A model for the String group

André Henriques

June 2005

The string group $String(n)$ is the 3-connected cover of $Spin(n)$. Given and compact simply connected group G, we will let $String_G$ be its 3-connected cover. The group $String_G$ is only defined up to homotopy, and various models have appeared in the literature. Stephan Stolz and Peter Teichner [7], [6] have a couple of models of $String_G$, one of which, inspired by Anthony Wassermann, is an extension of G by the group of projective unitary operators in a particular Von-Neuman algebra. Jean-Luc Brylinski [4] has a model which is a $U(1)$-gerbe with connection over the group G. More recently, John Baez et al [2] came up with a model of $String_G$ in their quest for a 2-Lie group integrating a given 2-Lie algebra. We show how to produce their model by applying a certain canonical procedure to their 2-Lie algebra.

A 2-Lie algebra is a two step L_∞-algebra. It consists of two vector spaces V_0 and V_1, and three brackets $[\,]$, $[\,,]$, $[\,,\,,]$ acting on $V := V_0 \oplus V_1$. They are of degree -1, 0, and 1 respectively and satisfy various axioms, see [1] for more details.

A 2-group is a group object in a 2-category [3]. It has a multiplication $\mu : G^2 \to G$, and an associator $\alpha : \mu \circ (\mu \times 1) \simeq \mu \circ (1 \times \mu)$ satisfying the pentagon axiom. There are strict and weak versions. If the 2-category is that of C^∞ Artin stacks, we get the notion of a 2-Lie group. Since Artin stacks are represented by Lie groupoids, we can think of (strict) 2-Lie group as group objects in Lie groupoids. Equivalently, these are crossed modules in the category of smooth manifolds [3].

It is also good to consider weak 2-groups. The classifying space of a weak 2-group contains (up to homotopy) the same amount of information as the 2-group itself. So we will replace 2-Lie groups with their classifying space. This also allows for an easy way to talk about n-Lie groups. The following definition was inspired by discussions with Jacob Lurie:

Definition 1 *The classifying space of a weak n-Lie group is a simplicial manifold*

$$X_\bullet = \big(X_0 \leftleftarrows X_1 \, \substack{\leftarrow\\[-1pt]\leftarrow\\[-1pt]\leftarrow} \, X_2 \cdots\big)$$

satisfying $X_0 = pt$, and the following version of the Kan condition:
Let $\Lambda^{m,j} \subset \partial\Delta^m$ be the jth horn. Then the restriction map

$$X_m = Hom(\Delta^m, X_\bullet) \to Hom(\Lambda^{m,j}, X_\bullet) \tag{1}$$

is a surjective fibration for all $m \leq n$ and a diffeomorphism for all $m > n$.

Given an n-Lie algebra, there exists a canonical procedure that produces the classifying space of an n-Lie group. The main idea goes back to Sullivan's work on rationnal homotopy theory [8]. A variant is further studied in [5].

Definition 2 *Let V be an n-Lie algebra with Chevaley-Eilenberg complex $C^*(V)$. The classifying space of the corresponding n-Lie group is then given by*

$$\big(\textstyle\int_n V\big)_m := Hom_{DGA}\big(C^*(V), \Omega^*(\Delta^m)\big)\big/\sim, \tag{2}$$

where $\sim$ identifies two m-simplicies if they are simplicially homotopic relatively to their $(n-1)$-skeleton.

Example 1 Let $\mathfrak{g}$ be a Lie algebra with corresponding Lie group G. A homomorphism from $C^*(\mathfrak{g})$ to $\Omega^*(\Delta^n)$ is the same thing as a flat connection on the trivial G-bundle $G \times \Delta^n$. These in turn correspond to maps $\Delta^n \to G$ modulo translation. Two n-simplicies are simplicially homotopic relatively to their 0-skeleton if their vertices agree. So we get

$$(\textstyle\int_1 \mathfrak{g})_n = Map\big(sk_0(\Delta^n), G\big)/G = G^n.$$

Therefore $\int_1 \mathfrak{g}$ is the standard simplicial model for BG. We can recover G along with its group structure by taking the simplicial π_1 of this simplicial manifold.

Now let us consider our motivating example. Let $\mathfrak{g}$ be a simple Lie algebra of compact type (defined over $\mathbb{R}$), and let $\langle , \rangle$ be the inner product on $\mathfrak{g}$ such that the norm of the short coroots is 1.

Definition 3 [2] *Let $\mathfrak{g}$ be a simple Lie algebra of compact type. Its string Lie algebra is the 2-Lie algebra $\mathfrak{str} = \mathfrak{str}(\mathfrak{g})$ given by*

$$\mathfrak{str}_0 = \mathfrak{g}, \qquad \mathfrak{str}_1 = \mathbb{R}$$

and brackets

$$[\,] = 0, \quad [(X_1, c_1), (X_2, c_2)] = ([X_1, X_2], 0),$$
$$[(X_1, c_1), (X_2, c_2), (X_3, c_3)] = (0, \langle [X_1, X_2], X_3\rangle).$$

The string Lie algebra should be thought as a central extension of the Lie algebra $\mathfrak{g}$, but which is controlled by $H^3(\mathfrak{g}, R)$ as opposed to $H^2(\mathfrak{g}, \mathbb{R})$. The Chevalley-Eilenberg complex of $\mathfrak{str}$ is then given by

$$C^*(\mathfrak{str}) = \mathbb{R} \oplus \Big[\mathfrak{g}^*\Big] \oplus \Big[\Lambda^2\mathfrak{g}^* \oplus \mathbb{R}\Big] \oplus \Big[\Lambda^3\mathfrak{g}^* \oplus \mathfrak{g}^*\Big] \oplus \Big[\Lambda^4\mathfrak{g}^* \oplus \Lambda^2\mathfrak{g}^* \oplus \mathbb{R}\Big] \oplus \ldots$$

Following (2), we study

$$\begin{aligned} Hom_{DGA}\big(C^*(\mathfrak{str}), \Omega^*(\Delta^n)\big) = \big\{&\alpha \in \Omega^1(\Delta^n; \mathfrak{g}), \beta \in \Omega^2(\Delta^n; \mathbb{R}) \,\big|\, \\ & d\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0, d\beta + \tfrac{1}{6}[\alpha, \alpha, \alpha] = 0\big\}. \end{aligned} \tag{3}$$

The 1-form α satisfies the Maurer Cartan equation, so we can integrate it to a map $f : \Delta^n \to G$, defined up to translation. This map satisfies $f^*(\theta_L) = \alpha$, where $\theta_L \in \Omega^1(G; \mathfrak{g})$ is the left invariant Maurer Cartan form on G. The 3-form $\frac{1}{6}[\alpha, \alpha, \alpha]$ is then the pullback of the Cartan 3-form

$$\eta = \tfrac{1}{6}\langle[\theta_L, \theta_L], \theta_L\rangle \in \Omega^3(G; \mathbb{R}),$$

which represents the generator of $H^3(G, \mathbb{Z})$. So we can rewrite (3) as

$$\big\{ f : \Delta^n \to G, \beta \in \Omega^2(\Delta^n) \,\big|\, d\beta = f^*(\eta) \big\} / G. \tag{4}$$

The set of n-simplices in $\textstyle\int_2 \mathfrak{str}$ is then the quotient of (4) by the relation of simplicial homotopy relative to the 1-skeleton. Applying this procedure, we get a simplicial manifold whose geometric realization has the homotopy type of $BString_G$ and which is equal to the nerve of the 2-group described in [2]. It is given by

$$\textstyle\int_2 \mathfrak{str} = \Big[* \leftleftarrows Path(G)/G \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \widetilde{Map(\partial\Delta^2, G)}/G \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \widetilde{Map(sk_1\Delta^3, G)}/G \cdots \Big],$$

where the tilde indicates that the group $Map(sk_1\Delta^i, G)$ has been centrally extended by $S^1 \otimes H_1(sk_1\Delta^i)$. Moreover, its simplicial homotopy groups are given by $\pi_1(\textstyle\int_2 \mathfrak{str}) = G$ and $\pi_2(\textstyle\int_2 \mathfrak{str}) = S^1$.

References

[1] John Baez and Alissa Crans. Higher-dimensional algebra VI. Lie 2-algebras. *Theory Appl. Categ.*, 12:492–538 (electronic), 2004.

[2] John Baez, Alissa Crans, Stevenson Danny, and Urs Schreiber. From loop groups to 2-groups. *math.QA/0504123*, 2005.

[3] John Baez and Aaron D. Lauda. Higher-dimensional algebra V. 2-groups. *Theory Appl. Categ.*, 12:423–491 (electronic), 2004.

[4] J.-L. Brylinski and D. A. McLaughlin. The geometry of degree-four characteristic classes and of line bundles on loop spaces. I & II. *Duke Math. J.*, 75 & 83(3 & 1):603–638 & 105–139, 1994 & 1996.

[5] Ezra Getzler. Lie theory for nilpotent L_∞-algebras. *math.AT/0404003*, 2004.

[6] Stephan Stolz. A conjecture concerning positive ricci curvature and the witten genus. *Math. Ann.*, 304(4):785–800, 1996.

[7] Stephan Stolz and Peter Teichner. What is an elliptic object? *Topology, geometry and quantum field theory*, 308:London Math. Soc. Lecture Note Ser. 247–343, 2004.

[8] Dennis Sullivan. Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math*, 47:269–331, 1997.