,\dots,1 \otimes e_m$ is a basis for $A \otimes U(0)$ as a module over A, hence $\phi(aa_i) = b(a)a_i$ for all i. Taking $i = 0$ we find $\phi(a) = b(a)$. Hence we have for all $a \in A$ and all i that $\phi(aa_i) = \phi(a)a_i$.
Let $c = \phi(1)$, then $\phi(a_i) = ca_i$ for all i and we see that ϕ is given by multiplication with $c \in A$. This shows that indeed $\mathrm{End}(V) = A$.

7. SIMPLE SINGULARITIES AND DYNKIN DIAGRAMS

7.1. Fintely determined map germs

Let $f: U \to \mathbb{C}$, $0 \in U \subset \mathbb{C}^{n+1}$ be a holomorphic mapping with isolated critical point in 0. I.e. 0 is critical (that is $df(0) = 0$) and there is a $\delta > 0$ such that for $\|z\| < \delta$, $df(z) \neq 0$ if $z \neq 0$. A critical point 0 is *nondegenerate* if

$$\det\left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0)\right) \neq 0.$$

PROPOSITION (Morse lemma). *If* f *has a nondegenerate critical point in* 0 *then there is a biholomorphic change of coordinates* ϕ *such that* $f\phi(z_0,\dots,z_n) = f(0) + z_0^2 + \dots + z_n^2$.

More generally one has

THEOREM. *if* 0 *is an isolated critical point of* f *then there is a local biholomorphic change of coordinates* ϕ *such that* $f\phi$ *is equal to a finite part of the Taylor expansion of* f *around* 0.

A proof can e.g. be found in [16], chapter 11. It uses the Nullstellensatz for holomorphic function germs, which shows that df is a finite mapping, and next a theorem of Tougeron. One can give a bound for the degree of the Taylor approximation in this theorem in terms of the ideal $(\partial_0 f,\ldots,\partial_n f) \subset \mathbb{C}\langle\langle z_0,\ldots,z_n\rangle\rangle$ generated by the partial derivatives of f. If the critical point is nondegenerate this number is 2 and one reobtains the Morse lemma.

From now on we consider *polynomials* with an isolated critical point in $0 \in \mathbb{C}^{n+1}$. (This is justified by the theorem above.)

7.2. Right equivalence and simple germs

Two germs of holomorphic mappings f,g are right *equivalent* (or *are of the same type*) if there exists a biholomorphic change of coordinates ϕ such that $g = f\phi$. A germ f is called *simple* if there is a finite list of germs such that every small perturbation of f is equivalent to a germ from this list.

THEOREM (ARNOLD [6]). *f*: $\mathbb{C}^{n+1} \supset U \to \mathbb{C}$ *is simple if* f *is equivalent to a germ in the following list:*

$x^{k+1} + y^2 + z_2^2 + \ldots + z_n^2$ *type* A_k $(k \geq 0)$

$x^2y + y^{k-1} + z_2^2 + \ldots + z_n^2$ *type* D_k $(k \geq 4)$

$x^3 + y^4 + z_2^2 + \ldots + z_n^2$ *type* E_6

$x^3 + xy^3 + z_2^2 + \ldots + z_n^2$ *type* E_7

$x^3 + y^5 + z_2^2 + \ldots + z_n^2$ *type* E_8 .

7.3. Morsifications

Let f be a polynomial with isolated critical point in $0 \in \mathbb{C}^{n+1}$. A morsification of f is a polynomial mapping F: $\mathbb{C}^{n+2} \to \mathbb{C}$ such that $F(z,0) = f(z)$ and $f_\lambda(z) = F(z,\lambda)$ has only nondegenerate critical points in a neighbourhood of $0 \in \mathbb{C}^{n+1}$ for small enough $\lambda \neq 0$. Morsifications always exist. In fact, one can take $F(z,\lambda) = f(z) + \sum_{i=0}^{n} \lambda_i z_i$ for suitable (generic) $\lambda_i = \lambda_i(\lambda)$.

7.4. Milnor number

For a small enough neighbourhood of 0 in $\mathbb{C}^{n+1}$ and small enough $\lambda \neq 0$ the number of critical points of f_λ in this neighbourhood is constant. This number $\mu(f)$ is called the *Milnor number* of f. This definition is independent of the choice of the Morsification. In fact

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\langle\langle z_0,\ldots,z_n\rangle\rangle}{(\partial_0 f,\ldots,\partial_n f)} ,$$

which is finite if and only if f has an isolated critical point. For different characterisations of $\mu(f)$ cf. [39].

7.5. Examples of Morsifications

We now give a number of examples of Morsifications of polynomials $\mathbb{C}^2 \to \mathbb{C}$ with real coefficients. The Morsifications given below all have the property that all critical points are real and all saddle points have the same critical value. Let

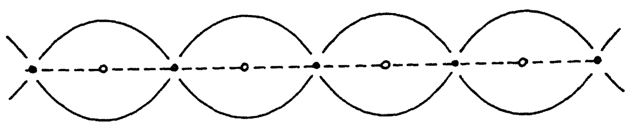
$$\phi_m(x,\lambda) = \begin{cases} (x+\lambda)^2(x+2\lambda)^2 \ldots (x+k\lambda)^2 & \text{if } m = 2k , \\ (x+\lambda)^2 \ldots (x+(k-1)\lambda)^2(x+k\lambda) & \text{if } m = 2k-1. \end{cases}$$

EXAMPLE (i). Morsifications for type A_m.

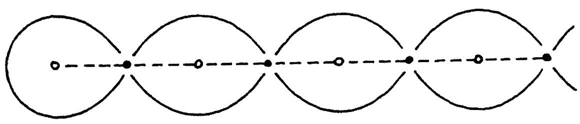
Polynomial : $x^{m+1} - y^2$

Morsification: $\phi_{m+1}(x,\lambda) - y^2$

Picture of the zero level



m = 9



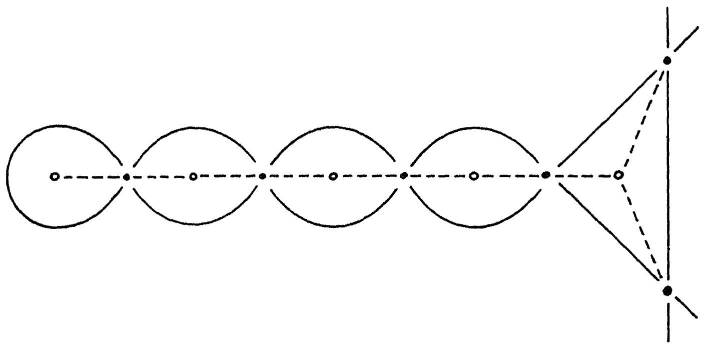
m = 8

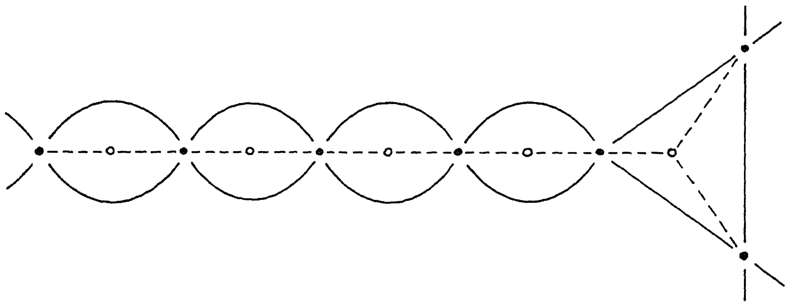
<u>EXAMPLE (ii)</u>. <u>Morsifications for type</u> D_m

Polynomial : $x^{m-1} - xy^2 = x(x^{m-2}-y^2)$

Morsification: $x(\phi_{m-2}(x,\lambda)-y^2)$

Picture of the zero level





In the following three examples one first constructs a deformation f_λ with one critical point in 0 and the other critical points nondegenerate. Moreover, the lowest degree part g_λ of f_λ is a polynomial of type D_4, which factors over $\mathbb{R}$ in three different linear factors. Let $g_{\lambda,\mu}$ be a Morsification for g_λ. Then for μ small enough $f_{\lambda,\mu} = g_{\lambda,\mu} + (f_\lambda - g_\lambda)$ is a Morsification of f_λ since the nondegenerate critical points of f_λ survive and stay approximately in the same place. For appropriate $\mu = \mu(\lambda)$ this gives us a Morsification for f.

EXAMPLE (iii). Morsification for type E_6

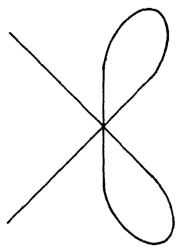
Polynomial : $x^3 + y^4$

Deformation : $x(x^2-\lambda y^2) + y^4$

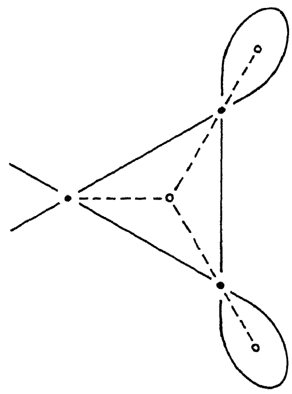
Morsification: $(x-\mu)(x^2-\lambda y^2) + y^4$

Pictures of the zero level for various λ,μ

$\lambda = 0, \mu = 0$



$\lambda > 0,\ \mu = 0$



$\lambda > 0,\ \lambda >> \mu > 0$

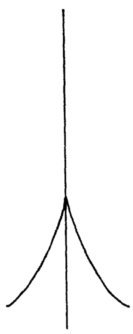
EXAMPLE (iv). Morsification for type E_7

Polynomial : $x^3 + xy^3 = x(x^2+y^3)$

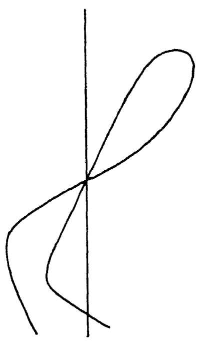
Deformation : $x(x^2+y^3+\lambda y^2-6\lambda xy)$

Morsification: $(x-\mu)(x^2+y^3+\lambda y^2-6\lambda xy)$

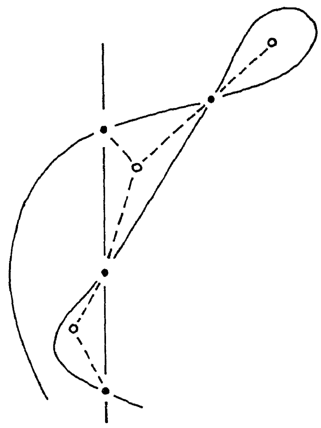
Pictures of the zero level for various λ,μ



$\lambda = 0$



$\lambda > 0,\ \mu = 0$



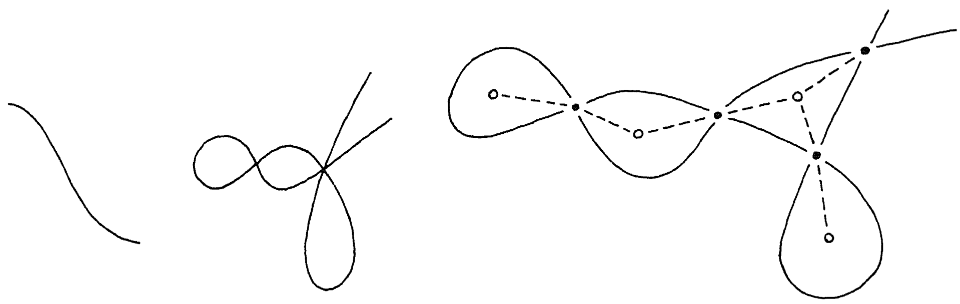
$\lambda > 0,\ \lambda >> \mu > 0$

EXAMPLE (v). Morsification for type E_8

Polynomial : $x^3 + y^5$

Deformation $x^3 + y^3(y-\beta)^2 + 3\alpha xy^2(y-\beta) + 2\alpha^2(x^2-y)$

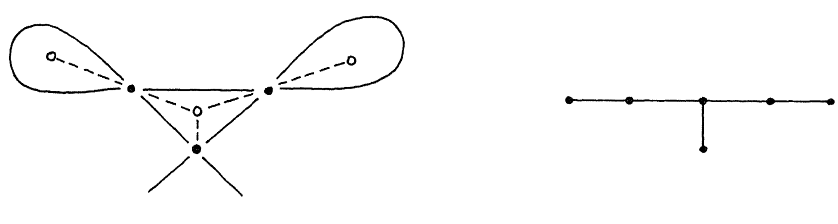
Pictures of the zero levels



$\alpha = \beta = 0$ $\alpha >> \beta > 0$ morsification

7.6. Separatrices

For the examples given above in 7.5 consider the gradient vector fields $(\frac{\partial f}{\partial x}(x,y,\lambda),\frac{\partial f}{\partial y}(x,y,\lambda))$, $\lambda \neq 0$ and construct the corresponding separatrix diagrams. These consist of a number of vertices, corresponding to the critical points of f and a number of lines, joining these vertices, where there is a line joining two given vertices if and only if there is an integral curve which joins the two corresponding critical points. An example is E_6:



In the examples (i) - (v) of 7.5 above the separatrix diagrams of the Morsifications of the polynomials given are precisely the Coxeter-Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

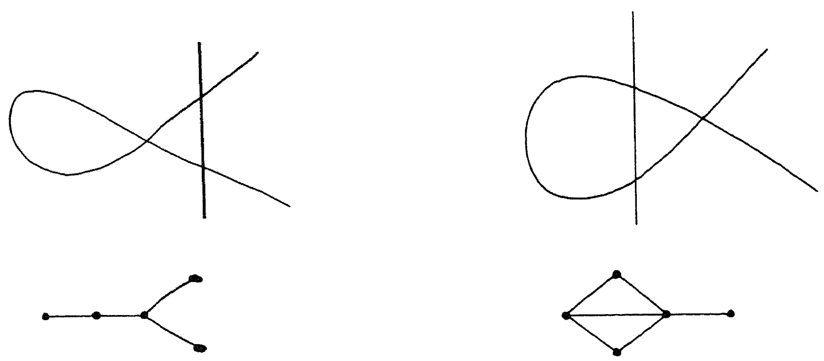
The following example shows that Morsifications and separatrix diagrams are not unique. Consider $x^4 - xy^2 = x(x^3-y^2)$ which is of type D_5. Two Morsifications of this polynomial are

$$x((x+\lambda)^2(x+2\lambda)-y^2)$$

and

$$(x+\tfrac{3}{2}\lambda)((x+\lambda)^2(x+2\lambda)-y^2)$$

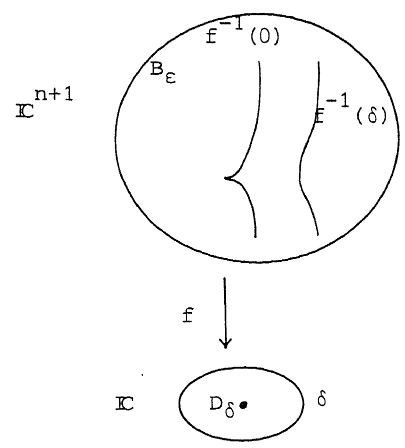
with zero level pictures and separatrix diagrams



7.7. The Milnor fibration

As above we consider a polynomial $f: \mathbb{C}^{n+1} \to \mathbb{C}$ with $f(0) = 0$ and isolated critical point in 0.

Let $X_{\varepsilon,\delta} = B_\varepsilon \cap f^{-1}(D_\delta \setminus \{0\})$, where $D_\delta = \{z \in \mathbb{C} \mid \|z\| \leq \delta\}$ and $B_\varepsilon = \{z \in \mathbb{C}^{n+1} \mid \|z\| \leq \varepsilon\}$. Let $F_{\varepsilon,t} = B_\varepsilon \cap f^{-1}(t)$. The restriction $f: X_{\varepsilon,\delta} \to D_\delta \setminus \{0\}$ is, for ε and δ sufficiently small, a locally trivial fibre bundle (cf. MILNOR [37]). Moreover, in the case of an isolated critical point at 0 the fibre $F = F_{\varepsilon,\delta}$ is homotopy equivalent to a wedge of μ (the Milnor number) copies of the n-sphere S^n. Thus $H_n(F) = \mathbb{Z}^\mu$, $H_0(F) = \mathbb{Z}$, $H_i(F) = 0$ for $i \neq 0,n$.



<u>EXAMPLE</u>. Let $f = z_0^2 + z_1^2$. The equations for the fibre F are $|z_0^2| + |z_1^2| \le \varepsilon$, $z_0^2 + z_1^2 = \delta$. Writing $z_j = x_j + iy_j$ we find $x_0^2 + x_1^2 + y_0^2 + y_1^2 \le \varepsilon$, $x_0^2 + x_1^2 - y_0^2 - y_1^2 = \delta$, $2x_0y_0 + 2x_1y_1 = 0$. Thus $x_0^2 + x_1^2 = \delta + y_0^2 + y_1^2$ which gives

$$((\delta+y_0^2+y_1^2)^{-\frac{1}{2}}x_0)^2 + ((\delta+y_0^2+y_1^2)^{-\frac{1}{2}}x_1)^2 = 1,$$

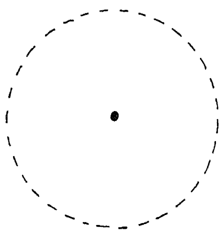
$$x_0y_0 + x_1y_1 = 0,$$

$$y_0^2 + y_1^2 \le 2^{-1}(\varepsilon^2-\delta).$$

Thus F is diffeomorphic to the bundle of tangent vectors to the circle S^1, the circle S^1 itself being obtained for $y_0 = y_1 = 0$.

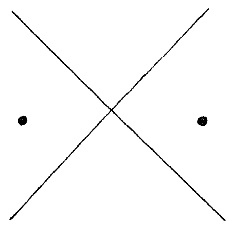
The pictures of the real points of the situation look as follows:

$f = x_0^2 + x_1^2$



S^1 is the level line $x_0^2 + x_1^2 = \delta$

$f = x_0^2 - x_1^2$



S^1 intersects $\mathbb{R}^2$ in two points of the level lines $x_0^2 - x_1^2 = \delta$

<u>THEOREM</u> (TJURINA [51], BRIESKORN [13]). *Let* $n = 2$ *and let* $\tilde{F}_{\varepsilon,0} \to F_{\varepsilon,0}$ *be the resolution of the isolated singularity at* 0 *of* $f^{-1}(0)$. *Then* f *is simple iff* $\tilde{F}_{\varepsilon,0}$ *is diffeomorphic with* $F_{\varepsilon,0}$.

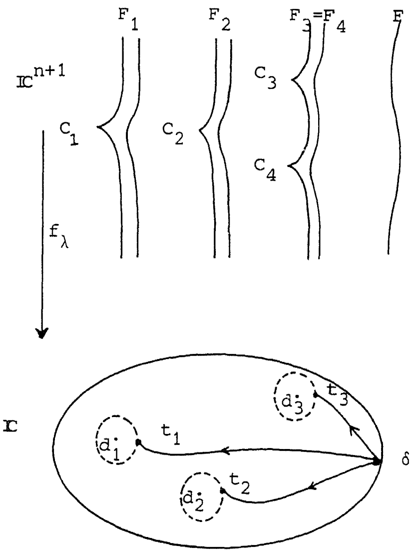
Cf. also section 5 above (especially 5.11 and 5.12) for a statement on the exceptional fibre of $\tilde{F}_{\varepsilon,0} \to F_{\varepsilon,0}$.

7.8. <u>Monodromy</u>

Using the local triviality of the fibre bundle $f\colon X_{\varepsilon,\delta} \to D_\delta\backslash\{0\}$, every piecewise smooth path $\omega\colon [0,1] \to D_\delta\backslash\{0\}$ can be made to induce a diffeomorphism $F_{\omega(0)} \to F_{\omega(1)}$. (In fact one defines a so-called connection.) Let $\omega(t) = \delta e^{2\pi it}$. The corresponding diffeomorphism $F \to F$ is called the geometric monodromy; the induced map on homology $h: H_n(F) \to H_n(F)$ is called the algebraic monodromy.

7.9. Vanishing cycles

Now let f_λ be a given Morsification of f. Let the critical points of f (for a given small λ) be $c_1,\dots,c_\mu$ and let the corresponding critical values be $d_1,\dots,d_\mu$. For small λ we obtain a fibration over $D\setminus\{d_1,\dots,d_\mu\}$, which, over ∂D, the boundary of D, is equivalent to the Milnor fibering of f. (Cf. 7.7 above.). Near every c_i we have again a Milnor fibration. Let $t_1,\dots,t_\mu$ be values near $d_1,\dots,d_\mu$ such that locally $f^{-1}(t_i)$ is a Milnor fibre near c_i. Set $F_i = F_{\varepsilon,t_i}$. Since c_i is nondegenerate each fibre F_i contains an n-sphere Z_i. And using paths (as in the picture) from δ to t_i we find embeddings



$$Z_i \hookrightarrow F_i \xrightarrow{\text{diffeo}} F.$$

In this way we find μ embedded n-spheres $S_1,\dots,S_\mu$ in F. These are called the *vanishing cycles*.

THEOREM (BRIESKORN [14]). *The homology classes* $[S_1],\dots,[S_\mu]$ *are a basis for* $H_n(F)$.

7.10. Intersection form

Now consider the intersection numbers $\langle S_i,S_j\rangle$ of the spheres S_i and S_j. The intersection form $\langle,\rangle$ is defined on $H_n(F)$ and (using small deformations of representing cycles if necessary) can be computed by counting intersection points (with multiplicities).

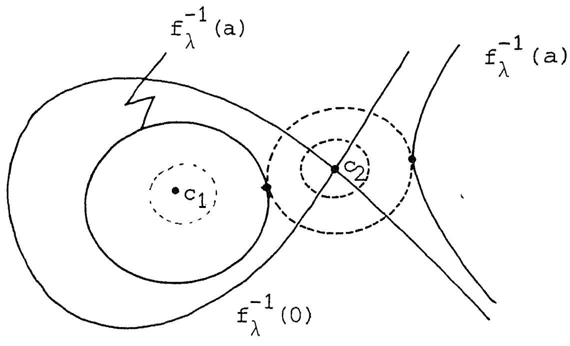
The intersection form is symmetric if n is even and antisymmetric if n is odd. If n is even then $\langle S_i, S_j\rangle = (-1)^k 2$. DURFEE [22] proved that the intersection matrix $(\langle S_i, S_j\rangle)$ determines the topology of the Milnor fibration.

THEOREM (TJURINA [51]). *Let* $n \equiv 2 \pmod 4$. *Then f is simple if and only if the intersection form is negative definite.*

7.11. Separatrix diagrams (contined)

We return to real Morsifications and the separatrix diagram.

EXAMPLE. $f = x^3 - y^2$ with Morsification f_λ. The intersection numbers of the vanishing cycles can be computed from the picture of the Morsification. In this example we have two vanishing cycles (one near c_1, the other near c_2). After transporting them to the same level curve $f_\lambda^{-1}(a)$ we see that their intersection number is one.



THEOREM (GUSEIN-ZADE [28], cf. also A'CAMPO [2] for a slightly different version). *Let f be a polynomial in two variables with real coefficients and let* f_λ *be a Morsification with real critical points and let all saddle points have the same critical value. Then*

(i) *if* c_i *is a saddle point and* c_j *a minimum, then* $\langle S_i, S_j\rangle$ *is equal to the number of integral curves joining* c_i *and* c_j;

(ii) *if* c_i *is a maximum and* c_j *a saddle point, then* $\langle S_i, S_j\rangle$ *is equal to the number of integral curves joining* c_i *and* c_j;

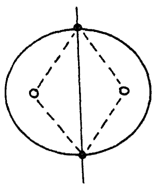
(iii) *if* c_i *is a minimum and* c_j *a maximum, then* $\langle S_i, S_j\rangle$ *is equal to the number of families of integral curves joining* c_i *with* c_j;

(iv) *in all other cases* $\langle S_i, S_j\rangle = 0$.

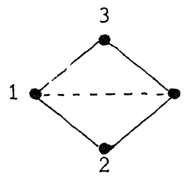
If $g(x,y,z) = f(x,y) + z^2$ one obtains almost the same result. In fact the critical points of g all satisfy $z = 0$ and (otherwise) coincide with those of f. Thus $\mu(f) = \mu(g)$. The intersection numbers are equal to $-|<S_i,S_j>|$ in the maximum-minimum case and equal to $|<S_i,S_j>|$ in all other cases except if $i = j$ then $<S_i,S_i> = -2$.

More or less as usual one represents the intersection matrix by a diagram of μ vertices, with two vertices joined by a number of lines equal to the intersection number of the corresponding vanishing cycles. Negative intersection numbers are represented by dotted lines (and no lines are drawn joining a vertex to itself).

EXAMPLE. $(x^2+y^2-\lambda)x$.



$$\begin{pmatrix} -2 & 2 & 1 & -1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & 1 & 1 & -2 \end{pmatrix}$$



If there are only saddle points and minima we do not find negative entries off the diagonal and we obtain exactly the separatrix diagram.

THEOREM (A'CAMPO [3]). *The following are equivalent:*
(i) f *has a Morsification with two critical values;*
(ii) *the diagram of the intersection matrix is a tree;*
(iii) f *is simple.*

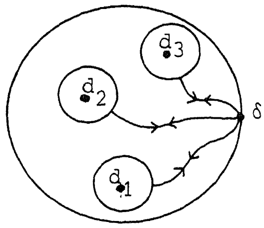
A'CAMPO [2] and GUSEIN-ZADE [29] have shown that for an arbitrar polynomial $f: \mathbb{C}^2 \to \mathbb{C}$ one can always find a $\tilde{f}: \mathbb{C}^2 \to \mathbb{C}$ with real coefficients, the same intersection matrix and admitting a Morsification $\tilde{f}_\lambda$, which satisfies the conditions of the theorem above. In fact f and $\tilde{f}$ can be joined by a family of constant Milnor number.

7.12. The monodromy group

The monodromy group W_f is the image of the mapping

$$\pi_1(D\backslash\{d_1,\ldots,d_\mu\}) \to \mathrm{Aut}(H_n(F)),$$

cf. 7.9 above. Given a Morsification of f one considers paths ω_i as indicated in the picture on the next page.



(First go from δ to t_i, then go around d_i, then back from t_i to δ.) Let $\sigma_i: H_n(F) \to H_n(F)$ be induced by the diffeomorphism corresponding to ω_i (cf. 7.8 above).

THEOREM (LAMOTKE [36]).

(i) $\sigma_1,\ldots,\sigma_\mu$ *generate* W_f;

(ii) $\sigma_i(x) = x - (-1)^{(n-1)n/2}\langle x,S_i\rangle S_i$;

(iii) $h = \sigma_\mu \circ \sigma_{\mu-1} \circ \ldots \circ \sigma_1$ *is the algebraic monodromy.*

Let $n \equiv 2 \pmod 4$. Then W_f is a Coxeter group if the intersection form $\langle,\rangle$ is negative definite.

THEOREM. f *is simple if and only if* W_f *is finite.*

7.13. Bibliographical note

General references for this section are [5], [6], [12] and the very recent survey paper [30]. These papers are suitable as introductions and summaries of the subject. Of these papers [12] also pays attention to singularities of differential equations.

8. CONCLUDING REMARKS AND ADDITIONAL BIBLIOGRAPHICAL NOTES

8.1. Systems of lines at angles of $\pi/3$ and $\pi/2$

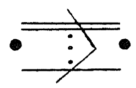
A *star* is a planar set of three lines which all make an angle of $\pi/3$ with each other. A set of lines in Euclidean n-space which mutually have the angles $\pi/3$ or $\pi/2$ is *star closed* if with any two it also contains the third line of a star. In [17] all such indecomposable (same notion as in 3.12 (iii)) sets of lines are determined. They are the root systems A_n, D_n, E_6, E_7, E_8. These are all maximal apart from $A_8 \subset E_8$, $D_8 \subset E_8$, $A_7 \subset E_7$.

8.2. Species and their representations

If one extends the notion of quiver a bit (cf. section 6 above) the "missing" Dynkin diagrams B_n, C_n, F_4, G_2 also appear. More precisely: let k be a field, a k-*species* (GABRIEL [24]), $(K_i, {}_iM_j)_{i,j\in I}$ is a finite set of fields K_i, which are finite dimensional over a common central subfield k, together with a set of $K_i - K_j$ bimodules ${}_iM_j$, such that for all $a \in k$, $m \in {}_iM_j$, $am = ma$, and such that ${}_iM_j$ is finite dimensional over k (for all i,j). The *diagram* of a species is defined as follows. The set of vertices is I, and there are

$$\dim_{K_i}({}_iM_j) \times \dim_{K_j}({}_iM_j) + \dim_{K_j}({}_jM_i) \times \dim_{K_i}({}_jM_i)$$

edges between the vertices i and j. In the special case ${}_jM_i = 0$ and $\dim_{K_i}({}_iM_j) < \dim_{K_j}({}_iM_j)$ we shall pictorially represent these facts by

i  j .

A *representation* $(V_i, {}_j\phi_i)$ of the k-species $(K_i, {}_iM_j)_{i,j\in I}$ is a set of right vector spaces V_i over K_i together with a set of K_j-linear mappings

$${}_j\phi_i\colon V_i \otimes_{K_i} {}_iM_j \to V_j, \qquad i,j \in I.$$

A homomorphism of representations $\alpha\colon (V_i, {}_j\phi_i) \to (V'_i, {}_j\phi'_i)$ is a set of K_i-linear mappings $\alpha_i\colon V_i \to V'_i$ such that

$${}_j\phi'_i(\alpha_i \otimes 1) = \alpha_j \circ {}_j\phi_i.$$

A k-species if a k-quiver if $K_i = k$ for all i. Such a quiver is completely determined by its diagram where the number of arrows going from i to j is equal to the k-dimension of ${}_jM_i$.

There is an obvious notion of direct sum and being indecomposable for representations of k-species. A k-species is of *finite type* if it has only finitely many non isomorphic indecomposable representations.

<u>THEOREM</u> (GABRIEL [24], DLAB-RINGEL [20]). *A* k-*species is of finite type if and only if its diagram is a finite disjoint union of the Dynkin diagrams* A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2. *Moreover the number of indecomposable representations of a* K-*species of the type of one of these Dynkin diagrams is equal to the number of positive roots of the corresponding root system.*

8.3. <u>Rational singularities</u>

Let V be the germ at v of a normal two-dimensional complex analytic space with singularity at v. (For definitions cf. [53]; for example $V = f^{-1}(0)$, where f(x,y,z) is the germ at 0 of a complex analytic function of 3 variables with isolated critical point at 0.) Let $\pi: M \to V$ be a resolution of the singularity. The genus p of V is the dimension of the complex vector space $H^1(M,\mathcal{O}_M)$ where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M. The analytic set V has a *rational* singularity at v if p = 0. There are many characterizations of rational singularities. One of them says that V has a rational singularity iff V is isomorphic (as a germ of a complex analytic space) to $f^{-1}(0)$ with f(x,y,z) one of the polynomials of type A_n, D_n, E_6, E_7, E_8 discussed above; cf. 7.2. For more characterizations, cf. [21], and also [53], [15], [56].

8.4. <u>Finite subgroups of SU(2)</u>

The group SU(2) acts linearly on $\mathbb{C}^2$. The discrete subgroups of SU(2) are the so-called binary cyclic, dihedral, tetrahedral octahedral and icosahedral groups. (By factoring out the centre {±I} one obtains the corresponding group of rotations of the sphere.) The quotient manifold $M = \mathbb{C}^2/\Gamma$ where Γ is a discrete subgroup of SU(2) is an algebraic surface with singularity. The ring of polynomials in two variables invariant under Γ has 3 generators. There is one relation (syzygy) connecting these 3 generators and this equation then is the equation of M as a surface in $\mathbb{C}^3$. The singularities (of polynomials) which one obtains in this way are respectively of type A_n (cyclic), D_n (dihedral), E_6 (tetrahedral), E_7 (octahedral), E_8 (isosahedral). Cf. [21], [5] and [15].

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