Levin-Wen Models and Tensor Categories

Liang Kong

CQT, National University of Singapore, Feb. 10, 2011

Institute for Advanced Study in Tsinghua University, Beijing

a joint work with Alexei Kitaev

Goals:

- to present a theory of boundary and defects of codimension 1,2,3 in non-chiral topological orders via Levin-Wen models;
- to show how the representation theory of tensor category enters the study of topological order at its full strength;
- to provide the physical foundation of the so-called extended Turaev-Viro topological field theories;

Kitaev's Toric Code Model

Levin-Wen models

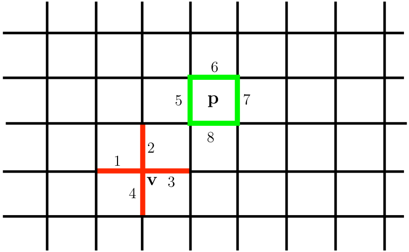
Extended Topological Field Theories

Outline

Kitaev's Toric Code Model

- Kitaev's Toric Code Model is equivalent to Levin-Wen model associated to the category $\mathrm{Rep}_{\mathbb{Z}_2}$ of representations of $\mathbb{Z}_2$.
- It is the simplest example that can illustrate the general features of Levin-Wen models.

Kitaev's Toric Code Model



$$\mathcal{H} = \otimes_{e \in \text{all edges}} \mathcal{H}_e; \qquad \mathcal{H}_e = \mathbb{C}^2.$$

$$H = -\sum_v A_v - \sum_p B_p.$$

$$A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4; \qquad B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8.$$

Vacuum properties of toric code model:

A vacuum state $|0\rangle$ is a state satisfying $A_v|0\rangle = |0\rangle, B_p|0\rangle = |0\rangle$ for all v and p.

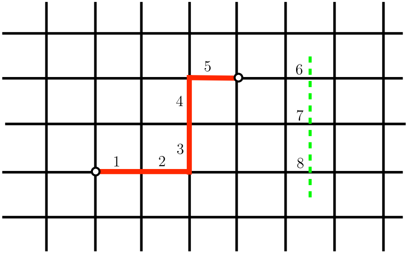
- If surface topology is trivial (a sphere, an infinite plane), the vacuum is unique.
- Vacuum is given by the condensation of closed strings, i.e.

$$|0\rangle = \sum_{c \in \text{all closed string configurations}} |c\rangle.$$

Excitations

- The "set" of excitations determines the topological phase.
- An excitation is defined to be super-selection sectors (irreducible modules) of a local operator algebra.
- There are four types of excitations: $1, e, m, \epsilon$. We denote the ground states of these sectors as $|0\rangle, |e\rangle, |m\rangle, |\epsilon\rangle$. We have

$$
\begin{aligned}
\exists v_0, \qquad & A_{v_0}|e\rangle = -|e\rangle, \\
\exists p_0, \qquad & B_{p_0}|m\rangle = -|m\rangle, \\
\exists v_1, p_1, \qquad & A_{v_1}|\epsilon\rangle = -|\epsilon\rangle,\ B_{p_1}|\epsilon\rangle = -|\epsilon\rangle.
\end{aligned}
$$



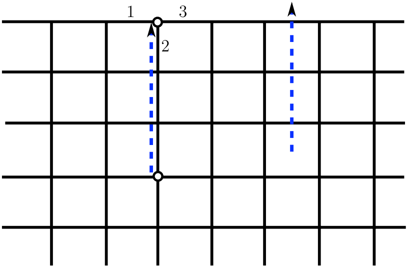
$$1 = e \otimes e \sim \sigma_z^1 \sigma_z^2 \sigma_z^3 \sigma_z^4 \sigma_z^5 |0\rangle,$$

$$1 = m \otimes m \sim \sigma_x^6 \sigma_x^7 \sigma_x^8 |0\rangle,$$

$$e \otimes m = \epsilon.$$

- ☞ $1, e, m, \epsilon$ are simple objects of a braided tensor category $Z(\mathrm{Rep}_{\mathbb{Z}_2})$ which is the monoidal center of $\mathrm{Rep}_{\mathbb{Z}_2}$.

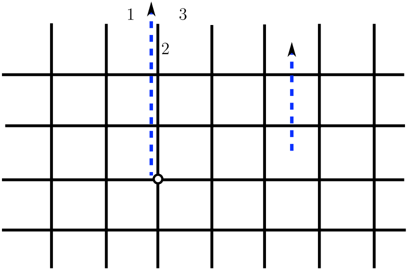
A smooth edge



$$\begin{aligned} 1 &\longrightarrow 1 \quad e \longrightarrow e \\ m &\longrightarrow 1 \quad \epsilon \longrightarrow e \end{aligned}$$

☞ This assignment actually gives a monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Rep}_{\mathbb{Z}_2}, \mathrm{Rep}_{\mathbb{Z}_2})$.

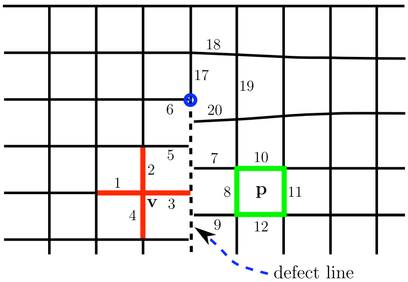
A rough edge



$$\begin{array}{ccccccc} 1 & \longrightarrow & 1 & & m & \longrightarrow & m \\ e & \longrightarrow & 1 & & \epsilon & \longrightarrow & m \end{array}$$

☞ This assignment gives another monoidal functor $Z(\mathsf{Rep}_{\mathbb{Z}_2}) \to \mathsf{Rep}_{\mathbb{Z}_2} = \mathsf{Fun}_{\mathsf{Rep}_{\mathbb{Z}_2}}(\mathsf{Hilb}, \mathsf{Hilb})$.

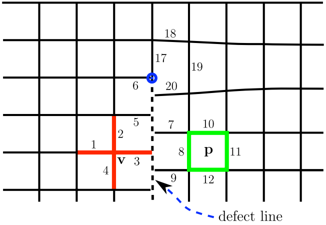
defects of codimension 1, 2



$$B_{p_1} = \sigma_x^7 \sigma_x^3 \sigma_x^2 \sigma_x^5; \quad B_{p_2} = \sigma_x^3 \sigma_x^7 \sigma_x^8 \sigma_x^9;$$

$$B_Q = \sigma_x^6 \sigma_y^{17} \sigma_z^{18} \sigma_z^{19} \sigma_z^{20}.$$

defects of codimension 1

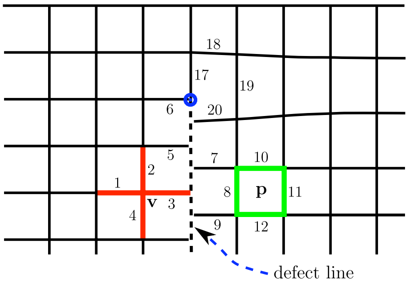


$$1 \mapsto 1 \mapsto 1, \qquad e \xrightarrow{\sigma_z^3} \mathrm{Ext}^{defect}_{3|7,8,9} \xrightarrow{\sigma_x^8} m,$$

$$m \mapsto \mathrm{Ext}^{defect}_{7|3,2,5} \mapsto e, \qquad \epsilon \mapsto \mathrm{Ext}^{defect}_{2,5,7,8,9,3} \mapsto \epsilon.$$

☞ This assignment gives an invertible monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}|\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Hilb}, \mathrm{Hilb}) \to Z(\mathrm{Rep}_{\mathbb{Z}_2})$.

defects of codimension 2



$$B_Q = \sigma_x^6 \sigma_y^{17} \sigma_z^{18} \sigma_z^{19} \sigma_z^{20}$$

☞ Two eigenstates of B_Q correspond to two simple $\mathrm{Rep}_{\mathbb{Z}_2}$-$\mathrm{Rep}_{\mathbb{Z}_2}$-bimodule functors $\mathrm{Hilb} \to \mathrm{Rep}_{\mathbb{Z}_2}$.

Outline

Basics of unitary tensor category

unitary tensor category $\mathcal{C}$ = unitary spherical fusion category

- semisimple: every object is a direct sum of simple objects;
- finite: there are only finite number of inequivalent simple objects, $i, j, k, l \in \mathcal{I}$, $|\mathcal{I}| < \infty$; $\dim \mathrm{Hom}(A, B) < \infty$.
- monoidal: $(i \otimes j) \otimes k \cong i \otimes (j \otimes k)$; $\mathbf{1} \in \mathcal{I}$, $\mathbf{1} \otimes i \cong i \cong i \otimes \mathbf{1}$;
- the fusion rule: $\dim \mathrm{Hom}(i \otimes j, k) = N_{ij}^{k}$ is finite;
- $\mathcal{C}$ is not assumed to be braided.

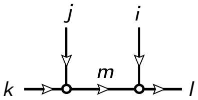
Theorem (Müger): The monoidal center $Z(\mathcal{C})$ of $\mathcal{C}$ is a modular tensor category.

Fusion matrices

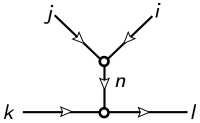
The associator $(i \otimes j) \otimes k \xrightarrow{\alpha} i \otimes (j \otimes k)$ induces an isomorphism:

$$\mathrm{Hom}((i \otimes j) \otimes k, l) \xrightarrow{\cong} \mathrm{Hom}(i \otimes (j \otimes k), l)$$

Writing in basis, we obtain the fusion matrices:



$$= \sum_n F^{ijk;l}_{mn}$$



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Levin-Wen models

We fix a unitary tensor category $\mathcal{C}$ with simple objects $i, j, k, l, m, n \in \mathcal{I}$.

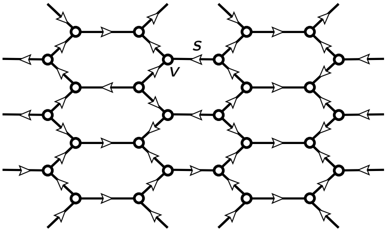


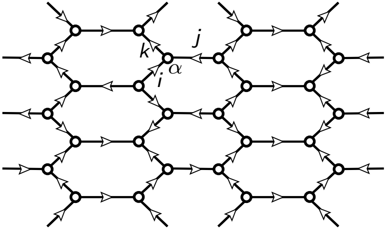
Figure: Levin-Wen model defined on a honeycomb lattice.

$$\mathcal{H}_s = \mathbb{C}^{\mathcal{I}}, \quad \mathcal{H}_v = \oplus_{i,j,k} \mathrm{Hom}_{\mathcal{C}}(i \otimes j, k).$$

$$\mathcal{H} = \otimes_s \mathcal{H}_s \otimes_v \mathcal{H}_v.$$

Hamiltonian

Chose a basis of $\mathcal{H}$, $i, j, k \in \mathcal{I}$ and $\alpha^{i'j';k'} \in \mathrm{Hom}_{\mathcal{C}}(i' \otimes j', k')$,



$$H = -\sum_{v} A_v - \sum_{p} B_p.$$

$$A_v|(i,j;k|\alpha^{i',j';k'})\rangle = \delta_{i,i'}\delta_{j,j'}\delta_{k,k'}|(i,j;k|\alpha^{i',j';k'})\rangle.$$

If the spin on v is such that A_v acts as 1, then it is called stable.

The definition of B_p operator

$$B_p := \sum_{i \in \mathcal{I}} \frac{d_i}{\sum_k d_k^2} B_p^i$$

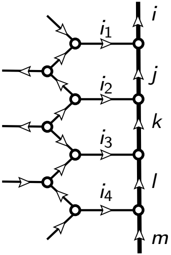
- If there are unstable spins around the plaquette p, B_p^i act on the plaquette as zero.
- If all spins around the plaquette p is stable, B_p^i acts by inserting a loop labeled by $s \in \mathcal{I}$ then evaluating the graph according to the composition of morphisms in $\mathcal{C}$.
- B_p is a projector. A_v and B_p commute.

Remark:

- Given a unitary tensor category $\mathcal{C}$, we obtain a lattice model.
- Conversely, Levin-Wen showed how the axioms of the unitary tensor category can be derived from the requirement to have a fix-point wave function of a string-net condensation state.

Edge theories

If we cut the lattice, we automatically obtain a lattice with a boundary with all boundary strings labeled by simple objects in $\mathcal{C}$.

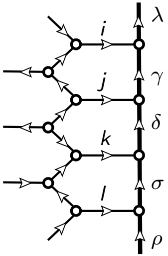


We will call such boundary as a $\mathcal{C}$-boundary or $\mathcal{C}$-edge.

Question: Are there any other possibilities?

$\mathcal{M}$-edge

It is possible to label the boundary strings by a different finite set $\{\lambda, \sigma, \dots\}$ which can be viewed as the set of inequivalent simples objects of another finite unitary semisimple category $\mathcal{M}$.

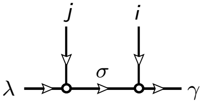


The requirement of giving a fix-point wave function of string-net condensation state is equivalent to require that $\mathcal{M}$ has a structure of $\mathcal{C}$-module. We call such boundary an ${}_{\mathcal{C}}\mathcal{M}$-boundary or ${}_{\mathcal{C}}\mathcal{M}$-edge.

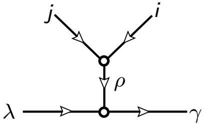
$\mathcal{C}$-module $\mathcal{M}$:

For $i \in \mathcal{C}$, $\gamma, \lambda \in \mathcal{M}$,

- $i \otimes \gamma$ is an object in $\mathcal{M}$ ($\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$)
- $\dim \mathrm{Hom}_{\mathcal{M}}(i \otimes \gamma, \lambda) = N_{i,\gamma}^{\lambda} < \infty$;
- $\mathbf{1} \otimes \gamma \cong \gamma$;
- associator $(i \otimes j) \otimes \lambda \xrightarrow{\alpha} i \otimes (j \otimes \lambda)$;
- fusion matrices:



$$= \sum_n F_{mn}^{ijk;l}$$



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Excitations on boundary:

Two approaches:

1. Kitaev: excitations are super-selection sectors of a local operator algebra;

2. Levin-Wen: excitations can be classified by closed string operator which commute with the Hamiltonian.

☞ Above two approaches lead to the same results.

Levin-Wen approach

Close the boundary to a circle, a closed string operator on it is nothing but a systematic reassignment of boundary string labels and spin labels:

$$\begin{aligned} \gamma &\mapsto F(\gamma) \in \mathcal{M}, \\ \mathrm{Hom}_{\mathcal{M}}(i \otimes \gamma, \lambda) &\mapsto \mathrm{Hom}_{\mathcal{M}}(i \otimes F(\gamma), F(\lambda)) \end{aligned}$$

This assignment is essentially the same data forming a functor from $\mathcal{M}$ to $\mathcal{M}$. Physical requirements (Levin-Wen) add certain consistency conditions which turn it into a $\mathcal{C}$-module functor.

Theorem: Excitations on a ${}_{\mathcal{C}}\mathcal{M}$-edge are given by simple objects in the category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ of $\mathcal{C}$-module functors.

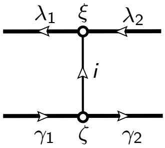
Kitaev's approach

We need construct the local operator algebra A.

$$A := \oplus_{i,\lambda_1,\lambda_2,\gamma_1,\gamma_2} \mathrm{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1) \otimes \mathrm{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2).$$

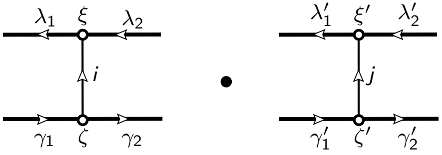
For $\xi \in \mathrm{Hom}_{\mathcal{M}}(i \otimes \lambda_2, \lambda_1)$ and $\zeta \in \mathrm{Hom}_{\mathcal{M}}(\gamma_1, i \otimes \gamma_2)$, the element $\xi \otimes \zeta \in A$ can be expressed by the following graph:

$$\xi \otimes \zeta =$$

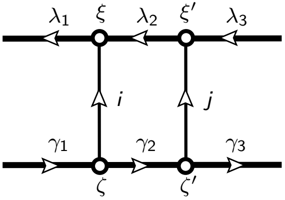


for $i \in \mathcal{C}$ and $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathcal{M}$.

The multiplication $A \otimes A \xrightarrow{\bullet} A$ is defined by



$$= \delta_{\lambda_2 \lambda_2'} \delta_{\gamma_2 \gamma_2'}$$



where the last graph is a linear span of graphs in A by applying F-moves twice and removing bubbles.

Action of A on excitations

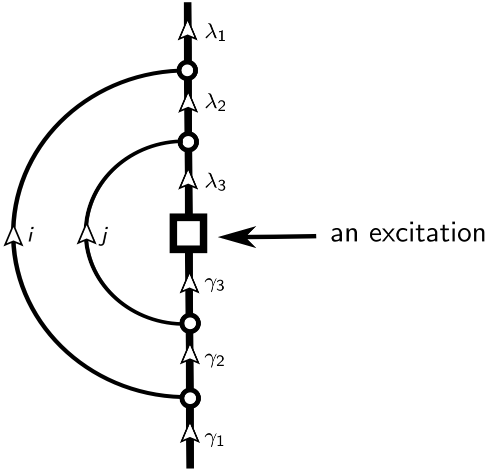
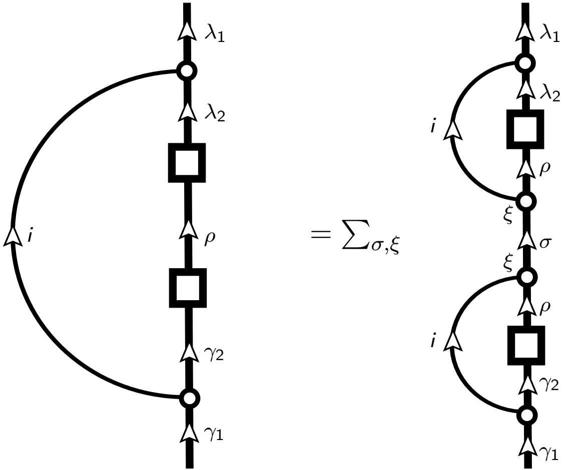
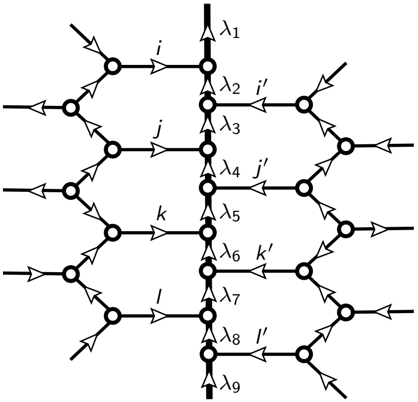


Figure: This picture show how two elements of local operator algebra A act on an edge excitation (up to an ambiguity of the excited region).



A is bialgebra with above comultiplication. With some small modification, one can turn it into a weak C^*-Hopf algebra so that the boundary excitations form a finite unitary fusion category.

a defect line or a domain wall



$i, j, k, l \in \mathcal{C}$, $\lambda_1, \ldots, \lambda_9 \in \mathcal{M}$, $i', j', k', l' \in \mathcal{D}$. $\mathcal{C}$ and $\mathcal{D}$ are unitary tensor categories and $\mathcal{M}$ is a $\mathcal{C}$-$\mathcal{D}$-bimodule. We call such defect ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-defect line or ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall.

- A $\mathcal{M}$-edge can be viewed as ${}_{\mathcal{C}}\mathcal{M}_{\mathsf{Hilb}}$-wall.

- Conversely, if we fold the system along the ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall, we obtain a doubled bulk system determined by $\mathcal{C} \boxtimes \mathcal{D}^{\mathsf{op}}$ with a single boundary determined by $\mathcal{M}$ which is viewed as a $\mathcal{C} \boxtimes \mathcal{D}^{\mathsf{op}}$-module.

$$\text{a } {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}\text{-wall} = \text{a } {}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathsf{op}}}\mathcal{M}\text{-edge}$$

Therefore, we have:

$$
\begin{aligned}
{}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}\text{-wall excitations} &= {}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{op}}}\mathcal{M}\text{-edge excitations} \\
&= \mathrm{Fun}_{\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{op}}}(\mathcal{M},\mathcal{M}) \\
&= \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})
\end{aligned}
$$

the category of $\mathcal{C}$-$\mathcal{D}$-bimodule.

As a special case, a line in $\mathcal{C}$-bulk $=$ a ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$-wall.

$$
\begin{aligned}
\mathcal{C}\text{-bulk excitations} &= {}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}\text{-wall excitations} \\
&= \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C},\mathcal{C}) = Z(\mathcal{C})
\end{aligned}
$$

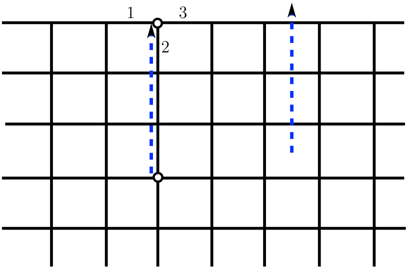
- A ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall can fuse with a ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$-wall into a ${}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})_{\mathcal{E}}$-wall.

- ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}$-wall (or ${}_{\mathcal{D}}\mathcal{N}_{\mathcal{E}}$-wall) excitations can fuse into ${}_{\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})_{\mathcal{E}}$-wall as follow:

$$(\mathcal{M} \xrightarrow{F} \mathcal{M}) \mapsto (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{F \boxtimes_{\mathcal{D}} \mathrm{id}_{\mathcal{M}}} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$$

$$(\mathcal{N} \xrightarrow{G} \mathcal{N}) \mapsto (\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \xrightarrow{\mathrm{id}_{\mathcal{M}} \boxtimes_{\mathcal{D}} G} \mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N})$$

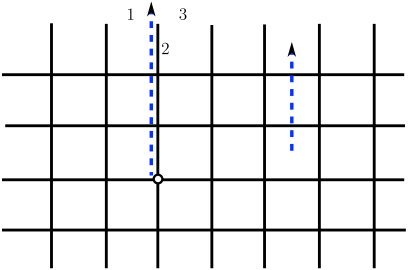
A smooth edge



$$\begin{array}{cccc} 1 & \longrightarrow & 1 & e \longrightarrow e \\ m & \longrightarrow & 1 & \epsilon \longrightarrow e \end{array}$$

☞ This assignment actually gives a monoidal functor $Z(\mathrm{Rep}_{\mathbb{Z}_2}) \to \mathrm{Rep}_{\mathbb{Z}_2} = \mathrm{Fun}_{\mathrm{Rep}_{\mathbb{Z}_2}}(\mathrm{Rep}_{\mathbb{Z}_2}, \mathrm{Rep}_{\mathbb{Z}_2})$.

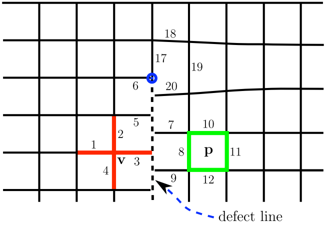
A rough edge



$$1 \longrightarrow 1 \quad m \longrightarrow m$$
$$e \longrightarrow 1 \quad \epsilon \longrightarrow m$$

☞ This assignment gives another monoidal functor $Z(\mathsf{Rep}_{\mathbb{Z}_2}) \to \mathsf{Rep}_{\mathbb{Z}_2} = \mathsf{Fun}_{\mathsf{Rep}_{\mathbb{Z}_2}}(\mathsf{Hilb}, \mathsf{Hilb})$.

defects of codimension 1



$$1 \mapsto 1 \mapsto 1, \qquad e \mapsto \mathrm{Ext}^{\mathit{defect}}_{3|7,8,9} \mapsto m,$$

$$m \mapsto \mathrm{Ext}^{\mathit{defect}}_{7|3,2,5} \mapsto e, \qquad \epsilon \mapsto \mathrm{Ext}^{\mathit{defect}}_{2,5,7,8,9,3} \mapsto \epsilon.$$

☞ This assignment gives an invertible monoidal functor $Z(\mathsf{Rep}_{\mathbb{Z}_2}) \to \mathsf{Fun}_{\mathsf{Rep}_{\mathbb{Z}_2}|\mathsf{Rep}_{\mathbb{Z}_2}}(\mathsf{Hilb}, \mathsf{Hilb}) \to Z(\mathsf{Rep}_{\mathbb{Z}_2})$.

Definition: If $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N} \cong \mathcal{C}$ and $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{M} \cong \mathcal{D}$, then $\mathcal{M}$ and $\mathcal{N}$ are called invertible; $\mathcal{C}$ and $\mathcal{D}$ are called Morita equivalent.

- $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent iff $Z(\mathcal{C})$ is equivalent to $Z(\mathcal{D})$ as braided tensor categories.
- Invertible $\mathcal{C}$-$\mathcal{C}$-defects form a group called Picard group $\mathrm{Pic}(\mathcal{C})$.
- We denote the auto-equivalence of $Z(\mathcal{C})$ as $\mathrm{Aut}(Z(\mathcal{C}))$.

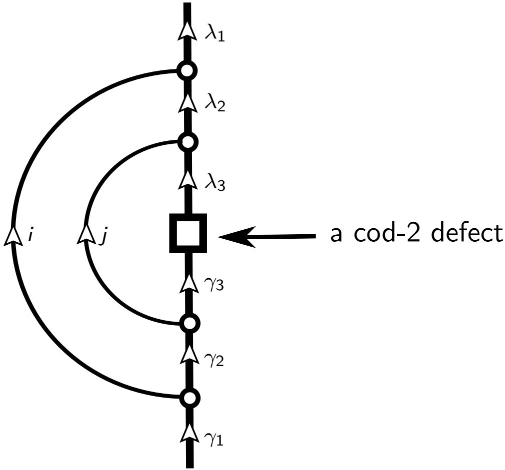
Theorem (Kitaev-K., Etingof-Nikshych-Ostrik):

$$\mathrm{Aut}(Z(\mathcal{C})) \cong \mathrm{Pic}(\mathcal{C}).$$

Defects of codimension 2

- A defect of codimension 2 is a junction between two defect lines. It is given by a module functor.
- An excitation can be viewed as a defect of codimension 2.
- Conversely, a defect of codimension 2 is an excitation in the sense that it can be realized as a super-selection sector of a local operator algebra A'.

Action of A' on defects of codimension 2



$$\lambda_1, \lambda_2, \lambda_3 \in \mathcal{M}, \gamma_1, \gamma_2, \gamma_3 \in \mathcal{N}$$

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Defects of codimension 3 (instantons)

If one takes into account the time direction, one can define a defect of codimension 3 by a natural transformation ϕ between module functors.

The Hamiltonian:

$$H \to H + H_t.$$

where H_t is a local operator defined using ϕ.

Dictionary 1:

Ingredients in LW-model	Tensor-categorical notions
a bulk lattice	a unitary tensor category $\mathcal{C}$
string labels in a bulk	simple objects in a unitary tensor category $\mathcal{C}$
excitations in a bulk	simple objects in $Z(\mathcal{C})$ the monoidal center of $\mathcal{C}$
an edge	a $\mathcal{C}$-module $\mathcal{M}$
string labels on an edge	simple objects in a $\mathcal{C}$-module $\mathcal{M}$
excitations on a $\mathcal{M}$-edge	$\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$: the category of $\mathcal{C}$-module functors
bulk-excitations fuse into an $\mathcal{M}$-edge	$Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}\|\mathcal{C}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ $(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \mapsto (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M} \xrightarrow{\mathcal{F}\boxtimes \mathrm{id}_{\mathcal{M}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{M})$.

Dictionary 2:

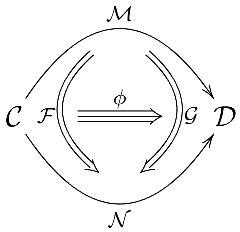
Ingredients in LW-model	Tensor-categorical notions
a domain wall	a $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{N}$
string labels on a $\mathcal{N}$-wall	simple objects in a $\mathcal{C}$-$\mathcal{D}$-bimodule ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$
excitations on a $\mathcal{N}$-wall	$\mathrm{Fun}_{\mathcal{C}\|\mathcal{D}}(\mathcal{N},\mathcal{N})$: the category of $\mathcal{C}$-$\mathcal{D}$-bimodule functors
fusion of two walls	$\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$
an invertible ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	$\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent, i.e. $\mathcal{N} \boxtimes_{\mathcal{D}} \mathcal{N}^{\mathrm{op}} \cong \mathcal{C}$, $\mathcal{N}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{N} \cong \mathcal{D}$.
bulk-excitation fuse into a ${}_{\mathcal{C}}\mathcal{N}_{\mathcal{D}}$-wall	$Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}\|\mathcal{C}}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}\|\mathcal{D}}(\mathcal{N},\mathcal{N})$ $(\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{C}) \mapsto (\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N} \xrightarrow{\mathcal{F}\boxtimes \mathrm{id}_{\mathcal{N}}} \mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{N})$.
defects of codimension 2: a $\mathcal{M}$-$\mathcal{N}$-excitation	simple objects $\mathcal{F}, \mathcal{G} \in \mathrm{Fun}_{\mathcal{C}\|\mathcal{D}}(\mathcal{M},\mathcal{N})$
a defect of codimesion 3 or an instanton	a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$

Outline

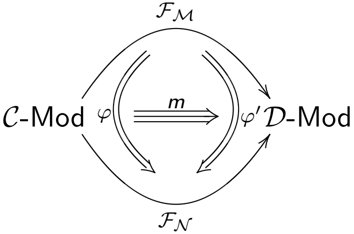
- Extended topological field theories was formulated by Baez and Dolan in terms of n-category in 90s. The classification was given in the so-called Baez-Dolan conjecture which was recently proved by Lurie.

- Levin-Wen models enriched by defects of codimension 1,2,3 provides a physical foundation behind the so-called extended Turaev-Viro topological field theories.

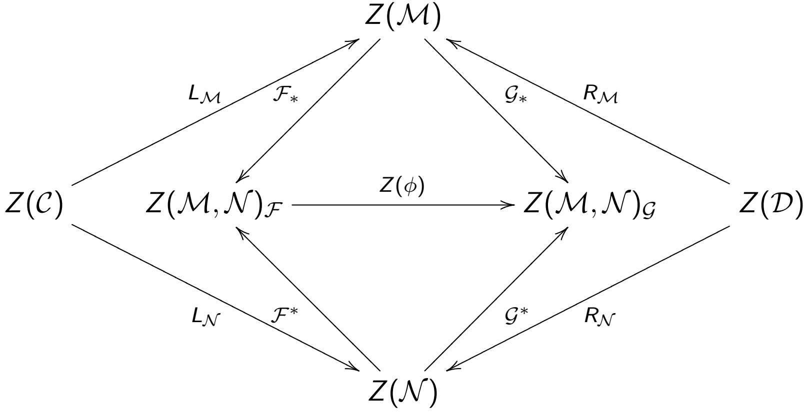
The building blocks of the lattice models:



which 0-1-2-3 cells of a tri-category, or “equivalently”,



Excitations (topological phases):



$Z(\mathcal{M}) := \mathsf{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{M})$, $Z(\mathcal{N}) := \mathsf{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{N},\mathcal{N})$,
$\mathcal{F},\mathcal{G},\in \mathsf{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{N})$, $Z(\mathcal{M},\mathcal{N})_{\mathcal{F}} := Z(\mathcal{N})\circ\mathcal{F}\circ Z(\mathcal{M})$,
$Z(\mathcal{M},\mathcal{N})_{\mathcal{G}} := Z(\mathcal{N})\circ\mathcal{G}\circ Z(\mathcal{M})$.

Conjecture (Functoriality of Holography): The assignment Z is a functor between two tricategories.

Remark: It also says that the notion of monoidal center is functorial.

General philosophy: for $n+1$-dim extended TQFT,

$$\text{pt} \mapsto n\text{-category of boundary conditions}.$$

Extended Turaev-Viro (2+1) TQFT: the bicategory of boundary conditions of LW-models = $\mathcal{C}$-Mod,

$$\text{pt}_{+,-} \quad \mapsto \quad \mathcal{C}, \mathcal{D} \ \text{ or } \ (\mathcal{C}\text{-Mod} \cong \mathcal{D}\text{-Mod}),$$

$$\text{an interval} \quad \mapsto \quad {}_{\mathcal{C}}\mathcal{M}_{\mathcal{D}}, {}_{\mathcal{D}}\mathcal{N}_{\mathcal{C}} \ \ (\text{invertible})$$

$$S^1 \quad \mapsto \quad Tr(\mathcal{C}) = Z(\mathcal{C}),$$

Conjecturely,

$$\text{Turaev-Viro}(\mathcal{C}) = \text{Reshtikin-Turaev}(Z(\mathcal{C})).$$

Thank you!