Higher-Dimensional Algebra VI: Lie 2-Algebras

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Abstract

The theory of Lie algebras can be categorified starting from a new notion of '2-vector space', which we define as an internal category in Vect. There is a 2-category 2Vect having these 2-vector spaces as objects, 'linear functors' as morphisms and 'linear natural transformations' as 2-morphisms. We define a 'semistrict Lie 2-algebra' to be a 2-vector space L equipped with a skew-symmetric bilinear functor $[\cdot,\cdot]\colon L \times L \to L$ satisfying the Jacobi identity up to a completely antisymmetric trilinear natural transformation called the 'Jacobiator', which in turn must satisfy a certain law of its own. This law is closely related to the Zamolodchikov tetrahedron equation, and indeed we prove that any semistrict Lie 2-algebra gives a solution of this equation, just as any Lie algebra gives a solution of the Yang–Baxter equation. We construct a 2-category of semistrict Lie 2-algebras and prove that it is 2-equivalent to the 2-category of 2-term L_∞-algebras in the sense of Stasheff. We also study strict and skeletal Lie 2-algebras, obtaining the former from strict Lie 2-groups and using the latter to classify Lie 2-algebras in terms of 3rd cohomology classes in Lie algebra cohomology. This classification allows us to construct for any finite-dimensional Lie algebra $\mathfrak{g}$ a canonical 1-parameter family of Lie 2-algebras $\mathfrak{g}_\hbar$ which reduces to $\mathfrak{g}$ at $\hbar = 0$. These are closely related to the 2-groups $G_\hbar$ constructed in a companion paper.

1 Introduction

One of the goals of higher-dimensional algebra is to 'categorify' mathematical concepts, replacing equational laws by isomorphisms satisfying new coherence laws of their own. By iterating this process, we hope to find n-categorical and eventually ω-categorical generalizations of as many mathematical concepts as possible, and use these to strengthen — and often simplify — the connections between different parts of mathematics. The previous paper of this series, HDA5 [6], categorified the concept of Lie group and began to explore the resulting theory of 'Lie 2-groups'. Here we do the same for the concept of Lie algebra, obtaining a theory of 'Lie 2-algebras'.

In the theory of groups, associativity plays a crucial role. When we categorify the theory of groups, this equational law is replaced by an isomorphism called the associator, which satisfies a new law of its own called the pentagon equation. The counterpart of the associative law in the theory of Lie algebras is the Jacobi identity. In a 'Lie 2-algebra' this is replaced by an isomorphism which we call the *Jacobiator*. This isomorphism satisfies an interesting new law of its own. As we shall see, this law, like the pentagon equation, can be traced back to Stasheff's work on homotopy-invariant algebraic structures — in this case, his work on L_∞-algebras, also known as strongly homotopy Lie algebras [24, 33]. This demonstrates yet again the close connection between categorification and homotopy theory.

To prepare for our work on Lie 2-algebras, we begin in Section 2 by reviewing the theory of internal categories. This gives a systematic way to categorify concepts: if K is some category of algebraic structures, a 'category in K' will be one of these structures but with categories taking the role of sets. Unfortunately, this internalization process only gives a 'strict' way to categorify, in which equations are replaced by identity morphisms. Nonetheless it can be a useful first step.

In Section 3, we focus on categories in Vect, the category of vector spaces. We boldly call these '2-vector spaces', despite the fact that this term is already used to refer to a very different categorification of the concept of vector space [22], for it is our contention that our 2-vector spaces lead to a more interesting version of categorified linear algebra than the traditional ones. For example, the tangent space at the identity of a Lie 2-group is a 2-vector space of our sort, and this gives a canonical representation of the Lie 2-group: its 'adjoint representation'. This is contrast to the phenomenon observed by Barrett and Mackaay [8], namely that Lie 2-groups have few interesting representations on the traditional sort of 2-vector space. One reason for the difference is that the traditional 2-vector spaces do not have a way to 'subtract' objects, while ours do. This will be especially important for finding examples of Lie 2-algebras, since we often wish to set $[x, y] = xy - yx$.

At this point we should admit that our 2-vector spaces are far from novel entities! In fact, a category in Vect is secretly just the same as a 2-term chain complex of vector spaces. While the idea behind this correspondence goes back to Grothendieck [21], and is by now well-known to the cognoscenti, we describe it carefully in Proposition 8, because it is crucial for relating 'categorified linear algebra' to more familiar ideas from homological algebra.

In Section 4.1 we introduce the key concept of 'semistrict Lie 2-algebra'. Roughly speaking, this is a 2-vector space L equipped with a bilinear functor

$$[\cdot,\cdot]\colon L \times L \to L,$$

the Lie bracket, that is skew-symmetric and satisfies the Jacobi identity up to a completely antisymmetric trilinear natural isomorphism, the 'Jacobiator' — which in turn is required to satisfy a law of its own, the 'Jacobiator identity'. Since we do not weaken the equation $[x, y] = -[y, x]$ to an isomorphism, we do not reach the more general concept of 'weak Lie 2-algebra': this remains a task for the future.

At first the Jacobiator identity may seem rather mysterious. As one might expect, it relates two ways of using the Jacobiator to rebracket an expression of the form $[[[w, x], y], z]$, just as the pentagon equation relates two ways of using the associator to reparenthesize an expression of the form $(((w \otimes x) \otimes y) \otimes z)$. But its detailed form seems complicated and not particularly memorable.

However, it turns out that the Jacobiator identity is closely related to the Zamolodchikov tetrahedron equation, familiar from the theory of 2-knots and braided monoidal 2-categories [5, 7, 15, 16, 22]. In Section 4.2 we prove that just as any Lie algebra gives a solution of the Yang–Baxter equation, every semistrict Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation! This pattern suggests that the theory of 'Lie n-algebras' — that is, structures like Lie algebras with $(n-1)$-categories taking the role of sets — is deeply related to the theory of $(n-1)$-dimensional manifolds embedded in $(n+1)$-dimensional space.

In Section 4.3, we recall the definition of an L_∞-algebra. Briefly, this is a chain complex V of vector spaces equipped with a bilinear skew-symmetric operation $[\cdot,\cdot]\colon V \times V \to V$ which satisfies the Jacobi identity up to an infinite tower of chain homotopies. We construct a 2-category of '2-term' L_∞-algebras, that is, those with $V_i = \{0\}$ except for $i = 0, 1$. Finally, we show this 2-category is equivalent to the previously defined 2-category of semistrict Lie 2-algebras.

In the next two sections we study *strict* and *skeletal* Lie 2-algebras, the former being those where the Jacobi identity holds 'on the nose', while in the latter, isomorphisms exist only between identical objects. Section 5 consists of an introduction to strict Lie 2-algebras and strict Lie 2-groups, together with the process for obtaining the strict Lie 2-algebra of a strict Lie 2-group. Section 6 begins with an exposition of Lie algebra cohomology and its relationship to skeletal Lie 2-algebras. We then show that Lie 2-algebras can be classified (up to equivalence) in terms of a Lie algebra $\mathfrak{g}$, a representation of $\mathfrak{g}$ on a vector space V, and an element of the Lie algebra cohomology group $H^3(\mathfrak{g}, V)$. With the help of this result, we construct from any finite-dimensional Lie algebra $\mathfrak{g}$ a canonical 1-parameter family of Lie 2-algebras $\mathfrak{g}_\hbar$ which reduces to $\mathfrak{g}$ at $\hbar = 0$. This is a new way of deforming a Lie algebra, in which the Jacobi identity is weakened in a manner that depends on the parameter $\hbar$. It is natural to suspect that this deformation is related to the theory of quantum groups and affine Lie algebras. In HDA5, we give evidence for this by using Chern–Simons theory to construct 2-groups $G_\hbar$ corresponding to the Lie 2-algebras $\mathfrak{g}_\hbar$ when $\hbar$ is an integer. However, it would be nice to find a more direct link between quantum groups, affine Lie algebras and the Lie 2-algebras $\mathfrak{g}_\hbar$.

In Section 7, we conclude with some guesses about how the work in this paper should fit into a more general theory of 'n-groups' and 'Lie n-algebras'.

Note: In all that follows, we denote the composite of morphisms $f: x \to y$ and $g: y \to z$ as $fg: x \to z$. All 2-categories and 2-functors referred to in this paper are *strict*, though sometimes we include the word 'strict' to emphasize this fact. We denote vertical composition of 2-morphisms by juxtaposition; we denote horizontal composition and whiskering by the symbol $\circ$.

2 Internal Categories

In order to create a hybrid of the notions of a vector space and a category in the next section, we need the concept of an 'internal category' within some category. The idea is that given a category K, we obtain the definition of a 'category in K' by expressing the definition of a usual (small) category completely in terms of commutative diagrams and then interpreting those diagrams within K. The same idea allows us to define functors and natural transformations in K, and ultimately to recapitulate most of category theory, at least if K has properties sufficiently resembling those of the category of sets.

Internal categories were introduced by Ehresmann [18] in the 1960s, and by now they are a standard part of category theory [10]. However, since not all readers may be familiar with them, for the sake of a self-contained treatment we start with the basic definitions.

Definition 1. *Let K be a category. An* **internal category** *or* **category in** K*, say X, consists of:*

- *an* **object of objects** $X_0 \in K$,
- *an* **object of morphisms** $X_1 \in K$,

together with

- **source** *and* **target** *morphisms* $s, t: X_1 \to X_0$,
- *a* **identity-assigning** *morphism* $i: X_0 \to X_1$,
- *a* **composition** *morphism* $\circ: X_1 \times_{X_0} X_1 \to X_1$

such that the following diagrams commute, expressing the usual category laws:

- *laws specifying the source and target of identity morphisms:*

$$
\begin{array}{ccc}
X_0 & \xrightarrow{i} & X_1 \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle s} \\
& & X_0
\end{array}
\qquad
\begin{array}{ccc}
X_0 & \xrightarrow{i} & X_1 \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle t} \\
& & X_0
\end{array}
$$

- *laws specifying the source and target of composite morphisms:*

$$
\begin{array}{ccc}
X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle s} \\
X_1 & \xrightarrow{s} & X_0
\end{array}
\qquad
\begin{array}{ccc}
X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1 \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle t} \\
X_1 & \xrightarrow{t} & X_0
\end{array}
$$

- *the associative law for composition of morphisms:*

$$
\begin{array}{ccc}
X_1 \times_{X_0} X_1 \times_{X_0} X_1 & \xrightarrow{\circ \times_{X_0} 1} & X_1 \times_{X_0} X_1 \\
{\scriptstyle 1 \times_{X_0} \circ}\downarrow & & \downarrow{\scriptstyle \circ} \\
X_1 \times_{X_0} X_1 & \xrightarrow{\circ} & X_1
\end{array}
$$

- *the left and right unit laws for composition of morphisms:*

$$
\begin{array}{ccccc}
X_0 \times_{X_0} X_1 & \xrightarrow{i \times 1} & X_1 \times_{X_0} X_1 & \xleftarrow{1 \times i} & X_1 \times_{X_0} X_0 \\
& {\scriptstyle p_2}\searrow & \downarrow{\scriptstyle \circ} & \swarrow{\scriptstyle p_1} & \\
& & X_1 & &
\end{array}
$$

The pullbacks referred to in the above definition should be clear from the usual definition of category; for instance, composition is defined on pairs of morphisms where the target of the first is the source of the second, so the pullback $X_1 \times_{X_0} X_1$ is defined via the square

$$
\begin{array}{ccc}
X_1 \times_{X_0} X_1 & \xrightarrow{p_2} & X_1 \\
\Big\downarrow {\scriptstyle p_1} & & \Big\downarrow {\scriptstyle s} \\
X_1 & \xrightarrow{t} & X_0
\end{array}
$$

Notice that inherent to this definition is the assumption that the pullbacks involved actually exist. This holds automatically when the 'ambient category' K has finite limits, but there are some important examples such as $K = \mathrm{Diff}$ where this is not the case. Throughout this paper, all of the categories considered have finite limits:

- Set, *the category whose objects are sets and whose morphisms are functions.*
- Vect, *the category whose objects are vector spaces over the field k and whose morphisms are linear functions.*
- Grp, *the category whose objects are groups and whose morphisms are homomorphisms.*
- Cat, *the category whose objects are small categories and whose morphisms are functors.*
- LieGrp, *the category whose objects are Lie groups and whose morphisms are Lie group homomorphisms.*
- LieAlg, *the category whose objects are Lie algebras over the field k and whose morphisms are Lie algebra homomorphisms.*

Having defined 'categories in K', we can now internalize the notions of functor and natural transformation in a similar manner. We shall use these to construct a 2-category $K\mathrm{Cat}$ consisting of categories, functors, and natural transformations in K.

Definition 2. *Let K be a category. Given categories X and X' in K, an* **internal functor** *or* **functor in** K *between them, say $F\colon X \to X'$, consists of:*

- *a morphism $F_0\colon X_0 \to X'_0$,*
- *a morphism $F_1\colon X_1 \to X'_1$*

such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

- *preservation of source and target:*

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X_0 \\ {\scriptstyle F_1}\big\downarrow & & \big\downarrow{\scriptstyle F_0} \\ X'_1 & \xrightarrow{s'} & X'_0 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{t} & X_0 \\ {\scriptstyle F_1}\big\downarrow & & \big\downarrow{\scriptstyle F_0} \\ X'_1 & \xrightarrow{t'} & X'_0 \end{array}$$

- *preservation of identity morphisms:*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ {\scriptstyle F_0}\big\downarrow & & \big\downarrow{\scriptstyle F_1} \\ X'_0 & \xrightarrow{i'} & X'_1 \end{array}$$

- *preservation of composite morphisms:*

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{F_1 \times_{X_0} F_1} & X'_1 \times_{X'_0} X'_1 \\ {\scriptstyle \circ}\big\downarrow & & \big\downarrow{\scriptstyle \circ'} \\ X_1 & \xrightarrow{F_1} & X'_1 \end{array}$$

Given two functors $F\colon X \to X'$ and $G\colon X' \to X''$ in some category K, we define their composite $FG\colon X \to X''$ by taking $(FG)_0 = F_0G_0$ and $(FG)_1 = F_1G_1$. Similarly, we define the identity functor in K, $1_X\colon X \to X$, by taking $(1_X)_0 = 1_{X_0}$ and $(1_X)_1 = 1_{X_1}$.

Definition 3. *Let K be a category. Given two functors $F, G\colon X \to X'$ in K, an* **internal natural transformation** *or* **natural transformation in** K *between them, say $\theta\colon F \Rightarrow G$, is a morphism $\theta\colon X_0 \to X'_1$ for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:*

- *laws specifying the source and target of a natural transformation:*

$$\begin{array}{ccc} X_0 & \xrightarrow{\theta} & X'_1 \\ & {\scriptstyle F_0}\searrow & \big\downarrow{\scriptstyle s} \\ & & X_0 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\theta} & X'_1 \\ & {\scriptstyle G_0}\searrow & \big\downarrow{\scriptstyle t} \\ & & X_0 \end{array}$$

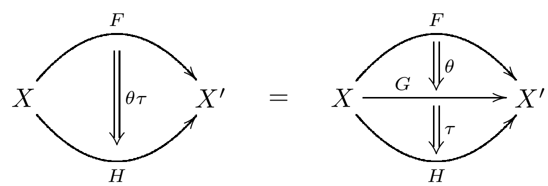
- *the commutative square law:*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\Delta(s\theta \times G)} & X'_1 \times_{X'_0} X'_1 \\
{\scriptstyle \Delta(F \times t\theta)} \downarrow & & \downarrow {\scriptstyle \circ'} \\
X'_1 \times_{X'_0} X'_1 & \xrightarrow{\circ'} & X'_1
\end{array}
$$

Just like ordinary natural transformations, natural transformations in K may be composed in two different, but commuting, ways. First, let X and X' be categories in K and let $F, G, H \colon X \to X'$ be functors in K. If $\theta \colon F \Rightarrow G$ and $\tau \colon G \Rightarrow H$ are natural transformations in K, we define their **vertical** composite, $\theta\tau \colon F \Rightarrow H$, by

$$\theta\tau := \Delta(\theta \times \tau) \circ' .$$

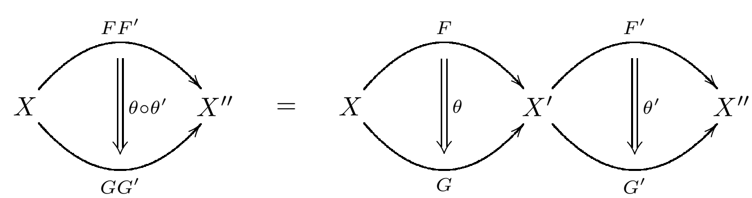
The reader can check that when $K = \mathrm{Cat}$ this reduces to the usual definition of vertical composition. We can represent this composite pictorially as:



Next, let X, X', X'' be categories in K and let $F, G \colon X \to X'$ and $F', G' \colon X' \to X''$ be functors in K. If $\theta \colon F \Rightarrow G$ and $\theta' \colon F' \Rightarrow G'$ are natural transformations in K, we define their **horizontal composite**, $\theta \circ \theta' \colon FF' \Rightarrow GG'$, in either of two equivalent ways:

$$
\begin{aligned}
\theta \circ \theta' &:= \Delta(F_0 \times \theta)(\theta' \times G'_1) \circ' \\
&= \Delta(\theta \times G_0)(F'_1 \times \theta') \circ' .
\end{aligned}
$$

Again, this reduces to the usual definition when $K = \mathrm{Cat}$. The horizontal composite can be depicted as:



It is routine to check that these composites are again natural transformations in K. Finally, given a functor $F \colon X \to X'$ in K, the identity natural transformation $1_F \colon F \Rightarrow F$ in K is given by $1_F = F_0 i$.

We now have all the ingredients of a 2-category:

Proposition 4. *Let K be a category. Then there exists a strict 2-category* K**Cat** *with categories in K as objects, functors in K as morphisms, and natural transformations in K as 2-morphisms, with composition and identities defined as above.*

Proof. It is straightforward to check that all the axioms of a 2-category hold; this result goes back to Ehresmann [18]. $\Box$

We now consider internal categories in Vect.

3 2-Vector spaces

Since our goal is to categorify the concept of a Lie algebra, we must first categorify the concept of a vector space. A categorified vector space, or '2-vector space', should be a category with structure analogous to that of a vector space, with functors replacing the usual vector space operations. Kapranov and Voevodsky [22] implemented this idea by taking a finite-dimensional 2-vector space to be a category of the form Vect^n, in analogy to how every finite-dimensional vector space is of the form k^n. While this idea is useful in contexts such as topological field theory [25] and group representation theory [3], it has its limitations. As explained in the Introduction, these arise from the fact that these 2-vector spaces have no functor playing the role of 'subtraction'.

Here we instead define a 2-vector space to be a category in Vect. Just as the main ingredient of a Lie algebra is vector space, a Lie 2-algebra will have an underlying 2-vector space of this sort. Thus, in this section we first define a 2-category of these 2-vector spaces. We then establish the relationship between these 2-vector spaces and 2-term chain complexes of vector spaces: that is, chain complexes having only two nonzero vector spaces. We conclude this section by developing some 'categorified linear algebra' — the bare minimum necessary for defining and working with Lie 2-algebras in the next section.

In the following we consider vector spaces over an arbitrary field, k.

Definition 5. *A* **2-vector space** *is a category in* Vect*.*

Thus, a 2-vector space V is a category with a vector space of objects V_0 and a vector space of morphisms V_1, such that the source and target maps $s, t\colon V_1 \to V_0$, the identity-assigning map $i\colon V_0 \to V_1$, and the composition map $\circ\colon V_1 \times_{V_0} V_1 \to V_1$ are all *linear*. As usual, we write a morphism as $f\colon x \to y$ when $s(f) = x$ and $t(f) = y$, and sometimes we write $i(x)$ as 1_x.

In fact, the structure of a 2-vector space is completely determined by the vector spaces V_0 and V_1 together with the source, target and identity-assigning maps. As the following lemma demonstrates, composition can always be expressed in terms of these, together with vector space addition:

Lemma 6. *When $K =$ Vect, one can omit all mention of composition in the definition of category in K, without any effect on the concept being defined.*

Proof. First, given vector spaces V_0, V_1 and maps $s, t\colon V_1 \to V_0$ and $i\colon V_0 \to V_1$, we will define a composition operation that satisfies the laws in Definition 1, obtaining a 2-vector space.

Given $f \in V_1$, we define the **arrow part** of f, denoted as $\vec{f}$, by

$$\vec{f} = f - i(s(f)).$$

Notice that $\vec{f}$ is in the kernel of the source map since

$$s(f - i(sf)) = s(f) - s(f) = 0.$$

While the source of $\vec{f}$ is always zero, its target may be computed as follows:

$$t(\vec{f}) = t(f - i(s(f)) = t(f) - s(f).$$

The meaning of the arrow part becomes clearer if we write $f\colon x \to y$ when $s(f) = x$ and $t(f) = y$. Then, given any morphism $f\colon x \to y$, we have $\vec{f}\colon 0 \to y - x$. In short, taking the arrow part of f has the effect of 'translating f to the origin'.

We can always recover any morphism from its arrow part together with its source, since $f = \vec{f} + i(s(f))$. We shall take advantage of this by identifying $f\colon x \to y$ with the ordered pair $(x, \vec{f})$. Note that with this notation we have

$$s(x, \vec{f}) = x, \qquad t(x, \vec{f}) = x + t(\vec{f}).$$

Using this notation, given morphisms $f\colon x \to y$ and $g\colon y \to z$, we define their composite by

$$fg := (x, \vec{f} + \vec{g}),$$

or equivalently,

$$(x, \vec{f})(y, \vec{g}) := (x, \vec{f} + \vec{g}).$$

It remains to show that with this composition, the diagrams of Definition 1 commute. The triangles specifying the source and target of the identity-assigning morphism do not involve composition. The second pair of diagrams commute since

$$s(fg) = x$$

and

$$t(fg) = x + t(\vec{f}) + t(\vec{g}) = x + (y - x) + (z - y) = z.$$

The associative law holds for composition because vector space addition is associative. Finally, the left unit law is satisfied since given $f\colon x \to y$,

$$i(x)f = (x, 0)(x, \vec{f}) = (x, \vec{f}) = f$$

and similarly for the right unit law. We thus have a 2-vector space.

Conversely, given a category V in Vect, we shall show that its composition must be defined by the formula given above. Suppose that $(f, g) = ((x, \vec{f}), (y, \vec{g}))$ and $(f', g') = ((x', \vec{f'}), (y', \vec{g'}))$ are composable pairs of morphisms in V_1. Since the source and target maps are linear, $(f+f', g+g')$ also forms a composable pair, and the linearity of composition gives

$$(f+f')(g+g') = fg + f'g'.$$

If we set $g = 1_y$ and $f' = 1_{y'}$, the above equation becomes

$$(f+1_{y'})(1_y+g') = f1_y + 1_{y'}g' = f + g'.$$

Expanding out the left hand side we obtain

$$((x,\vec{f}) + (y',0))((y,0)+(y',\vec{g'})) = (x+y',\vec{f})(y+y',\vec{g'}),$$

while the right hand side becomes

$$(x,\vec{f}) + (y,\vec{g'}) = (x+y',\vec{f}+\vec{g'}).$$

Thus we have $(x+y',\vec{f})(y+y',\vec{g'}) = (x+y',\vec{f}+\vec{g'})$, so the formula for composition in an arbitrary 2-vector space must be given by

$$fg = (x,\vec{f})(y,\vec{g}) = (x,\vec{f}+\vec{g})$$

whenever (f,g) is a composable pair. This shows that we can leave out all reference to composition in the definition of 'category in K' without any effect when $K = \mathrm{Vect}$. $\square$

In order to simplify future arguments, we will often use only the elements of the above lemma to describe a 2-vector space.

We continue by defining the morphisms between 2-vector spaces:

Definition 7. *Given 2-vector spaces V and W, a* **linear functor** $F\colon V \to W$ *is a functor in* Vect *from V to W.*

For now we let 2Vect stand for the category of 2-vector spaces and linear functors between them; later we will make 2Vect into a 2-category.

The reader may already have noticed that a 2-vector space resembles a **2-term chain complex** of vector spaces: that is, a pair of vector spaces with a linear map between them, called the 'differential':

$$C_1 \xrightarrow{\ d\ } C_0.$$

In fact, this analogy is very precise. Moreover, it continues at the level of morphisms. A **chain map** between 2-term chain complexes, say $\phi\colon C \to C'$, is simply a pair of linear maps $\phi_0\colon C_0 \to C'_0$ and $\phi_1\colon C_1 \to C'_1$ that 'preserves the differential', meaning that the following square commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{\ d\ } & C_0 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_0} \\ C'_1 & \xrightarrow{\ d'\ } & C'_0 \end{array}$$

There is a category 2Term whose objects are 2-term chain complexes and whose morphisms are chain maps. Moreover:

Proposition 8. *The categories* 2Vect *and* 2Term *are equivalent.*

Proof. We begin by introducing functors

$$S\colon \mathrm{2Vect} \to \mathrm{2Term}$$

and

$$T\colon \mathrm{2Term} \to \mathrm{2Vect}.$$

We first define S. Given a 2-vector space V, we define $S(V) = C$ where C is the 2-term chain complex with

$$\begin{aligned} C_0 &= V_0, \\ C_1 &= \ker(s) \subseteq V_1, \\ d &= t|_{C_1}, \end{aligned}$$

and $s, t\colon V_1 \to V_0$ are the source and target maps associated with the 2-vector space V. It remains to define S on morphisms. Let $F\colon V \to V'$ be a linear functor and let $S(V) = C$, $S(V') = C'$. We define $S(F) = \phi$ where ϕ is the chain map with $\phi_0 = F_0$ and $\phi_1 = F_1|_{C_1}$. Note that ϕ preserves the differential because F preserves the target map.

We now turn to the second functor, T. Given a 2-term chain complex C, we define $T(C) = V$ where V is a 2-vector space with

$$\begin{aligned} V_0 &= C_0, \\ V_1 &= C_0 \oplus C_1. \end{aligned}$$

To completely specify V it suffices by Lemma 6 to specify linear maps $s, t\colon V_1 \to V_0$ and $i\colon V_0 \to V_1$ and check that $s(i(x)) = t(i(x)) = x$ for all $x \in V_0$. To define s and t, we write any element $f \in V_1$ as a pair $(x, \vec{f}) \in C_0 \oplus C_1$ and set

$$\begin{aligned} s(f) &= s(x, \vec{f}) &= x, \\ t(f) &= t(x, \vec{f}) &= x + d\vec{f}. \end{aligned}$$

For i, we use the same notation and set

$$i(x) = (x, 0)$$

for all $x \in V_0$. Clearly $s(i(x)) = t(i(x)) = x$. Note also that with these definitions, the decomposition $V_1 = C_0 \oplus C_1$ is precisely the decomposition of morphisms into their source and 'arrow part', as in the proof of Lemma 6. Moreover, given any morphism $f = (x, \vec{f}) \in V_1$, we have

$$t(f) - s(f) = d\vec{f}.$$

Next we define T on morphisms. Suppose $\phi\colon C \to C'$ is a chain map between 2-term chain complexes:

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi_0} \\ C_1' & \xrightarrow{d'} & C_0' \end{array}$$

Let $T(C) = V$ and $T(C') = V'$. Then we define $F = T(\phi)$ where $F\colon V \to V'$ is the linear functor with $F_0 = \phi_0$ and $F_1 = \phi_0 \oplus \phi_1$. To check that F really is a linear functor, note that it is linear on objects and morphisms. Moreover, it preserves the source and target, identity-assigning and composition maps because all these are defined in terms of addition and the differential in the chain complexes C and C', and ϕ is linear and preserves the differential.

We leave it the reader to verify that T and S are indeed functors. To show that S and T form an equivalence, we construct natural isomorphisms $\alpha\colon ST \Rightarrow 1_{\mathrm{2Vect}}$ and $\beta\colon TS \Rightarrow 1_{\mathrm{2Term}}$.

To construct α, consider a 2-vector space V. Applying S to V we obtain the 2-term chain complex

$$\ker(s) \xrightarrow{t|_{\ker(s)}} V_0.$$

Applying T to this result, we obtain a 2-vector space V' with the space V_0 of objects and the space $V_0 \oplus \ker(s)$ of morphisms. The source map for this 2-vector space is given by $s'(x, \vec{f}) = x$, the target map is given by $t'(x, \vec{f}) = x + t(\vec{f})$, and the identity-assigning map is given by $i'(x) = (x, 0)$. We thus can define an isomorphism $\alpha_V\colon V' \to V$ by setting

$$\begin{aligned} (\alpha_V)_0(x) &= x, \\ (\alpha_V)_1(x, \vec{f}) &= i(x) + \vec{f}. \end{aligned}$$

It is easy to check that α_V is a linear functor. It is an isomorphism thanks to the fact, shown in the proof of Lemma 6, that every morphism in V can be uniquely written as $i(x) + \vec{f}$ where x is an object and $\vec{f} \in \ker(s)$.

To construct β, consider a 2-term chain complex, C, given by

$$C_1 \xrightarrow{d} C_0.$$

Then $T(C)$ is the 2-vector space with the space C_0 of objects, the space $C_0 \oplus C_1$ of morphisms, together with the source and target maps $s\colon (x, \vec{f}) \mapsto x$, $t\colon (x, \vec{f}) \mapsto x + d\vec{f}$ and the identity-assigning map $i\colon x \mapsto (x, 0)$. Applying the functor S to this 2-vector space we obtain a 2-term chain complex C' given by:

$$\ker(s) \xrightarrow{t|_{\ker(s)}} C_0.$$

Since $\ker(s) = \{(x, \vec{f})|x = 0\} \subseteq C_0 \oplus C_1$, there is an obvious isomorphism $ker(s) \cong C_1$. Using this we obtain an isomorphism $\beta_C\colon C' \to C$ given by:

$$\begin{array}{ccc} \ker(s) & \xrightarrow{t|_{\ker(s)}} & C_0 \\ \big\downarrow{\scriptstyle \sim} & & \big\downarrow{\scriptstyle 1} \\ C_1 & \xrightarrow{d} & C_0 \end{array}$$

where the square commutes because of how we have defined t.

We leave it to the reader to verify that α and β are indeed natural isomorphisms. $\square$

As mentioned in the Introduction, the idea behind Proposition 8 goes back at least to Grothendieck [21], who showed that groupoids in the category of abelian groups are equivalent to 2-term chain complexes of abelian groups. There are many elaborations of this idea, some of which we will mention later, but for now the only one we really *need* involves making 2Vect and 2Term into 2-categories and showing that they are 2-equivalent as 2-categories. To do this, we require the notion of a 'linear natural transformation' between linear functors. This will correspond to a chain homotopy between chain maps.

Definition 9. *Given two linear functors $F, G\colon V \to W$ between 2-vector spaces, a* **linear natural transformation** $\alpha\colon F \Rightarrow G$ *is a natural transformation in* Vect*.*

Definition 10. *We define* **2Vect** *to be* VectCat*, or in other words, the 2-category of 2-vector spaces, linear functors and linear natural transformations.*

Recall that in general, given two chain maps $\phi, \psi\colon C \to C'$, a **chain homotopy** $\tau\colon \phi \Rightarrow \psi$ is a family of linear maps $\tau\colon C_p \to C'_{p+1}$ such that $\tau_p d'_{p+1} + d_p \tau_{p-1} = \psi_p - \phi_p$ for all p. In the case of 2-term chain complexes, a chain homotopy amounts to a map $\tau\colon C_0 \to C'_1$ satisfying $\tau d' = \psi_0 - \phi_0$ and $d\tau = \psi_1 - \phi_1$.

Definition 11. *We define* **2Term** *to be the 2-category of 2-term chain complexes, chain maps, and chain homotopies.*

We will continue to sometimes use 2Term and 2Vect to stand for the underlying categories of these (strict) 2-categories. It will be clear by context whether we mean the category or the 2-category.

The next result strengthens Proposition 8.

Theorem 12. *The 2-category* 2Vect *is 2-equivalent to the 2-category* 2Term*.*

Proof. We begin by constructing 2-functors

$$S\colon \mathrm{2Vect} \to \mathrm{2Term}$$

and

$$T\colon \mathrm{2Term} \to \mathrm{2Vect}.$$

By Proposition 8, we need only to define S and T on 2-morphisms. Let V and V' be 2-vector spaces, $F, G\colon V \to V'$ linear functors, and $\theta\colon F \Rightarrow G$ a linear natural transformation. Then we define the chain homotopy $S(\theta)\colon S(F) \Rightarrow S(G)$ by

$$S(\theta)(x) = \vec{\theta}_x,$$

using the fact that a 0-chain x of $S(V)$ is the same as an object x of V. Conversely, let C and C' be 2-term chain complexes, $\phi, \psi\colon C \to C'$ chain maps and $\tau\colon \phi \Rightarrow \psi$ a chain homotopy. Then we define the linear natural transformation $T(\tau)\colon T(\phi) \Rightarrow T(\psi)$ by

$$T(\tau)(x) = (\phi_0(x), \tau(x)),$$

where we use the description of a morphism in $S(C')$ as a pair consisting of its source and its arrow part, which is a 1-chain in C'. We leave it to the reader to check that S is really a chain homotopy, T is really a linear natural transformation, and that the natural isomorphisms $\alpha\colon ST \Rightarrow 1_{\mathrm{2Vect}}$ and $\beta\colon TS \Rightarrow 1_{\mathrm{2Term}}$ defined in the proof of Proposition 8 extend to this 2-categorical context. □

We conclude this section with a little categorified linear algebra. We consider the direct sum and tensor product of 2-vector spaces.

Proposition 13. *Given 2-vector spaces $V = (V_0, V_1, s, t, i, \circ)$ and $V' = (V'_0, V'_1, s', t', i', \circ')$, there is a 2-vector space $V \oplus V'$ having:*

- *$V_0 \oplus V'_0$ as its vector space of objects,*
- *$V_1 \oplus V'_1$ as its vector space of morphisms,*
- *$s \oplus s'$ as its source map,*
- *$t \oplus t'$ as its target map,*
- *$i \oplus i'$ as its identity-assigning map, and*
- *$\circ \oplus \circ'$ as its composition map.*

Proof. The proof amounts to a routine verification that the diagrams in Definition 1 commute. □

Proposition 14. *Given 2-vector spaces $V = (V_0, V_1, s, t, i, \circ)$ and $V' = (V'_0, V'_1, s', t', i', \circ')$, there is a 2-vector space $V \otimes V'$ having:*

- *$V_0 \otimes V'_0$ as its vector space of objects,*

- $V_1 \otimes V_1'$ *as its vector space of morphisms,*
- $s \otimes s'$ *as its source map,*
- $t \otimes t'$ *as its target map,*
- $i \otimes i'$ *as its identity-assigning map, and*
- $\circ \otimes \circ'$ *as its composition map.*

Proof. Again, the proof is a routine verification. $\square$

We now check the correctness of the above definitions by showing the universal properties of the direct sum and tensor product. These universal properties only require the category structure of 2Vect, not its 2-category structure, since the necessary diagrams commute 'on the nose' rather than merely up to a 2-isomorphism, and uniqueness holds up to isomorphism, not just up to equivalence. The direct sum is what category theorists call a 'biproduct': both a product and coproduct, in a compatible way [26]:

Proposition 15. *The direct sum $V \oplus V'$ is the biproduct of the 2-vector spaces V and V', with the obvious inclusions*

$$i\colon V \to V \oplus V', \qquad i'\colon V' \to V \oplus V'$$

and projections

$$p\colon V \oplus V' \to V, \qquad p'\colon V \oplus V' \to V'.$$

Proof. A routine verification. $\square$

Since the direct sum $V \oplus V'$ is a product in the categorical sense, we may also denote it by $V \times V'$, as we do now in defining a 'bilinear functor', which is used in stating the universal property of the tensor product:

Definition 16. *Let V, V', and W be 2-vector spaces. A* **bilinear functor** $F\colon V \times V' \to W$ *is a functor such that the underlying map on objects*

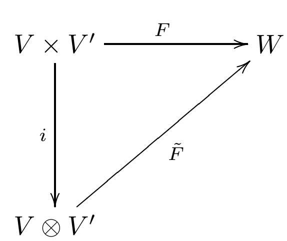
$$F_0\colon V_0 \times V_0' \to W_0$$

and the underlying map on morphisms

$$F_1\colon V_1 \times V_1' \to W_1$$

are bilinear.

Proposition 17. *Let V, V', and W be 2-vector spaces. Given a bilinear functor $F\colon V \times V' \to W$ there exists a unique linear functor $\tilde{F}\colon V \otimes V' \to W$ such that*



commutes, where $i\colon V \times V' \to V \otimes V'$ *is given by* $(v, w) \mapsto v \otimes w$ *for* $(v, w) \in (V \times V')_0$ *and* $(f, g) \mapsto f \otimes g$ *for* $(f, g) \in (V \times V')_1$*.*

Proof. The existence and uniqueness of $\tilde{F}_0\colon (V \otimes V')_0 \to W_0$ and $\tilde{F}_1\colon (V \otimes V')_1 \to W_1$ follow from the universal property of the tensor product of vector spaces, and it is then straightforward to check that $\tilde{F}$ is a linear functor. □

We can also form the tensor product of linear functors. Given linear functors $F\colon V \to V'$ and $G\colon W \to W'$, we define $F \otimes G\colon V \otimes V' \to W \otimes W'$ by setting:

$$\begin{aligned} (F \otimes G)_0 &= F_0 \otimes G_0, \\ (F \otimes G)_1 &= F_1 \otimes G_1. \end{aligned}$$

Furthermore, there is an 'identity object' for the tensor product of 2-vector spaces. In Vect, the ground field k acts as the identity for tensor product: there are canonical isomorphisms $k \otimes V \cong V$ and $V \otimes k \cong V$. For 2-vector spaces, a categorified version of the ground field plays this role:

Proposition 18. *There exists a unique* 2*-vector space* K*, the* **categorified ground field***, with* $K_0 = K_1 = k$ *and* $s, t, i = 1_k$*.*

Proof. Lemma 6 implies that there is a unique way to define composition in K making it into a 2-vector space. In fact, every morphism in K is an identity morphism. □

Proposition 19. *Given any* 2*-vector space* V*, there is an isomorphism* $\ell_V : K \otimes V \to V$*, which is defined on objects by* $a \otimes v \mapsto av$ *and on morphisms by* $a \otimes f \mapsto af$*. There is also an isomorphism* $r_V : V \otimes K \to V$*, defined similarly.*

Proof. This is straightforward. □

The functors ℓ_V and r_V are a categorified version of left and right multiplication by scalars. Our 2-vector spaces also have a categorified version of addition, namely a linear functor

$$+\colon V \oplus V \to V$$

mapping any pair (x, y) of objects or morphisms to $x + y$. Combining this with scalar multiplication by the object $-1 \in K$, we obtain another linear functor

$$-\colon V \oplus V \to V$$

mapping (x, y) to $x - y$. This is the sense in which our 2-vector spaces are equipped with a categorified version of subtraction. All the usual rules governing addition of vectors, subtraction of vectors, and scalar multiplication hold 'on the nose' as equations.

One can show that with the above tensor product, the category 2Vect becomes a symmetric monoidal category. One can go further and make the 2-category version of 2Vect into a symmetric monoidal 2-category [17], but we will not need this here. Now that we have a definition of 2-vector space and some basic tools of categorified linear algebra we may proceed to the main focus of this paper: the definition of a categorified Lie algebra.

4 Semistrict Lie 2-algebras

4.1 Definitions

We now introduce the concept of a 'Lie 2-algebra', which blends together the notion of a Lie algebra with that of a category. As mentioned previously, to obtain a Lie 2-algebra we begin with a 2-vector space and equip it with a bracket *functor*, which satisfies the Jacobi identity *up to a natural isomorphism*, the 'Jacobiator'. Then we require that the Jacobiator satisfy a new coherence law of its own, the 'Jacobiator identity'. We shall assume the bracket is bilinear in the sense of Definition 16, and also skew-symmetric:

Definition 20. *Let V and W be 2-vector spaces. A bilinear functor $F\colon V \times V \to W$ is* **skew-symmetric** *if $F(x,y) = -F(y,x)$ whenever (x,y) is an object or morphism of $V \times V$. If this is the case we also say the corresponding linear functor $\tilde{F}\colon V \otimes V \to W$ is skew-symmetric.*

We shall also assume that the Jacobiator is trilinear and completely antisymmetric:

Definition 21. *Let V and W be 2-vector spaces. A functor $F\colon V^n \to W$ is* **n-linear** *if $F(x_1,\ldots,x_n)$ is linear in each argument, where $(x_1,\ldots,x_n)$ is an object or morphism of V^n. Given n-linear functors $F,G\colon V^n \to W$, a natural transformation $\theta\colon F \Rightarrow G$ is* **n-linear** *if $\theta_{x_1,\ldots,x_n}$ depends linearly on each object x_i, and* **completely antisymmetric** *if the arrow part of $\theta_{x_1,\ldots,x_n}$ is completely antisymmetric under permutations of the objects.*

Since we do not weaken the bilinearity or skew-symmetry of the bracket, we call the resulting sort of Lie 2-algebra 'semistrict':

Definition 22. *A* **semistrict Lie 2-algebra** *consists of:*

- *a 2-vector space L*

equipped with

- *a skew-symmetric bilinear functor, the* **bracket**, $[\cdot,\cdot]\colon L \times L \to L$
- *a completely antisymmetric trilinear natural isomorphism, the* **Jacobiator**,

$$J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y],$$

that is required to satisfy

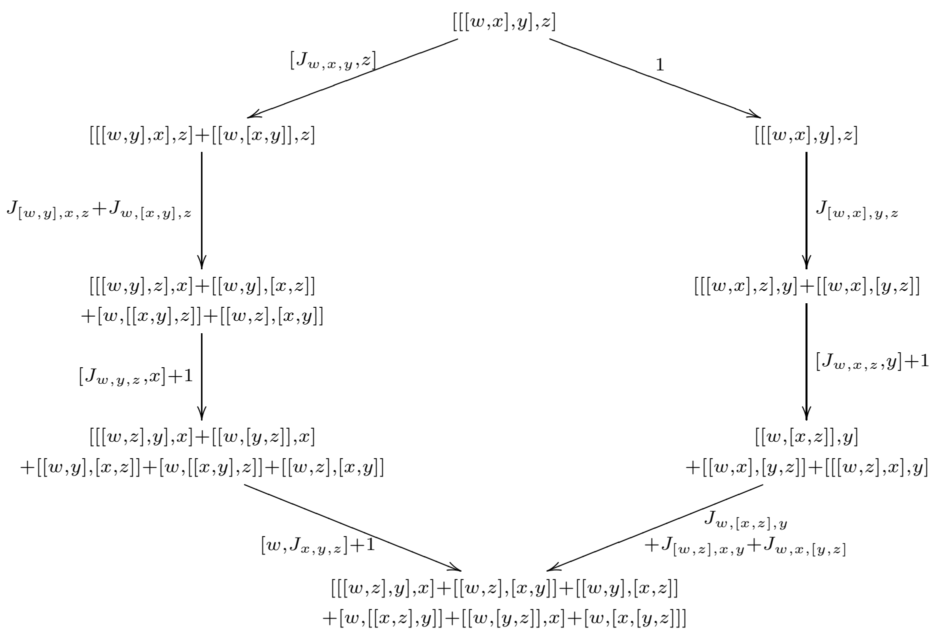
- *the* **Jacobiator identity**:

$$J_{[w,x],y,z}([J_{w,x,z},y]+1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) =$$

$$[J_{w,x,y},z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z},x]+1)([w,J_{x,y,z}]+1)$$

for all $w, x, y, z \in L_0$. (There is only one choice of identity morphism that can be added to each term to make the composite well-defined.)

The Jacobiator identity looks quite intimidating at first. But if we draw it as a commutative diagram, we see that it relates two ways of using the Jacobiator to rebracket the expression $[[[w, x], y], z]$:



Here the identity morphisms come from terms on which we are not performing any manipulation. The reader will surely be puzzled by the fact that we have included an identity morphism along one edge of this commutative octagon. This is explained in the next section, where we show that the Jacobiator identity is really just a disguised version of the 'Zamolodchikov tetrahedron equation', which plays an important role in the theory of higher-dimensional knots and braided monoidal 2-categories [7, 16, 17, 22]. The Zamolochikov tetrahedron equation says that two 2-morphisms are equal, each of which is the vertical composite of four factors. However, when we translate this equation into the language of Lie 2-algebras, one of these factors is an identity 2-morphism.

In the rest of this paper, the term 'Lie 2-algebra' will always refer to a semistrict one as defined above. We continue by setting up a 2-category of these Lie 2-algebras. A homomorphism between Lie 2-algebras should preserve both the 2-vector space structure and the bracket. However, we shall require that it preserve the bracket only *up to isomorphism* — or more precisely, up to a natural isomorphism satisfying a suitable coherence law. Thus, we make the following definition.

Definition 23. *Given Lie* 2*-algebras* L *and* L'*, a* **homomorphism** $F\colon L \to L'$ *consists of:*

- *A linear functor* F *from the underlying* 2*-vector space of* L *to that of* L'*, and*
- *a skew-symmetric bilinear natural transformation*

$$F_2(x,y)\colon [F_0(x), F_0(y)] \to F_0[x,y]$$

such that the following diagram commutes:

$$\begin{array}{ccc}
[[F_0(x),F_0(y)],F_0(z)] & \xrightarrow{J_{F_0(x),F_0(y),F_0(z)}} & [F_0(x),[F_0(y),F_0(z)]] + [[F_0(x),F_0(z)],F_0(y)] \\
\Big\downarrow {\scriptstyle [F_2,1]} & & \Big\downarrow {\scriptstyle [1,F_2]+[F_2,1]} \\
[F_0[x,y],F_0(z)] & & [F_0(x),F_0[y,z]] + [F_0[x,z],F_0(y)] \\
\Big\downarrow {\scriptstyle F_2} & & \Big\downarrow {\scriptstyle F_2+F_2} \\
F_0[[x,y],z] & \xrightarrow{F_1(J_{x,y,z})} & F_0[x,[y,z]] + F_0[[x,z],y]
\end{array}$$

Here and elsewhere we omit the arguments of natural transformations such as F_2 and G_2 when these are obvious from context.

We also have '2-homomorphisms' between homomorphisms:

Definition 24. *Let* $F, G\colon L \to L'$ *be Lie 2-algebra homomorphisms. A* **2-homomorphism** $\theta\colon F \Rightarrow G$ *is a linear natural transformation from* F *to* G *such that the following diagram commutes:*

$$\begin{array}{ccc}
[F_0(x),F_0(y)] & \xrightarrow{F_2} & F_0[x,y] \\
\Big\downarrow {\scriptstyle [\theta_x,\theta_y]} & & \Big\downarrow {\scriptstyle \theta_{[x,y]}} \\
[G_0(x),G_0(y)] & \xrightarrow{G_2} & G_0[x,y]
\end{array}$$

Definitions 23 and 24 are closely modelled after the usual definitions of 'monoidal functor' and 'monoidal natural transformation' [26].

Next we introduce composition and identities for homomorphisms and 2-homomorphisms. The composite of a pair of Lie 2-algebra homomorphisms

$F\colon L \to L'$ and $G\colon L' \to L''$ is given by letting the functor $FG\colon L \to L''$ be the usual composite of F and G:

$$L \xrightarrow{F} L' \xrightarrow{G} L''$$

while letting $(FG)_2$ be defined as the following composite:

$$\begin{array}{ccc} [(FG)_0(x), (FG)_0(y)] & \xrightarrow{(FG)_2} & (FG)_0[x, y] \; . \\ \big\downarrow{\scriptstyle G_2} & \nearrow{\scriptstyle F_2 \circ G} & \\ G_0[F_0(x), F_0(y)] & & \end{array}$$

where $F_2 \circ G$ is the result of whiskering the functor G by the natural transformation F_2. The identity homomorphism $1_L\colon L \to L$ has the identity functor as its underlying functor, together with an identity natural transformation as $(1_L)_2$. Since 2-homomorphisms are just natural transformations with an extra property, we vertically and horizontally compose these the usual way, and an identity 2-homomorphism is just an identity natural transformation. We obtain:

Proposition 25. *There is a strict 2-category* **Lie2Alg** *with semistrict Lie* 2-*algebras as objects, homomorphisms between these as morphisms, and* 2-*homomorphisms between those as 2-morphisms, with composition and identities defined as above.*

Proof. We leave it to the reader to check the details, including that the composite of homomorphisms is a homomorphism, this composition is associative, and the vertical and horizontal composites of 2-homomorphisms are again 2-homomorphisms. $\Box$

Finally, note that there is a forgetful 2-functor from Lie2Alg to 2Vect, which is analogous to the forgetful functor from LieAlg to Vect.

4.2 Relation to Topology

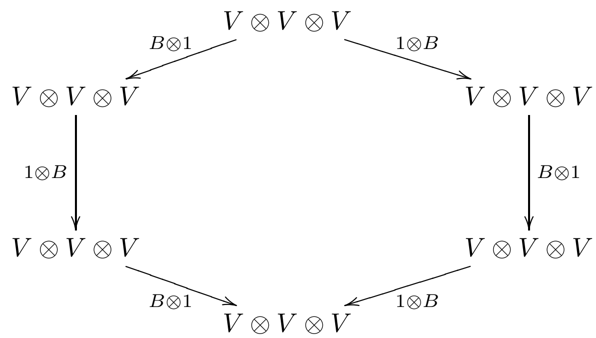
The key novel feature of a Lie 2-algebra is the coherence law for the Jacobiator: the so-called 'Jacobiator identity' in Definition 22. At first glance this identity seems rather arcane. In this section, we 'explain' this identity by showing its relation to the Zamolodchikov tetrahedron equation. This equation plays a role in the theory of knotted surfaces in 4-space which is closely analogous to that played by the Yang–Baxter equation, or third Reidemeister move, in the theory of ordinary knots in 3-space. In fact, we shall see that just as any Lie algebra gives a solution of the Yang–Baxter equation, any Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation.

We begin by recalling the Yang–Baxter equation:

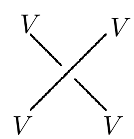
Definition 26. *Given a vector space V and an isomorphism $B\colon V \otimes V \to V \otimes V$, we say B is a* **Yang–Baxter operator** *if it satisfies the* **Yang–Baxter equation***, which says that:*

$$(B \otimes 1)(1 \otimes B)(B \otimes 1) = (1 \otimes B)(B \otimes 1)(1 \otimes B),$$

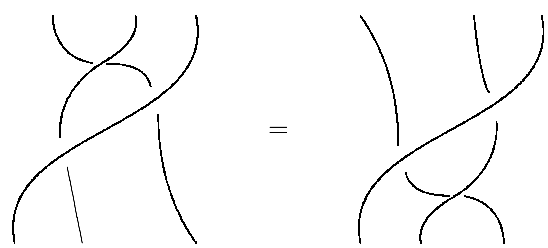
or in other words, that this diagram commutes:



If we draw $B\colon V \otimes V \to V \otimes V$ as a braiding:

$B =$ 

the Yang–Baxter equation says that:



This is called the 'third Reidemeister move' in knot theory [14], and it gives the most important relations in Artin's presentation of the braid group [9]. As a result, any solution of the Yang–Baxter equation gives an invariant of braids.

In general, almost any process of switching the order of two things can be thought of as a 'braiding'. This idea is formalized in the concept of a braided monoidal category, where the braiding is an isomorphism

$$B_{x,y}\colon x \otimes y \to y \otimes x.$$

Since the bracket $[x, y]$ in a Lie algebra measures the difference between xy and yx, it should not be too surprising that we can get a Yang–Baxter operator from any Lie algebra. And since the third Reidemeister move involves three strands, while the Jacobi identity involves three Lie algebra elements, it should also not be surprising that the Yang–Baxter equation is actually *equivalent* to the Jacobi identity in a suitable context:

Proposition 27. *Let L be a vector space equipped with a skew-symmetric bilinear operation $[\cdot,\cdot]\colon L \times L \to L$. Let $L' = k \oplus L$ and define the isomorphism $B\colon L' \otimes L' \to L' \otimes L'$ by*

$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y]).$$

Then B is a solution of the Yang–Baxter equation if and only if $[\cdot,\cdot]$ satisfies the Jacobi identity.

Proof. The proof is a calculation best left to the reader. □

The nice thing is that this result has a higher-dimensional analogue, obtained by categorifying everything in sight! The analogue of the Yang–Baxter equation is called the 'Zamolodchikov tetrahedron equation':

Definition 28. *Given a 2-vector space V and an invertible linear functor $B\colon V \otimes V \to V \otimes V$, a linear natural isomorphism*

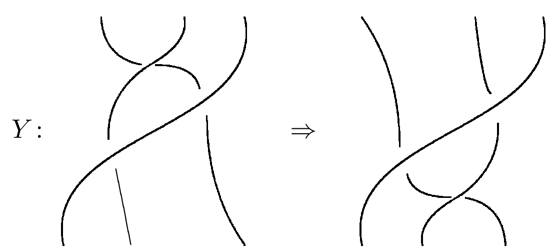
$$Y\colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

satisfies the **Zamolodchikov tetrahedron equation** *if*

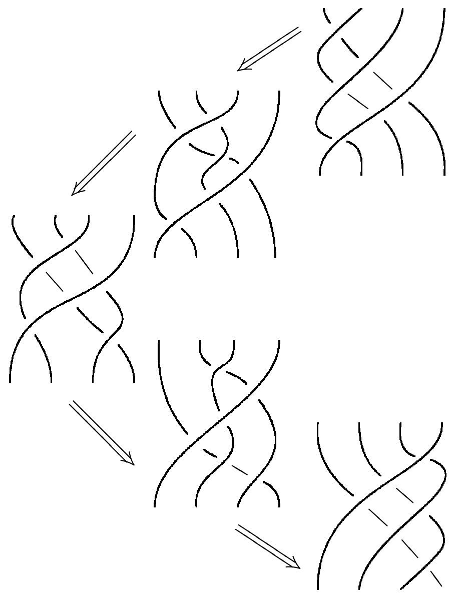
$$[(Y{\otimes}1)\circ(1{\otimes}1{\otimes}B)(1{\otimes}B{\otimes}1)(B{\otimes}1{\otimes}1)][(1{\otimes}B{\otimes}1)(B{\otimes}1{\otimes}1)\circ(1{\otimes}Y)\circ(B{\otimes}1{\otimes}1)]$$
$$[(1{\otimes}B{\otimes}1)(1{\otimes}1{\otimes}B)\circ(Y{\otimes}1)\circ(1{\otimes}1{\otimes}B)][(1{\otimes}Y)\circ(B{\otimes}1{\otimes}1)(1{\otimes}B{\otimes}1)(1{\otimes}1{\otimes}B)]$$
$$=$$
$$[(B{\otimes}1{\otimes}1)(1{\otimes}B{\otimes}1)(1{\otimes}1{\otimes}B)\circ(Y{\otimes}1)][(B{\otimes}1{\otimes}1)\circ(1{\otimes}Y)\circ(B{\otimes}1{\otimes}1)(1{\otimes}B{\otimes}1)]$$
$$[(1{\otimes}1{\otimes}B)\circ(Y{\otimes}1)\circ(1{\otimes}1{\otimes}B)(1{\otimes}B{\otimes}1)][(1{\otimes}1{\otimes}B)(1{\otimes}B{\otimes}1)(B{\otimes}1{\otimes}1)\circ(1{\otimes}Y)],$$

where $\circ$ represents the whiskering of a linear functor by a linear natural transformation.

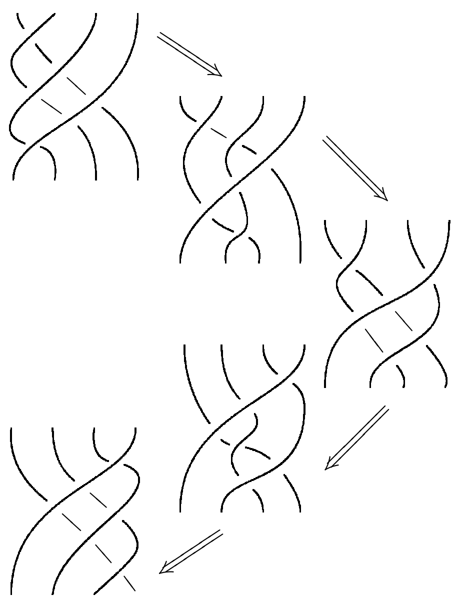
To see the significance of this complex but beautifully symmetrical equation, one should think of Y as the surface in 4-space traced out by the *process of performing* the third Reidemeister move:



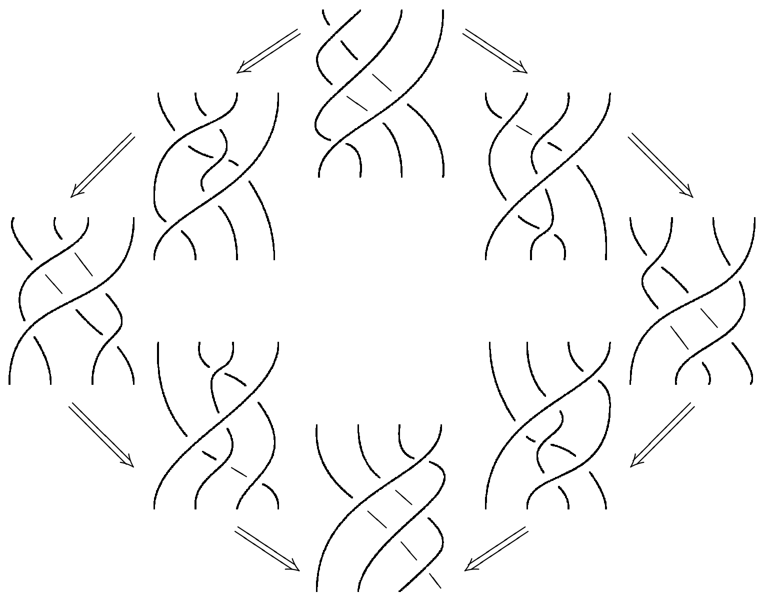
Then the Zamolodchikov tetrahedron equation says the surface traced out by first performing the third Reidemeister move on a threefold crossing and then sliding the result under a fourth strand:



is isotopic to that traced out by first sliding the threefold crossing under the fourth strand and then performing the third Reidemeister move:



In short, the Zamolodchikov tetrahedron equation is a formalization of this commutative octagon:



in a 2-category whose 2-morphisms are isotopies of surfaces in 4-space — or more precisely, '2-braids'. Details can be found in HDA1, HDA4 and a number of other references, going back to the work of Kapranov and Voevodsky [5, 7, 15, 16, 22].

In Section 4.1, we drew the Jacobiator identity as a commutative octagon. In fact, that commutative octagon becomes *equivalent* to the octagon for the Zamolodchikov tetrahedron equation in the following context:

Theorem 29. *Let L be a 2-vector space, let $[\cdot,\cdot]\colon L \times L \to L$ be a skew-symmetric bilinear functor, and let J be a completely antisymmetric trilinear natural transformation with $J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y]$. Let $L' = K \oplus L$, where K is the categorified ground field. Let $B\colon L' \otimes L' \to L' \otimes L'$ be defined as follows:*

$$B((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y]).$$

whenever (a,x) and (b,y) are both either objects or morphisms in L'. Finally, let

$$Y\colon (B \otimes 1)(1 \otimes B)(B \otimes 1) \Rightarrow (1 \otimes B)(B \otimes 1)(1 \otimes B)$$

be defined as follows:

$$Y = (p \otimes p \otimes p) \circ J \circ j$$

where $p\colon L' \to L$ is the projection functor given by the fact that $L' = K \oplus L$ and

$$j\colon L \to L' \otimes L' \otimes L'$$

is the linear functor defined by

$$j(x) = (1,0) \otimes (1,0) \otimes (0,x),$$

where x is either an object or morphism of L. Then Y is a solution of the Zamolodchikov tetrahedron equation if and only if J satisfies the Jacobiator identity.

Proof. Equivalently, we must show that Y satisfies the Zamolodchikov tetrahedron equation if and only if J satisfies the Jacobiator identity. Applying the left-hand side of the Zamolodchikov tetrahedron equation to an object $(a,w) \otimes (b,x) \otimes (c,y) \otimes (d,z)$ of $L' \otimes L' \otimes L' \otimes L'$ yields an expression consisting of various uninteresting terms together with one involving

$$J_{[w,x],y,z}([J_{w,x,z}, y] + 1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}),$$

while applying the right-hand side produces an expression with the same uninteresting terms, but also one involving

$$[J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1)$$

in precisely the same way. Thus, the two sides are equal if and only if the Jacobiator identity holds. The detailed calculation is quite lengthy. □

Corollary 30. *If L is a Lie 2-algebra, then Y defined as in Theorem 29 is a solution of the Zamolodchikov tetrahedron equation.*

We continue by exhibiting the correlation between our semistrict Lie 2-algebras and special versions of Stasheff's L_∞-algebras.

4.3 L_∞-algebras

An L_∞-algebra is a chain complex equipped with a bilinear skew-symmetric bracket operation that satisfies the Jacobi identity 'up to coherent homotopy'. In other words, this identity holds up to a specified chain homotopy, which in turn satisfies its own identity up to a specified chain homotopy, and so on *ad infinitum*. Such structures are also called are 'strongly homotopy Lie algebras' or 'sh Lie algebras' for short. Though their precursors existed in the literature beforehand, they made their first notable appearance in a 1985 paper on deformation theory by Schlessinger and Stasheff [33]. Since then, they have been systematically explored and applied in a number of other contexts [23, 24, 28, 31].

Since 2-vector spaces are equivalent to 2-term chain complexes, as described in Section 3, it should not be surprising that L_∞-algebras are related to the categorified Lie algebras we are discussing here. An elegant but rather highbrow way to approach this is to use the theory of operads [29]. An L_∞-algebra is actually an algebra of a certain operad in the symmetric monoidal category of chain complexes, called the 'L_∞ operad'. Just as categories in Vect are equivalent to 2-term chain complexes, strict ω-categories in Vect can be shown equivalent to general chain complexes, by a similar argument [12]. Using this equivalence, we can transfer the L_∞ operad from the world of chain complexes to the world of strict ω-category objects in Vect, and define a **semistrict Lie ω-algebra** to be an algebra of the resulting operad.

In more concrete terms, a semistrict Lie ω-algebra is a strict ω-category L having a vector space L_j of j-morphisms for all $j \geq 0$, with all source, target and composition maps being linear. Furthermore, it is equipped with a skew-symmetric bilinear bracket functor

$$[\cdot,\cdot]\colon L \times L \to L$$

which satisfies the Jacobi identity up to a completely antisymmetric trilinear natural isomorphism, the 'Jacobiator', which in turn satisfies the Jacobiator identity up to a completely antisymmetric quadrilinear modification... and so on. By the equivalence mentioned above, such a thing is really just *another way of looking at* an L_∞-algebra.

Using this, one can show that a semistrict Lie ω-algebra with only identity j-morphisms for $j > 1$ is the same as a semistrict Lie 2-algebra! But luckily, we can prove a result along these lines without using or even mentioning the concepts of 'operad', 'ω-category' and the like. Instead, for the sake of an accessible presentation, we shall simply recall the definition of an L_∞-algebra and prove that the 2-category of semistrict Lie 2-algebras is equivalent to a 2-category of '2-term' L_∞-algebras: that is, those having a zero-dimensional space of j-chains for $j > 1$.

Henceforth, all algebraic objects mentioned are considered over a fixed field k of characteristic other than 2. We make consistent use of the usual sign convention when dealing with graded objects. That is, whenever we interchange something of degree p with something of degree q, we introduce a sign of $(-1)^{pq}$. The following conventions regarding graded vector spaces, permutations, unshuffles, etc., follow those of Lada and Markl [23].

For graded indeterminates $x_1, \ldots, x_n$ and a permutation $\sigma \in S_n$ we define the **Koszul sign** $\epsilon(\sigma) = \epsilon(\sigma; x_1, \ldots, x_n)$ by

$$x_1 \wedge \cdots \wedge x_n = \epsilon(\sigma; x_1, \ldots, x_n) \cdot x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)},$$

which must be satisfied in the free graded-commutative algebra on $x_1, \ldots, x_n$. This is nothing more than a formalization of what has already been said above. Furthermore, we define

$$\chi(\sigma) = \chi(\sigma; x_1, \ldots, x_n) := \mathrm{sgn}(\sigma) \cdot \epsilon(\sigma; x_1, \ldots, x_n).$$

Thus, $\chi(\sigma)$ takes into account the sign of the permutation in S_n and the sign obtained from iteration of the basic convention.

If n is a natural number and $1 \leq j \leq n-1$ we say that $\sigma \in S_n$ is an $(j, n-j)$**-unshuffle** if

$$\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(j) \quad \text{ and } \quad \sigma(j+1) \leq \sigma(j+2) \leq \cdots \leq \sigma(n).$$

Readers familiar with shuffles will recognize unshuffles as their inverses. A *shuffle* of two ordered sets (such as a deck of cards) is a permutation of the ordered union preserving the order of each of the given subsets. An *unshuffle* reverses this process. A simple example should clear up any confusion:

Example 31. *When $n = 3$, the $(1, 2)$-unshuffles in S_3 are:*

$$\mathrm{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \text{and} \quad (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

The following definition of an L_∞-structure was formulated by Stasheff in 1985, see [33]. This definition will play an important role in what will follow.

Definition 32. *An* $\mathbf{L_\infty}$**-algebra** *is a graded vector space V equipped with a system $\{l_k | 1 \le k < \infty\}$ of linear maps $l_k \colon V^{\otimes k} \to V$ with $\deg(l_k) = k - 2$ which are totally antisymmetric in the sense that*

$$l_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \chi(\sigma) l_k(x_1, \ldots, x_n) \tag{1}$$

for all $\sigma \in S_n$ and $x_1, \ldots, x_n \in V$, and, moreover, the following generalized form of the Jacobi identity holds for $0 \le n < \infty$:

$$\sum_{i+j=n+1} \sum_{\sigma} \chi(\sigma)(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \tag{2}$$

where the summation is taken over all $(i, n-i)$-unshuffles with $i \ge 1$.

While somewhat puzzling at first, this definition truly does combine the important aspects of Lie algebras and chain complexes. The map l_1 makes V into a chain complex, since this map has degree -1 and equation (2) says its square is zero. Moreover, the map l_2 resembles a Lie bracket, since it is skew-symmetric in the graded sense by equation (1). In what follows, we usually denote $l_1(x)$ as dx and $l_2(x, y)$ as $[x, y]$. The higher l_k maps are related to the Jacobiator, the Jacobiator identity, and the higher coherence laws that would appear upon further categorification of the Lie algebra concept.

To make this more precise, let us refer to an L_∞-algebra V with $V_n = 0$ for $n \ge k$ as a ***k*-term** $\mathbf{L_\infty}$**-algebra.** Note that a 1-term L_∞-algebra is simply an ordinary Lie algebra, where $l_3 = 0$ gives the Jacobi identity. However, in a 2-term L_∞-algebra, we no longer have a trivial l_3 map. Instead, equation (2) says that the Jacobi identity for the 0-chains x, y, z holds up to a term of the form $dl_3(x, y, z)$. We do, however, have $l_4 = 0$, which provides us with the coherence law that l_3 must satisfy.

Since we will be making frequent use of these 2-term L_∞-algebras, it will be advantageous to keep track of their ingredients.

Lemma 33. *A 2-term L_∞-algebra, V, consists of the following data:*

- *two vector spaces V_0 and V_1 together with a linear map $d \colon V_1 \to V_0$,*

- *a bilinear map* $l_2 \colon V_i \times V_j \to V_{i+j}$*, where* $0 \leq i+j \leq 1$*, which we denote more suggestively as* $[\cdot,\cdot]$*,*

- *a trilinear map* $l_3 \colon V_0 \times V_0 \times V_0 \to V_1$*.*

These maps satisfy:

(a) $[x,y] = -[y,x]$*,*

(b) $[x,h] = -[h,x]$*,*

(c) $[h,k] = 0$*,*

(d) $l_3(x,y,z)$ *is totally antisymmetric in the arguments* x,y,z*,*

(e) $d([x,h]) = [x,dh]$*,*

(f) $[dh,k] = [h,dk]$*,*

(g) $d(l_3(x,y,z)) = -[[x,y],z] + [[x,z],y] + [x,[y,z]]$*,*

(h) $l_3(dh,x,y) = -[[x,y],h] + [[x,h],y] + [x,[y,h]]$*,*

(i) $[l_3(w,x,y),z] + [l_3(w,y,z),x] + l_3([w,y],x,z) + l_3([x,z],w,y) =$

$$[l_3(w,x,z),y] + [l_3(x,y,z),w] + l_3([w,x],y,z) +$$

$$l_3([w,z],x,y) + l_3([x,y],w,z) + l_3([y,z],w,x),$$

for all $w,x,y,z \in V_0$ *and* $h,k \in V_1$*.*

Proof. Note that $(a)-(d)$ hold by equation (1) of Definition 32 while $(e)-(i)$ follow from (2). □

We notice that (a) and (b) are the usual skew-symmetric properties satisfied by the bracket in a Lie algebra; (c) arises simply because there are no 2-chains. Equations (e) and (f) tell us how the differential and bracket interact, while (g) says that the Jacobi identity no longer holds on the nose, but up to chain homotopy. We will use (g) to define the Jacobiator in the Lie 2-algebra corresponding to a 2-term L_∞-algebra. Equation (h) will give the naturality of the Jacobiator. Similarly, (i) will give the Jacobiator identity.

We continue by defining homomorphisms between 2-term L_∞-algebras:

Definition 34. *Let* V *and* V' *be* 2*-term* L_∞*-algebras. An* $\mathbf{L}_\infty$***-homomorphism*** $\phi \colon V \to V'$ *consists of:*

- *a chain map* $\phi \colon V \to V'$ *(which consists of linear maps* $\phi_0 \colon V_0 \to V_0'$ *and* $\phi_1 \colon V_1 \to V_1'$ *preserving the differential),*

- *a skew-symmetric bilinear map* $\phi_2 \colon V_0 \times V_0 \to V_1'$*,*

such that the following equations hold for all $x,y,z \in V_0$*,* $h \in V_1$ *:*

- $d(\phi_2(x,y)) = \phi_0[x,y] - [\phi_0(x), \phi_0(y)]$
- $\phi_2(x, dh) = \phi_1[x,h] - [\phi_0(x), \phi_1(h)]$
- $[\phi_2(x,y), \phi_0(z)] + \phi_2([x,y],z) + \phi_1(l_3(x,y,z)) = l_3(\phi_0(x), \phi_0(y), \phi_0(z)) + [\phi_0(x), \phi_2(y,z)] + [\phi_2(x,z), \phi_0(y)] + \phi_2(x,[y,z]) + \phi_2([x,z],y)$

The reader should note the similarity between this definition and that of homomorphisms between Lie 2-algebras (Definition 23). In particular, the first two equations say that ϕ_2 defines a chain homotopy from $[\phi(\cdot), \phi(\cdot)]$ to $\phi[\cdot,\cdot]$, where these are regarded as chain maps from $V \otimes V$ to V'. The third equation in the above definition is just a chain complex version of the commutative square in Definition 23.

To make 2-term L_∞-algebras and L_∞-homomorphisms between them into a category, we must describe composition and identities. We compose a pair of L_∞-homomorphisms $\phi\colon V \to V'$ and $\psi\colon V' \to V''$ by letting the chain map $\phi\psi\colon V \to V''$ be the usual composite:

$$V \xrightarrow{\phi} V' \xrightarrow{\psi} V''$$

while defining $(\phi\psi)_2$ as follows:

$$(\phi\psi)_2(x,y) = \psi_2(\phi_0(x), \phi_0(y)) + \psi_1(\phi_2(x,y)).$$

This is just a chain complex version of how we compose homomorphisms between Lie 2-algebras. The identity homomorphism $1_V\colon V \to V$ has the identity chain map as its underlying map, together with $(1_V)_2 = 0$.

With these definitions, we obtain:

Proposition 35. *There is a category* $\mathbf{2TermL}_\infty$ *with 2-term L_∞-algebras as objects and L_∞-homomorphisms as morphisms.*

Proof. We leave this an exercise for the reader. □

Next we establish the equivalence between the category of Lie 2-algebras and that of 2-term L_∞-algebras. This result is based on the equivalence between 2-vector spaces and 2-term chain complexes described in Proposition 8.

Theorem 36. *The categories* Lie2Alg *and* $\mathrm{2TermL}_\infty$ *are equivalent.*

Proof. First we sketch how to construct a functor $T\colon \mathrm{2TermL}_\infty \to \mathrm{Lie2Alg}$. Given a 2-term L_∞-algebra V we construct the Lie 2-algebra $L = T(V)$ as follows.

We construct the underlying 2-vector space of L as in the proof of Proposition 8. Thus L has vector spaces of objects and morphisms

$$\begin{aligned} L_0 &= V_0, \\ L_1 &= V_0 \oplus V_1, \end{aligned}$$

and we denote a morphism $f\colon x \to y$ in L_1 by $f = (x, \vec{f})$ where $x \in V_0$ is the source of f and $\vec{f} \in V_1$ is its arrow part. The source, target, and identity-assigning maps in L are given by

$$\begin{aligned} s(f) &= s(x,\vec{f}) = x, \\ t(f) &= t(x,\vec{f}) = x + d\vec{f}, \\ i(x) &= (x, 0), \end{aligned}$$

and we have $t(f) - s(f) = d\vec{f}$. The composite of two morphisms in L_1 is given as in the proof of Lemma 6. That is, given $f = (x, \vec{f})\colon x \to y$, and $g = (y, \vec{g})\colon y \to z$, we have

$$fg := (x, \vec{f} + \vec{g}).$$

We continue by equipping $L = T(V)$ with the additional structure which makes it a Lie 2-algebra. First, we use the degree-zero chain map $l_2\colon V \otimes V \to V$ to define the bracket functor $[\cdot,\cdot]\colon L \times L \to L$. For a pair of objects $x, y \in L_0$ we define $[x, y] = l_2(x, y)$, where we use the 'l_2' notation in the L_∞-algebra V to avoid confusion with the bracket in L. The bracket functor is skew-symmetric and bilinear on objects since l_2 is. This is not sufficient, however. It remains to define the bracket functor on pairs of morphisms.

We begin by defining the bracket on pairs of morphisms where one morphism is an identity. We do this as follows: given a morphism $f = (x, \vec{f})\colon x \to y$ in L_1 and an object $z \in L_0$, we define

$$[1_z, f] := (l_2(z, x), l_2(z, \vec{f})),$$

$$[f, 1_z] := (l_2(x, z), l_2(\vec{f}, z)).$$

Clearly these morphisms have the desired sources; we now verify that they also have the desired targets. Using the fact that $t(f) = s(f) + d\vec{f}$ for any morphism $f \in L_1$, we see that:

$$\begin{aligned} t[1_z, f] &= s[1_z, f] + dl_2(z, \vec{f}) \\ &= l_2(z, x) + l_2(z, d\vec{f}) \quad \text{by } (e) \text{ of Lemma 33} \\ &= l_2(z, x) + l_2(z, y - x) \\ &= l_2(z, y) \end{aligned}$$

as desired, using the bilinearity of l_2. Similarly we have $t[f, 1_z] = l_2(y, z)$.

These definitions together with the desired functoriality of the bracket force us to define the bracket of an arbitrary pair of morphisms $f\colon x \to y$, $g\colon a \to b$ as follows:

$$\begin{aligned} [f, g] &= [f 1_y, 1_a g] \\ &:= [f, 1_a]\,[1_y, g] \\ &= (l_2(x, a), l_2(\vec{f}, a))\;(l_2(y, a), l_2(y, \vec{g})) \\ &= (l_2(x, a), l_2(\vec{f}, a) + l_2(y, \vec{g})). \end{aligned}$$

On the other hand, they also force us to define it as:

$$\begin{array}{rcl}[f,g] & = & [1_x f, g 1_b] \\ & := & [1_x, g]\,[f, 1_b] \\ & = & (l_2(x,a), l_2(x,\vec{g}))\;(l_2(x,b), l_2(\vec{f},b)) \\ & = & (l_2(x,a), l_2(x,\vec{g}) + l_2(\vec{f},b)).\end{array}$$

Luckily these are compatible! We have

$$l_2(\vec{f},a) + l_2(y,\vec{g}) = l_2(x,\vec{g}) + l_2(\vec{f},b) \qquad (3)$$

because the left-hand side minus the right-hand side equals $l_2(d\vec{f},\vec{g}) - l_2(\vec{f},d\vec{g})$, which vanishes by (f) of Lemma 33.

At this point we relax and use $[\cdot,\cdot]$ to stand both for the bracket on objects in L and the L_∞-algebra V. We will not, however, relax when it comes to the morphisms in L since $[\cdot,\cdot] \neq l_2(\cdot,\cdot)$ even on morphisms that are arrow parts, that is, morphisms in $\ker(s)$. By the above calculations, the bracket of morphisms $f\colon x \to y$, $g\colon a \to b$ in L is given by

$$\begin{array}{rcl}[f,g] & = & ([x,a], l_2(\vec{f},a) + l_2(y,\vec{g})) \\ & = & ([x,a], l_2(x,\vec{g}) + l_2(\vec{f},b)).\end{array}$$

The bracket $[\cdot,\cdot]\colon L \times L \to L$ is clearly bilinear on objects. Either of the above formulas shows it is also bilinear on morphisms, since the source, target and arrow part of a morphism depend linearly on the morphism, and the bracket in V is bilinear. The bracket is also skew-symmetric: this is clear for objects, and can be seen for morphisms if we use *both* the above formulas.

To show that $[\cdot,\cdot]\colon L \times L \to L$ is a functor, we must check that it preserves identities and composition. We first show that $[1_x, 1_y] = 1_{[x,y]}$, where $x, y \in L_0$. For this we use the fact that identity morphisms are precisely those with vanishing arrow part. Either formula for the bracket of morphisms gives

$$\begin{array}{rcl}[1_x, 1_y] & = & ([x,y], 0) \\ & = & 1_{[x,y]}.\end{array}$$

To show that the bracket preserves composition, consider the morphisms $f = (x, \vec{f})$, $f' = (y, \vec{f'})$, $g = (a, \vec{g})$, and $g' = (b, \vec{g'})$ in L_1, where $f\colon x \to y$, $f'\colon y \to z$, $g\colon a \to b$, and $g'\colon b \to c$. We must show

$$[ff', gg'] = [f,g][f',g'].$$

On the one hand, the definitions give

$$[ff', gg'] = ([x,a], l_2(\vec{f},a) + l_2(\vec{f'},a) + l_2(z,\vec{g}) + l_2(z,\vec{g'})),$$

while on the other, they give

$$[f,g][f',g'] = ([x,a], l_2(\vec{f},a) + l_2(y,\vec{g}) + l_2(\vec{f'},b) + l_2(z,\vec{g'}))$$

Therefore, it suffices to show that

$$l_2(\vec{f}', a) + l_2(z, \vec{g}) = l_2(y, \vec{g}) + l_2(\vec{f}', b).$$

After a relabelling of variables, this is just equation (3).

Next we define the Jacobiator for L and check its properties. We set

$$J_{x,y,z} := ([[x,y],z], l_3(x,y,z)).$$

Clearly the source of $J_{x,y,z}$ is $[[x,y],z]$ as desired, while its target is $[x,[y,z]] + [[x,z],y]$ by condition (g) of Lemma 33. To show $J_{x,y,z}$ is natural one can check that is natural in each argument. We check naturality in the third variable, leaving the other two as exercises for the reader. Let $f\colon z \to z'$. Then, $J_{x,y,z}$ is natural in z if the following diagram commutes:

$$\begin{array}{ccc} [[x,y],z] & \xrightarrow{[[1_x,1_y],f]} & [[x,y],z'] \\ \Big\downarrow{\scriptstyle J_{x,y,z}} & & \Big\downarrow{\scriptstyle J_{x,y,z'}} \\ [[x,z],y]+[x,[y,z]] & \xrightarrow{[[1_x,f],1_y]+[1_x,[1_y,f]]} & [[x,z'],y]+[x,[y,z']] \end{array}$$

Using the formula for the composition and bracket in L this means that we need

$$([[x,y],z], \vec{J}_{x,y,z'} + l_2([x,y],\vec{f})) = ([[x,y],z], l_2(l_2(x,\vec{f}),y) + (x, l_2(y,\vec{f})) + \vec{J}_{x,y,z}).$$

Thus, it suffices to show that

$$\vec{J}_{x,y,z'} + l_2([x,y],\vec{f}) = l_2(l_2(x,\vec{f}),y) + l_2(x, l_2(y,\vec{f})) + \vec{J}_{x,y,z}.$$

But $\vec{J}_{x,y,z}$ has been defined as $l_3(x,y,z)$ (and similarly for $\vec{J}_{x,y,z'}$), so now we are required to show that:

$$l_3(x,y,z') + l_2([x,y],\vec{f}) = l_3(x,y,z) + l_2(l_2(x,\vec{f}),y) + l_2(x, l_2(y,\vec{f})),$$

or in other words,

$$l_2([x,y],\vec{f}) + l_3(x,y,d\vec{f}) = l_2(l_2(x,\vec{f}),y) + l_2(x, l_2(y,\vec{f})).$$

This holds by condition (h) in Lemma 33 together with the complete antisymmetry of l_3.

The trilinearity and complete antisymmetry of the Jacobiator follow from the corresponding properties of l_3. Finally, condition (i) in Lemma 33 gives the Jacobiator identity:

$$J_{[w,x],y,z}([J_{w,x,z},y]+1)(J_{w,[x,z],y} + J_{[w,z],x,y} + J_{w,x,[y,z]}) =$$

$$[J_{w,x,y}, z](J_{[w,y],x,z} + J_{w,[x,y],z})([J_{w,y,z}, x] + 1)([w, J_{x,y,z}] + 1).$$

This completes the construction of a Lie 2-algebra $T(V)$ from any 2-term L_∞-algebra V. Next we construct a Lie 2-algebra homomorphism $T(\phi)\colon T(V) \to T(V')$ from any L_∞-homomorphism $\phi\colon V \to V'$ between 2-term L_∞-algebras.

Let $T(V) = L$ and $T(V') = L'$. We define the underlying linear functor of $T(\phi) = F$ as in Proposition 8. To make F into a Lie 2-algebra homomorphism we must equip it with a skew-symmetric bilinear natural transformation F_2 satisfying the conditions in Definition 23. We do this using the skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$. In terms of its source and arrow parts, we let

$$F_2(x, y) = ([\phi_0(x), \phi_0(y)], \phi_2(x, y)).$$

Computing the target of $F_2(x, y)$ we have:

$$\begin{aligned} tF_2(x,y) &= sF_2(x,y) + d\vec{F}_2(x,y) \\ &= [\phi_0(x), \phi_0(y)] + d\phi_2(x,y) \\ &= [\phi_0(x), \phi_0(y)] + \phi_0[x,y] - [\phi_0(x), \phi_0(y)] \\ &= \phi_0[x,y] \\ &= F_0[x,y] \end{aligned}$$

by the first equation in Definition 34 and the fact that $F_0 = \phi_0$. Thus we have $F_2(x, y)\colon [F_0(x), F_0(y)] \to F_0[x, y]$. Notice that $F_2(x, y)$ is bilinear and skew-symmetric since ϕ_2 and the bracket are. F_2 is a natural transformation by Theorem 12 and the fact that ϕ_2 is a chain homotopy from $[\phi(\cdot), \phi(\cdot)]$ to $\phi([\cdot, \cdot])$, thought of as chain maps from $V \otimes V$ to V'. Finally, the equation in Definition 34 gives the commutative diagram in Definition 23, since the composition of morphisms corresponds to addition of their arrow parts.

We leave it to the reader to check that T is indeed a functor. Next, we describe how to construct a functor $S\colon \mathrm{Lie2Alg} \to \mathrm{2TermL}_\infty$.

Given a Lie 2-algebra L we construct the 2-term L_∞-algebra $V = S(L)$ as follows. We define:

$$\begin{aligned} V_0 &= L_0 \\ V_1 &= \ker(s) \subseteq L_1. \end{aligned}$$

In addition, we define the maps l_k as follows:

- $l_1 h = t(h)$ for $h \in V_1 \subseteq L_1$.
- $l_2(x, y) = [x, y]$ for $x, y \in V_0 = L_0$.
- $l_2(x, h) = -l_2(h, x) = [1_x, h]$ for $x \in V_0 = L_0$ and $h \in V_1 \subseteq L_1$.
- $l_2(h, k) = 0$ for $h, k \in V_1 \subseteq L_1$.
- $l_3(x, y, z) = \vec{J}_{x,y,z}$ for $x, y, z \in V_0 = L_0$.

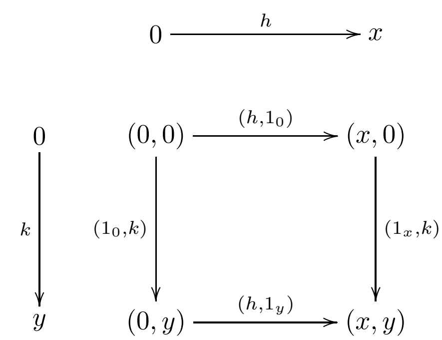
As usual, we abbreviate l_1 as d and l_2 on zero chains as $[\cdot,\cdot]$.

With these definitions, conditions (a) and (b) of Lemma 33 follow from the antisymmetry of the bracket functor. Condition (c) is automatic. Condition (d) follows from the complete antisymmetry of the Jacobiator.

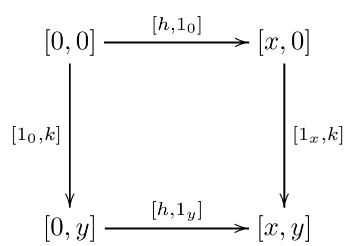
For $h \in V_1$ and $x \in V_0$, the functoriality of $[\cdot,\cdot]$ gives

$$\begin{array}{rcl} d(l_2(x,h)) & = & t([1_x,h]) \\ & = & [t(1_x),t(h)] \\ & = & [x,dh], \end{array}$$

which is (e) of Lemma 33. To obtain (f), 33, we let $h\colon 0 \to x$ and $k\colon 0 \to y$ be elements of V_1. We then consider the following square in $L \times L$,



which commutes by definition of a product category. Since $[\cdot,\cdot]$ is a functor, it preserves such commutative squares, so that



commutes. Since $[h,1_0]$ and $[1_0,k]$ are easily seen to be identity morphisms, this implies $[h,1_y] = [1_x,k]$. This means that in V we have $l_2(h,y) = l_2(x,k)$, or, since y is the target of k and x is the target of h, simply $l_2(h,dk) = l_2(dh,k)$, which is (f) of Lemma 33.

Since $J_{x,y,z}\colon [[x,y],z] \to [x,[y,z]] + [[x,z],y]$, we have

$$\begin{array}{rcl} d(l_3(x,y,z)) & = & t(\vec{J}_{x,y,z}) \\ & = & (t-s)(J_{x,y,z}) \\ & = & [x,[y,z]] + [[x,z],y] - [[x,y],z], \end{array}$$

which gives (g) of Lemma 33. The naturality of $J_{x,y,z}$ implies that for any $f\colon z \to z'$, we must have

$$[[1_x,1_y],f]\, J_{x,y,z'} = J_{x,y,z}\, ([[1_x,f],1_y] + [1_x,[1_y,f]]).$$

This implies that in V we have

$$l_2([x,y],\vec{f}) + l_3(x,y,z'-z) = l_2(l_2(x,\vec{f}),y) + l_2(x,l_2(y,\vec{f})),$$

for $x, y \in V_0$ and $\vec{f} \in V_1$, which is (h) of Lemma 33.

Finally, the Jacobiator identity dictates that the arrow part of the Jacobiator, l_3, satisfies the following equation:

$$l_2(l_3(w,x,y),z) + l_2(l_3(w,y,z),x) + l_3([w,y],x,z) + l_3([x,z],w,y) =$$

$$l_2(l_3(w,x,z),y) + l_2(l_3(x,y,z),w) + l_3([w,x],y,z) +$$
$$l_3([w,z],x,y) + l_3([x,y],w,z) + l_3([y,z],w,x).$$

This is (i) of Lemma 33.

This completes the construction of a 2-term L_∞-algebra $S(L)$ from any Lie 2-algebra L. Next we construct an L_∞-homomorphism $S(F)\colon S(L) \to S(L')$ from any Lie 2-algebra homomorphism $F\colon L \to L'$.

Let $S(L) = V$ and $S(L') = V'$. We define the underlying chain map of $S(F) = \phi$ as in Proposition 8. To make ϕ into an L_∞-homomorphism we must equip it with a skew-symmetric bilinear map $\phi_2\colon V_0 \times V_0 \to V_1'$ satisfying the conditions in Definition 34. To do this we set

$$\phi_2(x,y) = \vec{F}_2(x,y).$$

The bilinearity and skew-symmetry of ϕ_2 follow from that of F_2. Then, since ϕ_2 is the arrow part of F_2,

$$\begin{aligned} d\phi_2(x,y) &= (t-s)F_2(x,y) \\ &= F_0[x,y] - [F_0(x),F_0(y)] \\ &= \phi_0[x,y] - [\phi_0(x),\phi_0(y)], \end{aligned}$$

by definition of the chain map ϕ. The naturality of F_2 gives the second equation in Definition 34. Finally, since composition of morphisms corresponds to addition of arrow parts, the diagram in Definition 23 gives:

$$l_2(\phi_2(x,y),\phi_0(z)) + \phi_2([x,y],z) + \phi_1(l_3(x,y,z)) = l_3(\phi_0(x),\phi_0(y),\phi_0(z)) +$$

$$l_2(\phi_0(x),\phi_2(y,z)) + l_2(\phi_2(x,z),\phi_0(y)) + \phi_2(x,[y,z]) + \phi_2([x,z],y),$$

since $\phi_0 = F_0$, $\phi_1 = F_1$ on elements of V_1, and the arrow parts of J and F_2 are l_3 and ϕ_2, respectively.

We leave it to the reader to check that S is indeed a functor, and to construct natural isomorphisms $\alpha\colon ST \Rightarrow 1_{\mathrm{Lie2Alg}}$ and $\beta\colon TS \Rightarrow 1_{\mathrm{2TermL_\infty}}$. □

The above theorem also has a 2-categorical version. We have defined a 2-category of Lie 2-algebras, but not yet defined a 2-category of 2-term L_∞-algebras. For this, we need the following:

Definition 37. *Let V and V' be 2-term L_∞-algebras and let $\phi, \psi \colon V \to V'$ be L_∞-homomorphisms. An* $\mathbf{L_\infty}$**-2-homomorphism** *$\tau \colon \phi \Rightarrow \psi$ is a chain homotopy such that the following equation holds for all $x, y \in V_0$:*

- $\phi_2(x,y) - \psi_2(x,y) = [\phi_0(x), \tau(y)] + [\tau(x), \psi_0(y)] - \tau([x,y])$

We leave it as an exercise for the reader to define vertical and horizontal composition for these 2-homomorphisms, to define identity 2-homomorphisms, and to prove the following:

Proposition 38. *There is a strict 2-category* $\mathbf{2TermL_\infty}$ *with 2-term L_∞-algebras as objects, homomorphisms between these as morphisms, and 2-homomorphisms between those as 2-morphisms.*

With these definitions one can strengthen Theorem 36 as follows:

Theorem 39. *The 2-categories* Lie2Alg *and* $\mathrm{2TermL_\infty}$ *are 2-equivalent.*

The main benefit of the results in this section is that they provide us with another method to create examples of Lie 2-algebras. Instead of thinking of a Lie 2-algebra as a category equipped with extra structure, we may work with a 2-term chain complex endowed with the structure described in Lemma 33. In the next two sections we investigate the results of trivializing various aspects of a Lie 2-algebra, or equivalently of the corresponding 2-term L_∞-algebra.

5 Strict Lie 2-algebras

A 'strict' Lie 2-algebra is a categorified version of a Lie algebra in which all laws hold as equations, not just up to isomorphism. In a previous paper [2] one of the authors showed how to construct these starting from 'strict Lie 2-groups'. Here we describe this process in a somewhat more highbrow manner, and explain how these 'strict' notions are special cases of the semistrict ones described here.

Since we only weakened the Jacobi identity in our definition of 'semistrict' Lie 2-algebra, we need only require that the Jacobiator be the identity to recover the 'strict' notion:

Definition 40. *A semistrict Lie 2-algebra is* **strict** *if its Jacobiator is the identity.*

Similarly, requiring that the bracket be strictly preserved gives the notion of 'strict' homomorphism between Lie 2-algebras:

Definition 41. *Given semistrict Lie 2-algebras L and L', a homomorphism $F \colon L \to L'$ is* **strict** *if F_2 is the identity.*

Proposition 42. Lie2Alg *contains a sub-2-category* **SLie2Alg** *with strict Lie 2-algebras as objects, strict homomorphisms between these as morphisms, and arbitrary 2-homomorphisms between those as 2-morphisms.*

Proof. One need only check that if the homomorphisms $F\colon L \to L'$ and $G\colon L' \to L''$ have $F_2 = 1$, $G_2 = 1$, then their composite has $(FG)_2 = 1$. □

The following proposition shows that strict Lie 2-algebras as defined here agree with those as defined previously [2]:

Proposition 43. *The 2-category* SLie2Alg *is isomorphic to the 2-category* LieAlgCat *consisting of categories, functors and natural transformations in* LieAlg*.*

Proof. This is just a matter of unravelling the definitions. □

Just as Lie groups have Lie algebras, 'strict Lie 2-groups' have strict Lie 2-algebras. Before we can state this result precisely, we must recall the concept of a strict Lie 2-group, which was treated in greater detail in HDA5:

Definition 44. *We define* **SLie2Grp** *to be the strict 2-category* LieGrpCat *consisting of categories, functors and natural transformations in* LieGrp*. We call the objects in this 2-category* **strict Lie 2-groups***; we call the morphisms between these* **strict homomorphisms***, and we call the 2-morphisms between those* **2-homomorphisms***.*

Proposition 45. *There exists a unique 2-functor*

$$d\colon \mathrm{SLie2Grp} \to \mathrm{SLie2Alg}$$

such that:

1. *d maps any strict Lie 2-group C to the strict Lie 2-algebra $dC = \mathfrak{c}$ for which $\mathfrak{c}_0$ is the Lie algebra of the Lie group of objects C_0, $\mathfrak{c}_1$ is the Lie algebra of the Lie group of morphisms C_1, and the maps $s, t\colon \mathfrak{c}_1 \to \mathfrak{c}_0$, $i\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ and $\circ\colon \mathfrak{c}_1 \times_{\mathfrak{c}_0} \mathfrak{c}_1 \to \mathfrak{c}_1$ are the differentials of those for C.*

2. *d maps any strict Lie 2-group homomorphism $F\colon C \to C'$ to the strict Lie 2-algebra homomorphism $dF\colon \mathfrak{c} \to \mathfrak{c}'$ for which $(dF)_0$ is the differential of F_0 and $(dF)_1$ is the differential of F_1.*

3. *d maps any strict Lie 2-group 2-homomorphism $\alpha\colon F \Rightarrow G$ where $F, G\colon C \to C'$ to the strict Lie 2-algebra 2-homomorphism $d\alpha\colon dF \Rightarrow dG$ for which the map $d\alpha\colon \mathfrak{c}_0 \to \mathfrak{c}_1$ is the differential of $\alpha\colon C_0 \to C_1$.*

Proof. The proof of this long-winded proposition is a quick exercise in internal category theory: the well-known functor from LieGrp to LieAlg preserves pullbacks, so it maps categories, functors and natural transformations in LieGrp to those in LieAlg, defining a 2-functor $d\colon \mathrm{SLie2Grp} \to \mathrm{SLie2Alg}$, which is given explicitly as above. □

We would like to generalize this theorem and define the Lie 2-algebra not just of a strict Lie 2-group, but of a general Lie 2-group as defined in HDA5. However, this may require a weaker concept of Lie 2-algebra than that studied here.

A nice way to obtain strict Lie 2-algebras is from 'differential crossed modules'. This construction resembles one in HDA5, where we obtained strict Lie 2-groups from 'Lie crossed modules'. We recall that construction here before stating its Lie algebra analogue.

Starting with a strict Lie 2-group C, with Lie groups C_0 of objects and C_1 of morphisms, we construct a pair of Lie groups

$$G = C_0, \quad H = \ker(s) \subseteq C_1$$

where $s\colon C_1 \to C_0$ is the source map. We then restrict the target map to a homomorphism

$$t\colon H \to G.$$

In addition to the usual action of G on itself by conjugation, we have an action of G on H,

$$\alpha\colon G \to \mathrm{Aut}(H),$$

defined by

$$\alpha(g)(h) = i(g)hi(g)^{-1}.$$

where $i\colon C_0 \to C_1$ is the identity-assigning map. The target map is equivariant with respect to this action, meaning:

$$t(\alpha(g))(h) = gt(h)g^{-1}.$$

We also have the 'Peiffer identity':

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $h, h' \in H$. So, we obtain the Lie group version of a crossed module:

Definition 46. *A* **Lie crossed module** *is a quadruple* (G, H, t, α) *consisting of Lie groups* G *and* H*, a homomorphism* $t : H \to G$*, and an action* α *of* G *on* H *(that is, a homomorphism* $\alpha\colon G \to \mathrm{Aut}(H)$*) satisfying*

$$t(\alpha(g)(h)) = g\, t(h)\, g^{-1}$$

and

$$\alpha(t(h))(h') = hh'h^{-1}$$

for all $g \in G$ *and* $h, h' \in H$*.*

In Proposition 32 of HDA5 we sketched how one can reconstruct a strict Lie 2-group from its Lie crossed module.

For a Lie algebra analogue of this result, we should replace the Lie group $\mathrm{Aut}(H)$ by the Lie algebra $\mathfrak{der}(\mathfrak{h})$ consisting of all 'derivations' of $\mathfrak{h}$, that is, all linear maps $f\colon \mathfrak{h} \to \mathfrak{h}$ such that

$$f([y, y']) = [f(y), y'] + [y, f(y')].$$

Definition 47. *An* **infinitesimal crossed module** *is a quadruple* $(\mathfrak{g}, \mathfrak{h}, t, \alpha)$ *consisting of Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$*, a homomorphism* $t\colon \mathfrak{h} \to \mathfrak{g}$*, and an action* α *of* $\mathfrak{g}$ *as derivations of* $\mathfrak{h}$ *(that is, a homomorphism* $\alpha : \mathfrak{g} \to \mathfrak{Der}(\mathfrak{h})$*) satisfying*

$$t(\alpha(x)(y)) = [x, t(y)]$$

and

$$\alpha(t(y))(y') = [y, y']$$

for all $x \in \mathfrak{g}$ *and* $y, y' \in \mathfrak{h}$*.*

Infinitesimal crossed modules first appeared in the work of Gerstenhaber [20] where he classified them using the 3rd Lie algebra cohomology group of $\mathfrak{g}$. We shall see a similar classification of Lie 2-algebras in Corollary 56. Indeed, differential crossed modules are essentially the same as strict Lie 2-algebras:

Proposition 48. *Given a strict Lie 2-algebra* $\mathfrak{c}$*, there is a infinitesimal crossed module* $(\mathfrak{g}, \mathfrak{h}, t, \alpha)$ *where* $\mathfrak{g} = \mathfrak{c}_0$*,* $\mathfrak{h} = \ker(s)$*,* $t\colon \mathfrak{h} \to \mathfrak{g}$ *is the restriction of the target map from* $\mathfrak{c}_1$ *to* $\mathfrak{h}$*, and*

$$\alpha(x)(y) = [1_x, y].$$

Conversely, we can reconstruct any strict Lie 2-algebra up to isomorphism from its differential crossed module.

Proof. The proof is analogous to the standard one relating 2-groups and crossed modules [6, 19]. □

The diligent reader can improve this proposition by defining morphisms and 2-morphisms for infinitesimal crossed modules, and showing this gives a 2-category equivalent to the 2-category SLie2Alg.

Numerous examples of Lie crossed modules are described in Section 8.4 of HDA5. Differentiating them gives examples of infinitesimal crossed modules, and hence strict Lie 2-algebras.

6 Skeletal Lie 2-algebras

A semistrict Lie 2-algebra is *strict* when we assume the map l_3 vanishes in the corresponding L_∞-algebra, since this forces the Jacobiator to be the identity. We now investigate the consequences of assuming the differential d vanishes in the corresponding L_∞-algebra. Thanks to the formula

$$d\vec{f} = t(f) - s(f),$$

this implies that the source of any morphism in the Lie 2-algebra equals its target. In other words, the Lie 2-algebra is *skeletal:*

Definition 49. *A category is* **skeletal** *if isomorphic objects are equal.*

Skeletal categories are useful in category theory because every category is equivalent to a skeletal one formed by choosing one representative of each isomorphism class of objects [26]. The same sort of thing is true in the context of 2-vector spaces:

Lemma 50. *Any 2-vector space is equivalent, as an object of* 2Vect*, to a skeletal one.*

Proof: Using the result of Theorem 12 we may treat our 2-vector spaces as 2-term chain complexes. In particular, a 2-vector space is skeletal if the corresponding 2-term chain complex has vanishing differential, and two 2-vector spaces are equivalent if the corresponding 2-term chain complexes are chain homotopy equivalent. So, it suffices to show that any 2-term chain complex is chain homotopy equivalent to one with vanishing differential. This is well-known, but the basic idea is as follows. Given a 2-term chain complex

$$C_1 \xrightarrow{\quad d \quad} C_0$$

we express the vector spaces C_0 and C_1 as $C_0 = \mathrm{im}(d) \oplus C_0'$ and $C_1 = \ker(d) \oplus X$ where X is a vector space complement to $\ker(d)$ in C_1. This allows us to define a 2-term chain complex C' with vanishing differential:

$$C_1' = \ker(d) \xrightarrow{\quad 0 \quad} C_0' \ .$$

The inclusion of C' in C can easily be extended to a chain homotopy equivalence. □

Using this fact we obtain a result that will ultimately allow us to classify Lie 2-algebras:

Proposition 51. *Every Lie 2-algebra is equivalent, as an object of* Lie2Alg*, to a skeletal one.*

Proof: Given a Lie 2-algebra L we may use Lemma 50 to find an equivalence between the underlying 2-vector space of L and a skeletal 2-vector space L'. We may then use this to transport the Lie 2-algebra structure from L to L', and obtain an equivalence of Lie 2-algebras between L and L'. □

It is interesting to observe that a skeletal Lie 2-algebra that is also strict amounts to nothing but a Lie algebra L_0 together with a representation of L_0 on a vector space L_1. This is the infinitesimal analogue of how a strict skeletal 2-group G consists of a group G_0 together with an action of G_0 as automorphisms of an abelian group G_1. Thus, the representation theory of groups and Lie algebras is automatically subsumed in the theory of 2-groups and Lie 2-algebras!

To generalize this observation to other skeletal Lie 2-algebras, we recall some basic definitions concerning Lie algebra cohomology:

Definition 52. *Let $\mathfrak{g}$ be a Lie algebra and ρ a representation of $\mathfrak{g}$ on the vector space V. Then a* ***V*-valued *n*-cochain** *ω on $\mathfrak{g}$ is a totally antisymmetric map*

$$\omega\colon \mathfrak{g}^{\otimes n} \to V.$$

The vector space of all n-cochains is denoted by $C^n(\mathfrak{g}, V)$. The **coboundary operator** *$\delta\colon C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ is defined by:*

$$\begin{aligned}(\delta\omega)(v_1, v_2, \ldots, v_{n+1}) \quad := \quad & \sum_{i=1}^{n+1} (-1)^{i+1} \rho(v_i)\omega(v_1, \ldots, \hat{v_i}, \ldots, v_{n+1}) \\ + \quad & \sum_{1 \le j < k \le n+1} (-1)^{j+k} \omega([v_j, v_k], v_1, \ldots, \hat{v_j}, \ldots, \hat{v_k}, \ldots, v_{n+1}),\end{aligned}$$

Proposition 53. *The Lie algebra coboundary operator δ satisfies $\delta^2 = 0$.*

Definition 54. *A V-valued n-cochain ω on $\mathfrak{g}$ is an* ***n*-cocycle** *when $\delta\omega = 0$ and an* ***n*-coboundary** *if there exists an $(n-1)$-cochain θ such that $\omega = \delta\theta$. We denote the groups of n-cocycles and n-coboundaries by $Z^n(\mathfrak{g}, V)$ and $B^n(\mathfrak{g}, V)$ respectively. The nth* **Lie algebra cohomology group** *$H^n(\mathfrak{g}, V)$ is defined by*

$$H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).$$

The following result illuminates the relationship between Lie algebra cohomology and L_∞-algebras.

Theorem 55. *There is a one-to-one correspondence between L_∞-algebras consisting of only two nonzero terms V_0 and V_n, with $d = 0$, and quadruples $(\mathfrak{g}, V, \rho, l_{n+2})$ where $\mathfrak{g}$ is a Lie algebra, V is a vector space, ρ is a representation of $\mathfrak{g}$ on V, and l_{n+2} is a $(n+2)$-cocycle on $\mathfrak{g}$ with values in V.*

Proof.
($\Rightarrow$) Given such an L_∞-algebra V we set $\mathfrak{g} = V_0$. V_0 comes equipped with a bracket as part of the L_∞-structure, and since d is trivial, this bracket satisfies the Jacobi identity on the nose, making $\mathfrak{g}$ into a Lie algebra. We define $V = V_n$, and note that the bracket also gives a map $\rho\colon \mathfrak{g} \otimes V \to V$, defined by $\rho(x)f = [x, f]$ for $x \in \mathfrak{g}, f \in V$. We have

$$\begin{aligned}\rho([x,y])f &= [[x,y],f] \\ &= [[x,f],y] + [x,[y,f]] \quad \text{by (2) of Definition 32} \\ &= [\rho(x)f, y] + [x, \rho(y)f]\end{aligned}$$

for all $x, y \in \mathfrak{g}$ and $f \in V$, so that ρ is a representation. Finally, the L_∞ structure gives a map $l_{n+2}\colon \mathfrak{g}^{\otimes(n+2)} \to V$ which is in fact a $(n+2)$-cocycle. To see this, note that

$$0 = \sum_{i+j=n+4} \sum_{\sigma} l_j(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n+2)})$$

where we sum over $(i,(n+3)-i)$-unshuffles $\sigma \in S_{n+3}$. However, the only choices for i and j that lead to nonzero l_i and l_j are $i = n+2, j = 2$ and $i = 2, j = n+2$. In addition, notice that in this situation, $\chi(\sigma)$ will consist solely of the sign of the permutation because all of our x_i's have degree zero. Thus, the above becomes, with σ a $(n+2,1)$-unshuffle and τ a $(2,n+1)$-unshuffle:

$$\begin{aligned}
0 &= \sum_{\sigma} \chi(\sigma)(-1)^{n+2}[l_{n+2}(x_{\sigma(1)},\ldots,x_{\sigma(n+2)}),x_{\sigma(n+3)}] \\
&\quad + \sum_{\tau} \chi(\tau) l_{n+2}([x_{\tau(1)},x_{\tau(2)}],x_{\tau(3)},\ldots,x_{\tau(n+3)}) \\
&= \sum_{i=1}^{n+3}(-1)^{n+3-i}(-1)^{n+2}[l_{n+2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3}),x_i] \\
&\quad + \sum_{1\le i<j\le n+3}(-1)^{i+j+1} l_{n+2}([x_i,x_j],x_1,\ldots,\hat{x_i},\ldots,\hat{x_j},\ldots,x_{n+3}) \qquad (\dagger) \\
&= \sum_{i=1}^{n+3}(-1)^{i+1}[l_{n+2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3}),x_i] \\
&\quad + \sum_{1\le i<j\le n+3}(-1)^{i+j+1} l_{n+2}([x_i,x_j],x_1,\ldots,\hat{x_i},\ldots,\hat{x_j},\ldots,x_{n+3}) \\
&= -\sum_{i=1}^{n+3}(-1)^{i+1}[x_i,l_{n+2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+3})] \\
&\quad - \sum_{1\le i<j\le n+3}(-1)^{i+j} l_{n+2}([x_i,x_j],x_1,\ldots,\hat{x_i},\ldots,\hat{x_j},\ldots,x_{n+3}) \\
&= -\delta l_{n+2}(x_1,x_2,\ldots,x_{n+3}).
\end{aligned}$$

The first sum in (†) follows because we have $(n+3)$ $(n+2,1)$-unshuffles and the sign of any such unshuffle is $(-1)^{n+3-i}$. The second sum follows similarly because we have $(n+3)$ $(2,n+1)$-unshuffles and the sign of a $(2,n+1)$-unshuffle is $(-1)^{i+j+1}$. Therefore, l_{n+2} is a $(n+2)$-cocycle.

($\Leftarrow$) Conversely, given a Lie algebra $\mathfrak{g}$, a representation ρ of $\mathfrak{g}$ on a vector space V, and an $(n+2)$-cocycle l_{n+2} on $\mathfrak{g}$ with values in V, we define our L_∞-algebra V by setting $V_0 = \mathfrak{g}$, $V_n = V$, $V_i = \{0\}$ for $i \neq 0, n$, and $d = 0$. It remains to define the system of linear maps l_k, which we do as follows: Since $\mathfrak{g}$ is a Lie algebra, we have a bracket defined on V_0. We extend this bracket to define the map l_2, denoted by $[\cdot,\cdot]\colon V_i \otimes V_j \to V_{i+j}$ where $i, j = 0, n$, as follows:

$$[x,f] = \rho(x)f,$$

$$[f,y] = -\rho(y)f,$$

$$[f,g] = 0$$

for $x, y \in V_0$ and $f, g \in V_n$. With this definition, the map $[\cdot,\cdot]$ satisfies condition (1) of Definition 32 . We define $l_k = 0$ for $3 \leq k \leq n+1$ and $k > n+2$, and take l_{n+2} to be the given $(n+2)$ cocycle, which satisfies conditions (1) and (2) of Definition 32 by the cocycle condition. □

We can classify skeletal Lie 2-algebras using the above construction with $n = 1$:

Corollary 56. *There is a one-to-one correspondence between isomorphism classes of skeletal Lie 2-algebras and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space V, a representation ρ of $\mathfrak{g}$ on V, and a 3-cocycle on $\mathfrak{g}$ with values in V.*

Proof. This is immediate from Theorem 36 and Theorem 55. □

Since every Lie 2-algebra is equivalent as an object of Lie2Alg to a skeletal one, this in turn lets us classify *all* Lie 2-algebras, though only up to equivalence:

Theorem 57. *There is a one-to-one correspondence between equivalence classes of Lie 2-algebras (where equivalence is as objects of the 2-category* Lie2Alg*) and isomorphism classes of quadruples consisting of a Lie algebra $\mathfrak{g}$, a vector space V, a representation ρ of $\mathfrak{g}$ on V, and an element of $H^3(\mathfrak{g}, V)$.*

Proof. This follows from Theorem 51 and Corollary 56; we leave it to the reader to verify that equivalent skeletal Lie 2-algebras give cohomologous 3-cocycles and conversely. □

We conclude with perhaps the most interesting examples of finite-dimensional Lie 2-algebras coming from Theorem 56. These make use of the following identities involving the Killing form $\langle x, y\rangle := \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$ of a finite-dimensional Lie algebra:

$$\langle x, y\rangle = \langle y, x\rangle,$$

and

$$\langle [x, y], z\rangle = \langle x, [y, z]\rangle.$$

Example 58. *There is a skeletal Lie 2-algebra built using Theorem 56 by taking $V_0 = \mathfrak{g}$ to be a finite-dimensional Lie algebra over the field k, V_1 to be k, ρ the trivial representation, and $l_3(x, y, z) = \langle x, [y, z]\rangle$. We see that l_3 is a 3-cocycle using the above identities as follows:*

$$\begin{aligned} (\delta l_3)(w,x,y,z) &= \rho(w)l_3(x,y,z) - \rho(x)l_3(w,y,z) + \rho(y)l_3(w,x,z) - \rho(z)l_3(w,x,y) \\ &\quad -l_3([w,x],y,z) + l_3([w,y],x,z) - l_3([w,z],x,y) \\ &\quad -l_3([x,y],w,z) + l_3([x,z],w,y) - l_3([y,z],w,x) \\ &= -\langle [w,x],[y,z]\rangle + \langle [w,y],[x,z]\rangle - \langle [w,z],[x,y]\rangle \\ &\quad -\langle [x,y],[w,z]\rangle + \langle [x,z],[w,y]\rangle - \langle [y,z],[w,x]\rangle \end{aligned}$$

This second step above follows because we have a trivial representation. Continuing on, we have

$$(\delta l_3)(w,x,y,z) = -2\langle [w,x],[y,z]\rangle + 2\langle [w,y],[x,z]\rangle - 2\langle [w,z],[x,y]\rangle$$

$$\begin{aligned} &= -2\langle w,[x,[y,z]]\rangle + 2\langle w,[y,[x,z]]\rangle - 2\langle w,[z,[x,y]]\rangle \\ &= -2\langle w,[x,[y,z]] + [y,[z,x]] + [z,[x,y]]\rangle \\ &= -2\langle w,0\rangle \\ &= 0. \end{aligned}$$

More generally, we obtain a Lie 2-algebra this way taking $l_3(x,y,z) = \hbar\langle x,[y,z]\rangle$ *where* $\hbar$ *is any element of* k*. We call this Lie 2-algebra* $\mathfrak{g}_\hbar$*.*

It is well known that the Killing form of $\mathfrak{g}$ is nondegenerate if and only if $\mathfrak{g}$ is semisimple. In this case the 3-cocycle described above represents a nontrivial cohomology class when $\hbar \neq 0$, so by Theorem 57 the Lie 2-algebra $\mathfrak{g}_\hbar$ is not equivalent to a skeletal one with vanishing Jacobiator. In other words, we obtain a Lie 2-algebra that is not equivalent to a skeletal strict one.

Suppose the field k has characteristic zero, the Lie algebra $\mathfrak{g}$ is finite dimensional and semisimple, and V is finite dimensional. Then a version of Whitehead's Lemma [1] says that $H^3(\mathfrak{g},V) = \{0\}$ whenever the representation of $\mathfrak{g}$ on V is nontrivial and irreducible. This places some limitations on finding interesting examples of nonstrict Lie 2-algebras other than those of the form $\mathfrak{g}_\hbar$.

In HDA5 we show how the Lie 2-algebras $\mathfrak{g}_\hbar$ give rise to 2-groups when $\hbar$ is an integer. The construction involves Chern–Simons theory. Since Chern–Simons theory is also connected to the theory of quantum groups and affine Lie algebras, it is natural to hope for a more direct link between these structures and the Lie 2-algebras $\mathfrak{g}_\hbar$. After all, they are all 'deformations' of more familiar algebraic structures which take advantage of the 3-cocycle $\langle x,[y,z]\rangle$ and the closely related 2-cocycle on $C^\infty(S^1,\mathfrak{g})$.

The smallest nontrivial example of the Lie 2-algebras $\mathfrak{g}_\hbar$ comes from $\mathfrak{g} = \mathfrak{su}(2)$. Since $\mathfrak{su}(2)$ is isomorphic to $\mathbb{R}^3$ with its usual vector cross product, and its Killing form is proportional to the dot product, this Lie 2-algebra relies solely on familiar properties of the dot product and cross product:

$$x \times y = -y \times x,$$

$$x \cdot y = y \cdot x,$$

$$x \cdot (y \times z) = (x \times y) \cdot z,$$

$$x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0.$$

It will be interesting to see if this Lie 2-algebra, where the Jacobiator comes from the triple product, has any applications to physics. Just for fun, we work out the details again in this case:

Example 59. *There is a skeletal Lie* 2*-algebra built using Theorem 56 by taking* $V_0 = \mathbb{R}^3$ *equipped with the cross product,* $V_1 = \mathbb{R}$*,* ρ *the trivial representation, and* $l_3(x,y,z) = x \cdot (y \times z)$*. We see that* l_3 *is a 3-cocycle as follows:*

$$\begin{aligned} (\delta l_3)(w,x,y,z) \quad &= \quad -l_3([w,x],y,z) + l_3([w,y],x,z) - l_3([w,z],x,y) \\ & \quad\; -l_3([x,y],w,z) + l_3([x,z],w,y) - l_3([y,z],w,x) \end{aligned}$$

$$
\begin{aligned}
&= -(w \times x) \cdot (y \times z) + (w \times y) \cdot (x \times z) - (w \times z) \cdot (x \times y) \\
&\quad -(x \times y) \cdot (w \times z) + (x \times z) \cdot (w \times y) - (y \times z) \cdot (w \times x) \\
&= -2(w \times x) \cdot (y \times z) + 2(w \times y) \cdot (x \times z) - 2(w \times z) \cdot (x \times y) \\
&= -2w \cdot (x \times (y \times z)) + 2w \cdot (y \times (x \times z)) - 2w \cdot (z \times (x \times y)) \\
&= -2w \cdot (x \times (y \times z) + y \times (z \times x) + z \times (x \times y)) \\
&= 0.
\end{aligned}
$$

7 Conclusions

In HDA5 and the present paper we have seen evidence that the theory of Lie groups and Lie algebras can be categorified to give interesting theories of Lie 2-groups and Lie 2-algebras. We expect this pattern to continue as shown in the following tables.

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	manifolds	Lie groupoids	Lie 2-groupoids
$k = 1$	Lie groups	Lie 2-groups	Lie 3-groups
$k = 2$	abelian Lie groups	braided Lie 2-groups	braided Lie 3-groups
$k = 3$	‘ ’	symmetric Lie 2-groups	sylleptic Lie 3-groups
$k = 4$	‘ ’	‘ ’	symmetric Lie 3-groups
$k = 5$	‘ ’	‘ ’	‘ ’

1. k-tuply groupal Lie n-groupoids: expected results

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	vector bundles	Lie algebroids	Lie 2-algebroids
$k = 1$	Lie algebras	Lie 2-algebras	Lie 3-algebras
$k = 2$	abelian Lie algebras	braided Lie 2-algebras	braided Lie 3-algebras
$k = 3$	‘ ’	symmetric Lie 2-algebras	sylleptic Lie 3-algebras
$k = 4$	‘ ’	‘ ’	symmetric Lie 3-algebras
$k = 5$	‘ ’	‘ ’	‘ ’

2. k-tuply stabilized Lie n-algebroids: expected results

Table 1 gives names for k-tuply groupal n-groupoids [4] for which the set of j-morphisms is a smooth manifold for each j, and for which the operations are all smooth. Manifolds, Lie groups and abelian Lie groups are well-understood; Lie groupoids have also been intensively investigated [27], but the study of Lie

2-groups has just barely begun, and the other entries in the chart are still *terra incognita*: they seem not to have even been defined yet, although this should be easy for the entries in the second column.

Table 2 gives names for the 'infinitesimal versions' of the entries in the first chart. The classic example is that of a Lie algebra, which can be formed by taking the tangent space of a Lie group at the identity element. Similarly, we have seen that the tangent 2-vector space at the identity object of a strict Lie 2-group becomes a strict Lie 2-algebra; we also expect a version of this result to hold for the more general Lie 2-groups defined in HDA5, though our 'semistrict Lie 2-algebras' may not be sufficiently general for this task.

The $k = 0$ row of Table 2 is a bit different from the rest. For example, a manifold does not have a distinguished identity element at which to take the tangent space. To deal with this we could work instead with pointed manifolds, but another option is to take the tangent space at *every* point of a manifold and form the tangent bundle, which is a vector bundle. Similarly, a Lie groupoid does not have a distinguished 'identity object', so the concept of 'Lie algebroid' [27] must be defined a bit subtly. The same will be true of Lie n-groupoids and their Lie n-algebroids. For this reason it may be useful to treat the $k = 0$ row separately and use the term 'k-tuply stabilized Lie n-algebra' for what we are calling a $(k + 1)$-tuply stabilized Lie n-algebroid.

The general notion of 'k-tuply stabilized Lie n-algebra' has not yet been defined, but at least we understand the 'semistrict' ones: as explained in Section 4.3, these are just various sorts of L_∞-algebra with their underlying n-term chain complexes reinterpreted as strict $(n - 1)$-categories in Vect. More precisely, we define a **semistrict k-tuply stabilized Lie n-algebra** to be the result of taking an L_∞-algebra V with $V_i = 0$ when $i < k$ or $i \geq n + k$ and transferring all the structure on its underlying n-term chain complex to the corresponding strict $(n - 1)$-category in Vect.

In this language, Theorem 55 gives a way of constructing a semistrict Lie n-algebra with only nontrivial objects and n-morphisms from an $(n+2)$-cocycle on the Lie algebra of objects. This can be seen as an infinitesimal version of the usual 'Postnikov tower' construction of a connected homotopy $(n + 1)$-type with only π_1 and π_{n+1} nonzero from an $(n + 2)$-cocycle on the group π_1. The analogy comes into crisper focus if we think of a connected homotopy $(n + 1)$-type as an 'n-group'. Then the Postnikov construction gives an n-group with only nontrivial objects and n-morphisms from an $(n + 2)$-cocycle on the group of objects; now the numbering scheme perfectly matches that for Lie n-algebras. For $n = 2$ we described how this works more explicitly in HDA5. One of the goals of the present paper was to show that just as group cohomology arises naturally in the classification of n-groups, Lie algebra cohomology arises in the classification of Lie n-algebras.

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