Kostant on E_8

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Here are my lecture notes from Bertram Kostant's talk on E_8, kindly rendered into LaTeX by Adler Santos.

1 Chopping e_8 into 31 pieces of dimension 8

In what follows we work with the complex form of E_8 and its Lie algebra e_8. Choose a root system Δ for e_8. Each root ψ in Δ gives a root vector e_ϕ in e_8, and e_8 is then a direct sum of its Cartan subalgebra h and the spans of these vectors e_ϕ. The Cartan subalgebra is 8-dimensional, and there are 240 roots, so the dimension of e_8 is 248.

Choose simple roots $\alpha_1, ..., \alpha_8$. There is a special element w in h such that

$$w \cdot \alpha_i = 1$$

for $i = 1, ..., 8$. This gives a 1-parameter subgroup $\exp(sw)$ in E_8, where $s \in C$. An element of E_8 is called *principal* if it is conjugate to one of the form $\exp(sw)$. There is a unique element of order 2 of the form $\exp(sw)$, so we get a distinguished conjugacy class of order-2 elements in E_8, namely the principal ones. There's a copy of the group $(Z/2)^5$ sitting inside this conjugacy class. The group $(Z/2)^5$ has 32 characters, that is, homomorphisms

$$\chi : (Z/2)^5 \longrightarrow U(1).$$

31 of these characters are nontrivial. Note that

$$248 = 31 \times 8.$$

The nontrivial characters $\chi_j (j = 1, ..., 31)$ have an interesting property. Let

$$h_j = \{x \in e_8 : Ad(a)x = \chi_j(a)x \text{ for all } a \in (Z/2)^5\}$$

Then h_j is a Cartan subalgebra of e_8! And,

$$e_8 = h_1 \oplus \ldots \oplus h_{31}.$$

So, the 248-dimensional Lie algebra e_8 is a direct sum of 31 8-dimensional Cartan algebras.

2 How $SL(2, 32)$ acts to permute the 31 pieces of e_8

E_8 has a finite subgroup called the Dempwolf group, F_{Demp}, with the above $(Z/2)^5$ as a normal subgroup. In fact we have an exact sequence:

$$1 \longrightarrow (Z/2)^5 \longrightarrow F_{Demp} \longrightarrow SL(2, 32) \longrightarrow 1.$$

Starting from the above $(Z/2)^5$ subgroup of E_8, we get a chain of subgroups

$$(Z/2) \subset (Z/2)^2 \subset (Z/2)^3 \subset (Z/2)^4 \subset (Z/2)^5.$$

The centralizer of $Z/2$ in E_8 is $D_8 = Spin(16)$.

The centralizer of $(Z/2)^2$ is $D_4 \times D_4$, or more precisely, $Spin(8) \times Spin(8)$ mod the diagonal copy of $Z/2$.

We have a vector space decomposition

$$e_8 = (so(8) \oplus so(8)) \;\oplus\; V_8 \otimes V_8 \;\oplus\; S_8^+ \otimes S_8^+ \;\oplus\; S_8^- \otimes S_8^-$$

where V_8, S_8^+ and S_8^- are the 8-dimensional "vector", "right-handed spinor" and "left-handed spinor" representations of $Spin(8)$, respectively. These three representations are related by triality.

The elements of the Dempwolf group permute the 64-dimensional subspaces $V_8 \otimes V_8$, $S_8^+ \otimes S_8^+$ and $S_8^- \otimes S_8^-$ of e_8.

Griess and Ryba have been studying finite subgroups of E_8, for example:

- Robert L. Griess, Jr. and A. J. E. Ryba, Embeddings of $PGL(2, 31)$ and $SL(2, 32)$ in $E8(C)$, with appendices by Michael Larsen and Jean Pierre Serre, *Duke Math. J.* **94** (1998), 181-211.

Serre has too:

- Jean-Pierre Serre, Sous-groupes finis des groupes de Lie, Sminaire Bourbaki vol. 1998/99, *Astrisque* **266** (2000), 415-230.

The 5-dimensional vector space $(Z/2)^5$ can be made into a field, F_{32}, and invertible 2×2 matrices with determinant 1 having entries in this field form the group $SL(2, 32)$. This whole group sits inside E_8. It contains the aforementioned copy of $(Z/2)^5$ as matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Proving this was hard!

The group $SL(2, 32)$ has cardinality

$$3 \times 11 \times 31 \times 32$$

corresponding to something analogous to an Iwasawa decomposition $G = KAN$: the subgroup K has cardinality 3×11, the subgroup A has cardinality 31, and the subgroup N (consisting of the upper triangular matrices shown above) has cardinality 32.

The solvable subgroup of $SL(2,32)$ has one irreducible 31-dimensional representation. The adjoint action of $SL(2,32)$ on e_8 acts to permute its 31 Cartan subalgebras according to this representation!

3 A product of two copies of the Standard Model gauge group in E_8

The gauge group of the Standard model is usually said to be $SU(3) \times SU(2) \times U(1)$, but this group has a $Z/6$ subgroup that acts trivially on all known particles. The quotient $(SU(3) \times SU(2) \times U(1))/(Z/6)$ is isomorphic to $S(U(3) \times U(2))$ - that is, the subgroup of $SU(5)$ consisting of block diagonal matrices with a 3×3 block and a 2×2 block. So, this could be called the "true" gauge group of the Standard Model. Each generation of fermions transforms according to the obvious representation of $S(U(3) \times U(2))$ on the exterior algebra ΛC^5. This idea is the basis of the $SU(5)$ grand unified theory.
There is an element of order 11 in $SL(2,32) \subset E_8$. The centralizer of this element in E_8 is a product of two copies of the Standard Model gauge group:

$$S(U(3) \times U(2)) \times S(U(3) \times U(2))$$

Let g_i be the subspace of e_8 consisting of linear combinations of root vectors e_phi where the root ϕ is a sum of i simple roots. Let g_{-i} be the analogous thing defined using the negatives of the simple roots. Then, the Lie algebra of the $S(U(3) \times U(2)) \times S(U(3) \times U(2))$ subgroup of E_8 is

$$g_{-22} \oplus g_{-11} \oplus g_{11} \oplus g_{22}$$

(There is also an element of order 3 in $SL(2,32)$, whose centralizer in E_8 is $SU(9)$.)

4 A product of two copies of $su(5)$ in e_8

There's also a copy of $(Z/5)^3$ in E_8. If we think of this as a 3-dimensional vector space containing "lines" (1-dimensional subspaces), then it contains

$$1 + 5 + 5^2 = 31$$

lines. The centralizer in E_8 of any such line is the group

$$(SU(5) \times SU(5))/(Z/5)$$

This group is 48-dimensional. It has a 248-dimensional representation coming from the adjoint action on e_8. This is the direct sum of the 48-dimensional subrepresentation coming from $su(5) \oplus su(5) \subset e_8$ and a representation of dimension

$$248 - 48 = 200$$

This 200-dimensional representation is

$$5 \otimes 10 \;\oplus\; 5^* \otimes 10^* \;\oplus\; 10 \otimes 5^* \;\oplus\; 10^* \otimes 5$$

where 5 is the defining representation of $SU(5)$, 10 is its second exterior power, and 5^* and 10^* are the duals of these.

Is any of this useful in particle physics? In particular, what could the "doubling" mean?