Differential Calculi on Quantum Vector Spaces with Hecke-type Relations

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Abstract

From a vector space V equipped with a Yang-Baxter operator R one may form the r-symmetric algebra $S_RV = TV/\langle v\otimes w - R(v\otimes w)\rangle$, which is a quantum vector space in the sense of Manin, and the associated quantum matrix algebra $M_RV = T(End(V))/\langle f\otimes g - R(f\otimes g)R^{-1}\rangle$. In the case when R satisfies a Hecke-type identity $R^2 = (1-q)R + q$, we construct a differential calculus Ω_RV for S_RV which agrees with that constructed by Pusz, Woronowicz, Wess, and Zumino when R is essentially the R-matrix of $GL_q(n)$. Elements of Ω_RV may be regarded as differential forms on the quantum vector space S_RV. We show that Ω_RV is M_RV-covariant, in the sense that there is a coaction $\Phi_*\colon \Omega_RV \to M_RV\otimes\Omega_RV$ with $\Phi_* d = (1\otimes d)\Phi_*$ extending the natural coaction $\Phi\colon S_RV \to M_RV\otimes S_RV$.

1 Introduction

The study of fermions, starting roughly with Pauli's formulation of the exclusion principle, led eventually to the realization that supercommutative algebra is a natural generalization of commutative algebra, and that one may consider manifolds as a special case of supermanifolds. The generalization of concepts to supercommutative algebra is based on the rule that the classical twist map

$$a \otimes b \mapsto b \otimes a$$

should be replaced everywhere by the map

$$a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a \, .$$

The study of "anyons" (and related 2-dimensional physics), in which the notion of interchanging identical particles is formalized using the braid group rather than the symmetric group, leads to a promising further generalization of commutative algebra, which we call "r-commutative algebra." Here the role of the twist map is replaced by any Yang-Baxter operator

$$a \otimes b \mapsto R(a \otimes b)$$

that is compatible in a certain sense with the multiplication map $m: A \otimes A \to A$. An algebra A equipped with such an "r-structure" is said to be "r-commutative" if $m = mR$. Many interesting noncommutative analogs of manifolds are in fact r-commutative. These include noncommutative tori [3, 10], quantum groups, quantum matrix algebras, and quantum vector spaces [4, 5, 7, 8, 13-15 as well as the Clifford and Weyl algebras [2, 11].

For a noncommutative algebra A, the natural analog of the notion of differential forms is a *differential calculus*, that is, a differential graded algebra Ω that is generated as such by $\Omega^0 \cong A$. In the case when the r-commutative algebra A has $R^2 = 1$, the author has constructed a differential calculus $\Omega_R A$ for A which reduces to the usual differential forms when A is the algebra of smooth functions on a manifold or supermanifold [1]. Here we address a more general case, which occurs in the theory of quantum groups. Our construction generalizes the work of Pusz, Woronowicz, Wess and Zumino [9, 12, 16] on $GL_q(n)$-covariant differential calculi on certain quantum vector spaces, and "explains" it from the viewpoint of r-commutative algebra.

2 The differential calculus $\Omega_R V$

We work over an arbitrary field k, and by an "algebra" we mean a unital associative algebra over k. Let V be a vector space over k. Given $R \in End(V \otimes V)$, define $R_i \in End(V^{\otimes n})$ by

$$R(v_1 \otimes \cdots v_n) = v_1 \otimes \cdots \otimes v_{i-1} \otimes R(v_i \otimes v_{i+1}) \otimes v_{i+1} \otimes \cdots \otimes v_n \ .$$

Given $R \in End(V \otimes V)$, we say that R is a *Yang-Baxter operator* on V if R is invertible and $R_1R_2R_1 = R_2R_1R_2$ on $V^{\otimes 3}$. Let s_i, $1 \le i \le n-1$, denote the standard generators of the braid group B_n, which satisfy the relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ , \quad s_i s_j = s_j s_i \ , \ |i-j| \ge 2 \ .$$

Then $R \in End(V \otimes V)$ is a Yang-Baxter operator if and only if for all n the map $s_i \mapsto R_i$ extends to a representation ρ of B_n on $V^{\otimes n}$.

Define the element $s_{nm} \in B_{n+m}$ by

$$s_{nm} = (s_m \cdots s_1)(s_{m+1} \cdots s_2) \cdots (s_{n+m-1} \cdots s_n) \ .$$

Pictorially, this element can be represented as the braid in Figure 1. Let $m\colon A \otimes A \to A$ be the multiplication map. An *r-algebra* is an algebra A equipped with a Yang-Baxter operator such that

$$R(1 \otimes a) = a \otimes 1 \ , \quad R(a \otimes 1) = 1 \otimes a$$

for all $a \in A$, and the following diagram commutes:

$$\begin{tikzcd}
A^{\otimes 4} \arrow[r, "\rho(s_{22})"] \arrow[d, "m\otimes m"'] & A^{\otimes 4} \arrow[d, "m\otimes m"] \\
A^{\otimes 2} \arrow[r, "R"] & A^{\otimes 2}
\end{tikzcd}$$

This diagram has the pictorial interpretation given in Figure 2, where the joining of two strands represents the multiplication map. A Yang-Baxter operator on an algebra A satisfying these conditions is called an *r-structure* for A.

An *r-ideal* of an r-algebra A is defined to be a two-sided ideal $I \subseteq A$ such that R preserves $I \otimes A + A \otimes I$. Given vector spaces V, V' with Yang-Baxter operators R, R', a linear map $f: V \to V'$ is called an *r-map* if $(f \otimes f)R = R'(f \otimes f)$. Given r-algebras A and B, a homomorphism that is also an r-map is said to be an *r-morphism.* The kernel of an r-morphism is an r-ideal, and conversely, the quotient of an r-algebra by an r-ideal has a unique r-structure such that the quotient map is an r-morphism.

We say that A, or alternatively its r-structure R, is *r-commutative* if $m = mR$. The quotient of an r-commutative algebra by an r-ideal is again r-commutative. Given a vector space V equipped with a Yang-Baxter operator R, the quotient of the tensor algebra TV by the ideal generated by the relations

$$v \otimes w = R(v \otimes w)$$

is called the *r-symmetric algebra* $S_R V$. There is a natural inclusion $V \hookrightarrow S_R V$ and a unique r-structure on $S_R V$ extending the Yang-Baxter operator R on V, and $S_R V$ is r-commutative relative to this r-structure (which we also denote as R). An r-map $f: V \to V'$ extends to a unique r-morphism $f_*: S_R V \to S_{R'} V'$.

We say that R is of *type* q, where q is a nonzero element of k, if R satisfies the Hecke-type relation

$$R^2 = (1 - q)R + q\ .$$

This implies that the only possible eigenvalues of R are 1 and $-q$. For example, if V has the basis e_i, $1 \le i \le n$, and

$$R(e_i \otimes e_j) = \begin{cases} e_j \otimes e_i & \text{if } i = j \\ q^{-1} e_j \otimes e_i & \text{if } i < j \\ q^{-1} e_j \otimes e_i + (1 - q^{-2}) e_i \otimes e_j & \text{if } i > j \end{cases}$$

then R is of type q^{-2}; up to various conventional normalizations this R is the R-matrix for $GL_q(n)$.

Recall that for any algebra A, the universal differential calculus $\Omega_u(A)$ is a differential calculus over A defined by the property that any differential calculus for

A is, up to isomorphism, a quotient of $\Omega_u(A)$ [3, 6, 15]. Let V be a vector space equipped with a Yang-Baxter operator R of type q. Given any $v, w \in V$, we write $R(v \otimes w) = w^i \otimes v_i$, using the Einstein summation convention. The differential calculus of *differential forms* over $S_R V$, denoted $\Omega_R V$, is defined as the quotient of $\Omega_u(S_R V)$ by the differential ideal generated by $d1$ together with all elements of the form $qvdw - (dw^i)v_i$. In the example given above $\Omega_R V$ is precisely the differential calculus studied by Pusz, Woronowicz, Wess and Zumino. For example, in the case $n = 2$, letting $e_1 = x$, $e_2 = y$, we obtain the relations

$$xy = qyx \ , \ xdx = q^2(dx)x \ , \ ydy = q^2(dy)y \ ,$$

$$ydx = q(dx)y \ , \ xdy = q(dy)x + (q^2 - 1)(dx)y \ ,$$

and differentiating,

$$(dx)^2 = -q^2(dx)^2 \ , \ (dy)^2 = -q^2(dy)^2 \ , \ dx\,dy = -q^{-1}dy\,dx \ .$$

The factor of q^{-1} in the relation

$$vdw = q^{-1}(dw^i)v_i \tag{1}$$

may seem odd at first glance, but is easily understood. Note that since R is of type q,

$$R(w^i \otimes v_i) = (1 - q)w^i \otimes v_i \ + \ qv \otimes w \ ,$$

so that

$$w^i dv_i = (q^{-1} - 1)(dw^i)v_i \ + \ (dv)w \ . \tag{2}$$

The point is that a linear combination of equations (1) and (2) gives the relation

$$(dv)w + vdw = (dw^i)v_i + w^i\,dv_i \ ,$$

which must hold in any differential calculus over $S_R V$. Without the factor of q^{-1} in relation (1), this would not be the case.

Since the algebraic differential forms on a vector space V form a graded commutative algebra isomorphic to $SV \otimes \Lambda V$, one could hope that the differential forms on the sort of "quantum vector space" we are considering here form a "graded r-commutative algebra." This is essentially the case, although as r-commutativity subsumes graded commutativity, the notion of "graded r-commutativity" is redundant. We shall show that $\Omega_R V$ has a natural r-commutative structure. Let us write $i: V \to \Omega_R V$ for the natural inclusion, and write $d: V \to \Omega_R V$ for the map $v \mapsto dv$. Let $m: \Omega_R V \otimes \Omega_R V \to \Omega_R V$ denote the multiplication map. Then (1) may be written as

$$m(i \otimes d) = q^{-1} m(d \otimes i) R ,$$

and (2) may be written as

$$m(d \otimes i) = m(i \otimes d + (1 - q^{-1}) d \otimes i) R .$$

Along these lines, we also have

$$m(i \otimes i) = m(i \otimes i) R ,$$

and, applying d to both sides of equation (1),

$$m(d \otimes d) = -q^{-1} (d \otimes d) R .$$

These equations motivate the following:

Theorem 1. *Let V be a vector space equipped with a Yang-Baxter operator R of type q. There is a unique r-structure $\widetilde{R}$ on $\Omega_R V$ such that the following hold:*

$$\widetilde{R}(i \otimes i) = (i \otimes i) R , \tag{3}$$

$$\widetilde{R}(i \otimes d) = q^{-1} (d \otimes i) R , \tag{4}$$

$$\widetilde{R}(d \otimes i) = (i \otimes d + (1 - q^{-1}) d \otimes i) R , \tag{5}$$

$$\widetilde{R}(d \otimes d) = -q^{-1} (d \otimes d) R . \tag{6}$$

Moreover, this r-structure is r-commutative.

Proof - Let W be the subspace of $\Omega_R V$ spanned by elements of the form v and dv, where $v \in V$. Note that $W \otimes W$ is invariant under the proposed r-structure $\widetilde{R}$ and generates $\Omega_R V$ as an algebra. Thus Lemma 1 of [1] implies that $\widetilde{R}$, if it exists, is uniquely determined by its action on $W \otimes W$, which is given by equations (3-6).

To prove existence, now let W denote the space $V \oplus V$, and let $\iota v = (v, 0) \in V \oplus V$, and $\partial v = (0, v) \in V \oplus V$. The universal differential calculus $\Omega_u(S_R V)$ is naturally isomorphic to the quotient of the tensor algebra TW by the relations

$$(\iota v)(\iota w) = (\iota w^i)(\iota v_i)\ , \ (\partial v)(\iota w) + (\iota v)(\partial w) = (\partial w^i)(\iota v_i) + (\iota w^i)(\partial v_i)\ ,$$

the isomorphism being given by $v \mapsto [\iota v]$, $dv \mapsto [\partial v]$. It follows from the calculations above that $\Omega_R V$ is naturally isomorphic to the quotient of the algebra TW by the relations

$$m(\iota \otimes \iota)(v \otimes w) = m(\iota \otimes \iota)R(v \otimes w)\ ,$$

$$m(\iota \otimes \partial)(v \otimes w) = q^{-1}(\partial \otimes \iota)R(v \otimes w)\ ,$$

$$m(\partial \otimes \iota)(v \otimes w) = (\iota \otimes \partial + (1 - q^{-1})\partial \otimes \iota)R(v \otimes w)\ ,$$

and

$$m(\partial \otimes \partial)(v \otimes w) = -q^{-1}(\partial \otimes \partial)R(v \otimes w)\ ,$$

where m denotes the multiplication map for TW. We claim that there is a Yang-Baxter operator $\widetilde{R}$ on W such that equations like (3-6) hold, but with i replaced by ι and d by ∂. If this is the case, the description of $\Omega_R V$ as a quotient of TW shows that there is an algebra isomorphism $\Omega_R V \cong S_{\widetilde{R}} W$ given by $v \mapsto \iota v$, $dv \mapsto \partial v$. Using this isomorphism to transfer the r-structure on $S_{\widetilde{R}} W$ to $\Omega_R V$, we obtain the desired r-structure on $\Omega_R W$, which is clearly r-commutative.

Checking that $\widetilde{R} \in End(W \otimes W)$ actually satisfies the Yang-Baxter equations consists of checking eight cases. Thus, for example, showing that

$$\widetilde{R}_1 \widetilde{R}_2 \widetilde{R}_1(\partial u \otimes \partial v \otimes \iota w) = \widetilde{R}_2 \widetilde{R}_1 \widetilde{R}_2(\partial u \otimes \partial v \otimes \iota w)$$

for all $u, v, w \in V$ is equivalent to showing that $\tilde{R}_1\tilde{R}_2\tilde{R}_1(\partial\otimes\partial\otimes\iota) = \tilde{R}_2\tilde{R}_1\tilde{R}_2(\partial\otimes\partial\otimes\iota)$. Indeed,

$$\tilde{R}_1\tilde{R}_2\tilde{R}_1(\partial\otimes\partial\otimes\iota) =$$

$$-q^{-1}\tilde{R}_1\tilde{R}_2(\partial\otimes\partial\otimes\iota)R_1 =$$

$$-q^{-1}\tilde{R}_1(\partial\otimes\iota\otimes\partial + (1-q^{-1})\partial\otimes\partial\otimes\iota)R_2R_1 =$$

$$-q^{-1}(\iota\otimes\partial\otimes\partial + (1-q^{-1})\partial\otimes\iota\otimes\partial - q^{-1}(1-q^{-1})\partial\otimes\partial\otimes\iota)R_1R_2R_1$$

is equal to

$$\tilde{R}_2\tilde{R}_1\tilde{R}_2(\partial\otimes\partial\otimes\iota) =$$

$$\tilde{R}_2\tilde{R}_1(\partial\otimes\iota\otimes\partial + (1-q^{-1})\partial\otimes\partial\otimes\iota)R_2 =$$

$$\tilde{R}_2(\iota\otimes\partial\otimes\partial + (1-q^{-1})\partial\otimes\iota\otimes\partial - q^{-1}(1-q^{-1})\partial\otimes\partial\otimes\iota)R_1R_2 =$$

$$(-q^{-1}\iota\otimes\partial\otimes\partial - q^{-1}(1-q^{-1})\partial\otimes\iota\otimes\partial + q^{-2}(1-q^{-1})\partial\otimes\partial\otimes\iota)R_2R_1R_2\ .$$

The other seven cases are similar, and for the most part simpler. □

Corollary 1. *As an r-algebra, $\Omega_R V$ is isomorphic to the r-symmetric algebra generated by $S_{\tilde{R}}(V\oplus V)$, where $R \in End(V\oplus V)$ is given by equations (3-6), and $iv = (v, 0)$, $dv = (0, v)$.*

Proof - This follows from the proof of Theorem 1. □

3 Covariance of the Differential Calculus

We begin by reviewing some results on quantum matrix algebras from the viewpoint of r-commutative algebra. If R a Yang-Baxter operator on the vector space V, we define the Yang-Baxter operator $\tilde{R}$ on $End(V)$ as follows, identifying $End(V)\otimes End(V)$ with $End(V\otimes V)$:

$$\tilde{R}(f\otimes g) = R(f\otimes g)R^{-1}\ ,$$

where $f, g \in End(V)$. We call the r-symmetric algebra $S_{\widetilde{R}}(End(V))$ the *quantum matrix algebra* $M_R V$. Up to minor changes in notation, this is the same as the quantum matrix algebra discussed by Faddeev, Reshetikhin and Takhtajan [5].

We will describe a coaction of $M_R V$ on $\Omega_R V$ in the case when V is finite-dimensional and R is of type q. First, though, we need a lemma on tensor products. If V and V' are vector spaces equipped with Yang-Baxter operators R and R', there is a Yang-Baxter operator $R \hat{\otimes} R'$ on $V \otimes V'$ given by $R \hat{\otimes} R' = \sigma(R \otimes R')\sigma^{-1}$, where $\sigma\colon (V \otimes V')^{\otimes 2} \to V^{\otimes 2} \otimes V'^{\otimes 2}$ is given by $\sigma((v \otimes v') \otimes (w \otimes w')) = (v \otimes w) \otimes (v' \otimes w')$. We call $R \hat{\otimes} R'$ the *tensor product* Yang-Baxter operator.

Lemma 1. *Let V and V' be vector spaces equipped with Yang-Baxter operators R and R'. Then there is a unique homomorphism $i\colon S_{R \hat{\otimes} R'}(V \otimes V') \to S_R V \otimes S_R V'$ extending the identity map $V \otimes V' \to V \otimes V'$.*

Proof - Let $\iota\colon T(V \otimes W) \to TV \otimes TW$ be the algebra homomorphism extending the identity map on $V \otimes W$. We will show that there is a unique homomorphism i such that the following diagram commutes:

$$\begin{array}{ccc} T(V \otimes V') & \xrightarrow{\iota} & TV \otimes TV' \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\ S_{R \hat{\otimes} R'}(V \otimes V') & \xrightarrow{i} & S_R V \otimes S_{R'} V' \end{array}$$

where π_1, π_2 are the natural quotient maps. First, i is clearly unique if it exists since π_1 is onto. For existence it suffices to prove that $\ker \pi_1 \subseteq \ker \pi_2 \iota$. Since ι is 1-1, it suffices to show that $\iota \ker \pi_1 \subseteq \ker \pi_2$. Recall that $\ker \pi_1$ is generated by elements of the form $(R \hat{\otimes} R' - 1)((v \otimes v') \otimes (w \otimes w'))$, where $v, w \in V$ and $v', w' \in V'$. Applying ι to such an element we obtain

$$(R \otimes R' - 1)(v \otimes w \otimes v' \otimes w') =$$
$$(R - 1)(v \otimes w) \otimes R'(v' \otimes w') + (v \otimes w) \otimes (R' - 1)(v' \otimes w') \ .$$

Similarly, $\ker \pi_2$ is generated by elements of the form $(R-1)(v \otimes w) \otimes 1$ and $1 \otimes (R' - 1)(v' \otimes w')$. It follows that the generators (7) of $\iota \ker \pi_1$ lie in $\ker \pi_2$. $\square$

If W is a vector space with Yang-Baxter operator R we equip the dual W^* with the Yang-Baxter operator $(R^{-1})^*$. This makes the pairing $W^* \otimes W \to k$ into an r-map when $W^* \otimes W$ is given the tensor product Yang-Baxter operator and k is given the trivial Yang-Baxter operator (the identity on $k \otimes k$). Moreover, identifying $End(W)$ with $W \otimes W^*$, we have $\widetilde{R} = R \hat{\otimes} (R^{-1})^*$. As a consequence, the natural action $\alpha: End(W) \otimes W \to W$ is an r-map, where $End(W) \otimes W$ is given the tensor product Yang-Baxter operator. Now suppose that W is finite-dimensional. Using the natural isomorphism $End(W)^* \cong End(W^*)$, we may dualize the action α to obtain an r-map

$$\alpha^*: V \to End(V) \otimes V$$

where $V = W^*$. In fact this r-map is well-defined for any finite-dimensional vector space V equipped with a Yang-Baxter operator (since we may take $W = V^*$). Quite concretely, if e_i is a basis for V and $e^i_j \in End(V)$ are such that $e^i_j e_k = \delta^i_k e_j$, we have $\alpha^* e_j = e^i_j \otimes e_i$. The r-map α^* induces an r-morphism from $S_R V$ to $S_{\widetilde{R} \hat{\otimes} R}(End(V) \otimes V)$, which we compose with the map given in Lemma 1 to obtain an r-morphism

$$\Phi: S_R V \to M_R V \otimes S_R V \ .$$

In a similar manner, the multiplication map $\mu: End(W) \otimes End(W) \to End(W)$ dualizes to give the r-map $\mu^*: End(V) \to End(V) \otimes End(V)$; explicitly, we have $\mu^*(e^i_j) = e^k_j \otimes e^i_k$. This gives rise to an r-morphism

$$\Delta: M_R V \to M_R V \otimes M_R V \ .$$

Since $S_R V$ is generated by V and $M_R V$ is generated by $End(V)$, we may easily check that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \ ,$$

$$(\Delta \otimes 1)\Phi = (1 \otimes \Phi)\Phi \ ,$$

using the properties of α^* and μ^*. In short, Δ is a coproduct on $M_R V$ and Φ is a coaction of $M_R V$ on $S_R V$. Moreover, the map $\epsilon\colon M_R V \to k$ given by $\epsilon(e^i_j) = \delta^j_i$ is a counit for $M_R V$.

Now that we have translated some results on quantum matrix algebras into the language of r-commutative algebra, we can prove the covariance of the differential calculus $\Omega_R V$, where V is a finite-dimensional vector space with a Yang-Baxter operator R of type q. In the following theorem, we regard $M_R V \otimes \Omega_R V$ as a differential graded algebra with the differential $D = 1 \otimes d$.

Theorem 2. *The homomorphism $\Phi\colon S_R V \to M_R V \otimes S_R V$ extends uniquely to a differential graded algebra morphism $\Phi_*\colon \Omega_R V \to M_R V \otimes \Omega_R V$. Moreover, Φ_* is a coaction, that is,* $(\Delta \otimes 1)\Phi_* = (1 \otimes \Phi_*)\Phi_*$ *.*

Proof - The uniqueness of Φ_* follows from the fact that $\Omega_R V$ is a differential calculus for $S_R V$, so that any differential graded algebra morphism from $\Omega_R V$ is determined uniquely by its restriction to $S_R V$. For existence, note first that Φ extends uniquely to a differential graded algebra morphism $\Psi\colon \Omega_u(S_R V) \to M_R V \otimes \Omega_R V$, by the universal property of the universal differential calculus. It suffices, therefore, to show that $\ker \Psi$ contains $\ker j$, where $j\colon \Omega_u(S_R V) \to \Omega_R V$ is the quotient map. Recall that $\ker j$ is generated as a differential ideal by $d1$ and the elements of form $m(q(i \otimes d) - (d \otimes i)R)(v \otimes w)$ for $v, w \in V$. We shall show that Ψ annihilates these elements. First, $\Psi(d1) = D\Psi 1 = D(1 \otimes 1) = 0$. Second, using the fact that Ψ extends Φ and that Φ is an r-map,

$$\Psi m(q(i \otimes d) - (d \otimes i)R)(v \otimes w) = m(q(1 \otimes D) - (D \otimes 1)(\widetilde{R} \hat{\otimes} R))(\Phi \otimes \Phi)(v \otimes w)\ .$$

Unraveling the definitions and using the fact that $m(q(i \otimes d) - (d \otimes i)R) = 0$ as a map from $V \otimes V$ to $\Omega_R V$, it follows that the right side vanishes. The fact that Φ_* is a coaction follows from the fact that Φ is a coaction and Φ_* is a differential graded algebra morphism extending Φ to the differential calculus $\Omega_R V$. $\square$

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