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ON THE HOPF TERM IN A TWO-DIMENSIONAL σ MODEL FOR ANTIFERROMAGNETS

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We discuss the possible topological terms in $(2+1)d$ sigma models for antiferromagnets and point out errors in some of the 'proofs' of the nonexistence of the Hopf term.

In considering the high T_c superconductivity which occurs in materials with antiferromagnetic 2-dimensional layers, it is common to use a continuum approximation in terms of a spin field $\mathbf{n}(t, x, y)$ where $\mathbf{n} = (n_x\, n_y\, n_t)$ satisfies the constraint $\mathbf{n} \cdot \mathbf{n} = 1$. The classical action in this approximation is:

$$s \int (\partial_\mu n_i)^2 dt dx dy \,,$$

where s is the spin. Differentiating the constraint on $\mathbf{n}$ implies that the two-form F defined by $F_{\nu\lambda} = \varepsilon^{ijk} n_i \partial_\nu n_j \partial_\lambda n_k$ is closed: $dF = 0$. It follows immediately that the 'topological current' $J^\mu = \epsilon^{\mu\nu\lambda} F\nu\lambda$ is conserved: $\partial_\mu J^\mu = 0$. It also follows from DeRahm theory that $F = dA$ for some one-form A: $\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$. Dzialoshinskii, Polyakov, and Weigmann[1] suggested that to the above action for the $2+1$ dimensional Heisenberg antiferromagnetic field should be added the Hopf term:

$$\frac{k}{8\pi} \int A_\mu J^\mu dt dx dy \,.$$

The Hopf term assumes nonzero integer values on "Hopf textures", or homotopically nontrivial maps from S^3 to S^2. Since the Hopf term is constant for each homotopy class of maps, it does not affect the classical equations of motion, but in the quantum theory it endows the skyrmions described by the model with fractional statistics.

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Since then several authors have claimed that the Hopf term in the action for the Heisenberg antiferromagnet is actually zero.[2,3,4,5] At present the general belief seems to be that there are several proofs that the Hopf term is zero.

In the work of Haldane,[2] the $(2+1)d$ problem is considered as a generalisation of the $(1+1)d$ problem. This topological term that appears in the $(1+1)d$ problem is the Pontryagin index:

$$Q = \frac{1}{4\pi} \int \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_t \mathbf{n}) dt dx \,.$$

If we consider the $(1+1)d$ problem as imbedded in the $(2+1)d$ problem as the plane ($y = \text{const}$) then the integrand of the above expression is the y-component of the topological current J_y: $Q = (1/4\pi) \int J_y dx dt$. If one imposes the boundary condition that $\mathbf{n}(t, x, y) = \mathbf{n}_0$ at space-time infinity then Q is an integer independent of the plane $\{y = \text{const}\}$. Thus

$$\int \partial_y J_y dt dx dy = 0 \text{ and similarly } \int \partial_x J_x dt dx dy = 0 \,. \tag{1}$$

These expressions are part of the topological term that Haldane[2] calculates in $(2+1)d$ as a direct generalization of the topological term in $(1+1)d$. The fact that both expressions are zero leads him to conclude that there is no topological term in $(2+1)d$ or any higher dimension. It is clear, however, that these terms are *not* part of the Hopf term. They are part of $\int \partial_\mu J^\mu dx dy dt = 0$.

Dombre and Read[3] expand the Lagrangian for the lattice $(2+1)d$ problem in powers of the space-time derivatives up to the third order. The terms with third-order derivatives are essentially the same as the integrals in Eq. (1), and vanish as in Ref. 2 because of the boundary conditions. Here it is important to note that the Hopf invariant is nonlocal in terms of $\mathbf{n}$ and an expansion of the Lagrangian in powers of the space-time derivatives cannot give a nonlocal function of $\mathbf{n}(x, y, t)$. The third order terms considered in Ref. 3 do not represent the Hopf term.

In both works,[2,3] the derivation of the continuum limit requires that the number of spins along a line in the (x, y) plane be even. This seemingly unphysical assumption indicates that the passage to the continuum limit for antiferromagnetic theories is not well understood.

In general, in a system that may be separated into 'slow' and 'fast' variables, topological terms may appear as a Berry's phase. Let S denote the phase space of the slow variables (which are treated classically) and let $\mathbf{H}$ denote the Hilbert space of states for the fast variables (which are treated quantum-mechanically). The state of the slow variables defines an external Hamiltonian for the fast variables, i.e., for each $s \in S$ there is a Hamiltonian H_s on $\mathbf{H}$. A smoothly varying family of nondegenerate eigenvalues, $\Psi_s \in H$ with $H_s \Psi_s = E_s \Psi_s$, defines a complex line bundle on S, and adiabatic evolution of the system corresponds to parallel transport in this bundle with respect to the Berry connection. The holonomy around a loop is known as Berry's phase. It is worth noting that there are two sources of Berry's phase: by the Ambrose-Singer theorem[6] the holonomy of a connection arises from curvature (a local effect) and from transport around noncontractible loops (a global effect). It is the latter that should be the natural source of Hopf term.

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