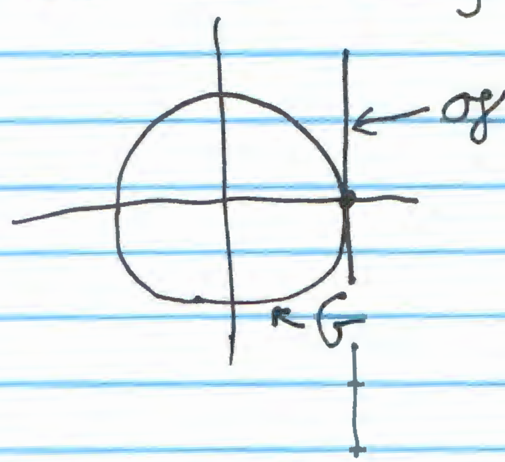
LIE GROUPS = SYMMETRY + CALCULUS
(groups) (manifolds)

A Lie group is a group that's a manifold.

Its tangent space at the identity is called a "Lie algebra"



Here

$$G = \{ e^{i\theta} : \theta \in \mathbb{R} \}$$

$$\mathfrak{g} \cong \{ i\theta : \theta \in \mathbb{R} \}$$

Lie algebras let us reduce problems about Lie groups to linear algebra.

I want to cover:

1) general theorems about Lie groups & Lie algebras — but skipping many proofs!

2) examples — the key to really understanding Lie groups

3) applications ... to:

A) geometry: Felix Klein's idea that a type of geometrical figure (point, line, etc.) is really an action of a Lie group on a manifold;

B) quantum physics: the idea that a type of particle is really an action (aka "representation") of a Lie group on a Hilbert space.

A Lie group is a group in the category of (smooth) manifolds.

Recall:

Def. – A topological manifold is a topological space X such that each point $x \in X$ has an open neighborhood $U_x \ni x$ that is homeomorphic to $\mathbb{R}^n$.

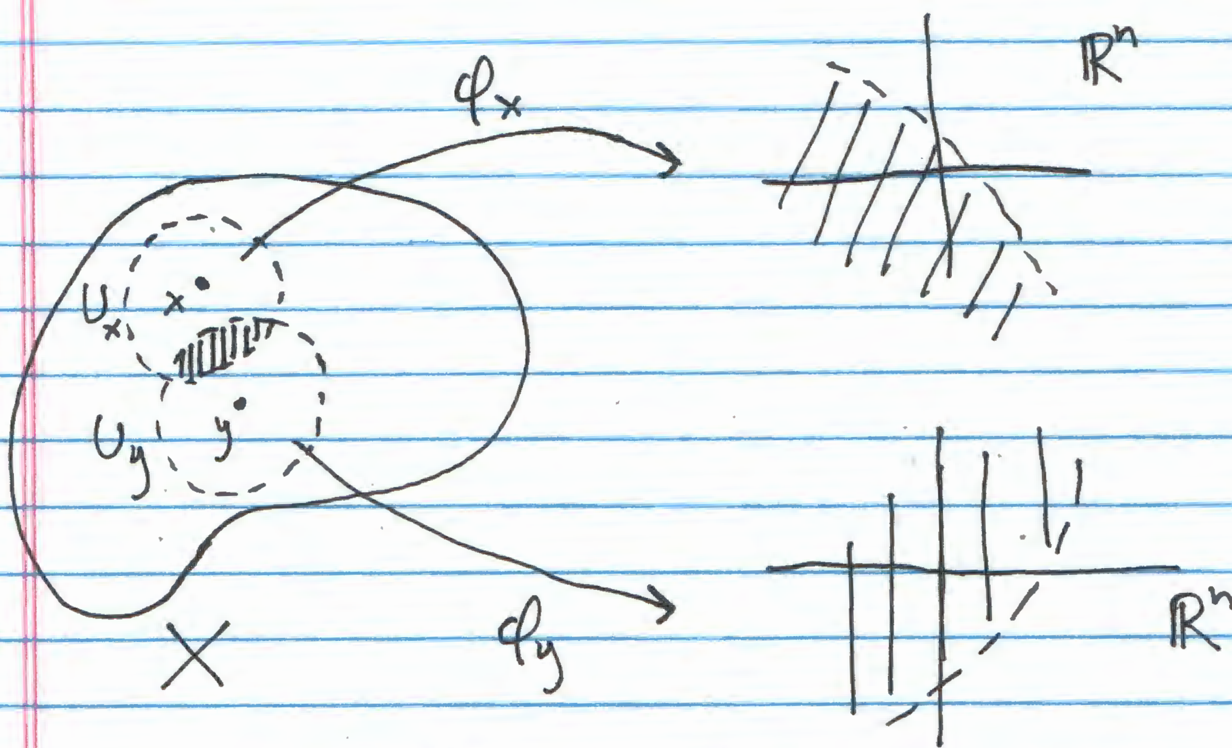
We call n the dimension of X, and we say X is an n-manifold.

We call a homeomorphism $\varphi_x : U_x \longrightarrow \mathbb{R}^n$ a chart.

Def. – A (smooth) manifold is a topological space $x \in X$ such that each point $x \in X$ is equipped with an open neighborhood $U_x \ni x$ and a homeomorphism $\varphi_x : U_x \to \mathbb{R}^n$ s.t. $\forall x, y \in X$

$$\varphi_x \circ \varphi_y^{-1}$$

is smooth (infinitely differentiable) where defined.



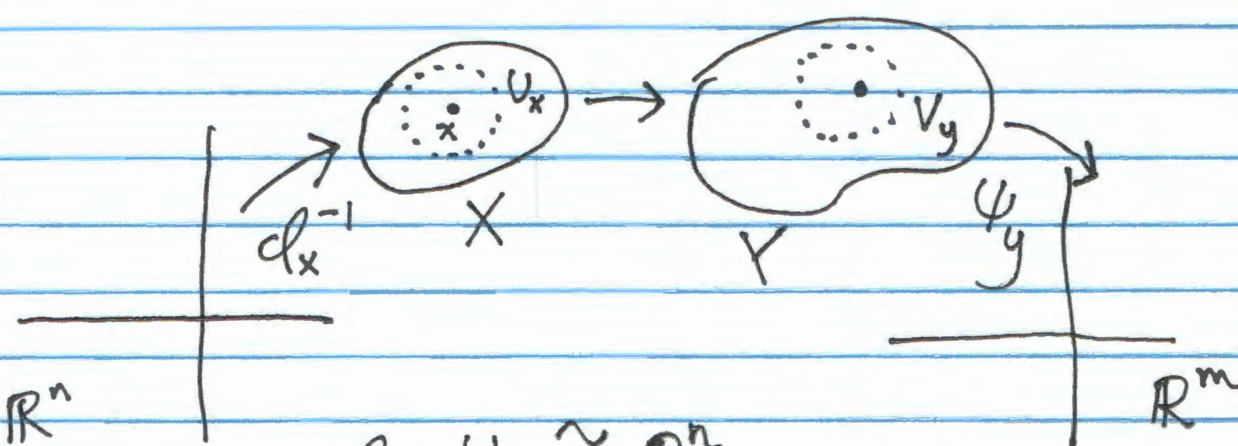
$\varphi_y \circ \varphi_x^{-1}$ maps the upper shaded region onto the lower shaded region. We demand that it be smooth.

So for us, a manifold is really a pair $(X, \{\varphi_x\}_{x \in X})$, though we usually write X.

Def.— A (smooth) map from a manifold $(X, \{\varphi_x\}_{x \in X})$ to a manifold $(Y, \{\psi_y\}_{y \in Y})$ is a function $f: X \to Y$ such that $\forall x, y$

$$\psi_y \circ f \circ \varphi_x^{-1}$$

is smooth where defined.



$$\varphi_x : U_x \xrightarrow{\sim} \mathbb{R}^n$$

$$\psi_y : V_y \xrightarrow{\sim} \mathbb{R}^m$$

There is a category Diff of (smooth) manifolds & (smooth) maps.

Two isomorphic manifolds are called <u>diffeomorphic</u>:

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{aligned} g \circ f &= 1_X \\ f \circ g &= 1_Y \end{aligned}$$

We can often change the charts on a manifold & get a diffeomorphic manifold, which we consider "the same".

Def. – A <u>Lie group</u> $(G, \cdot)$ is a manifold that is a group such that multiplication

$$\begin{aligned} \cdot : G \times G &\to G \\ (g,h) &\mapsto g \cdot h \end{aligned}$$

and taking inverses

$$\begin{aligned} ^{-1} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth maps.

Why is $G \times G$ a manifold?

Lemma - Given manifolds

$$(X, \{\varphi_x : U_x \to \mathbb{R}^n\}_{x \in X})$$

and

$$(Y, \{\psi_y : V_y \to \mathbb{R}^m\}_{y \in Y})$$

there is an $(n+m)$-dimensional manifold called their <u>product</u>:

$$(X \times Y, \{\varphi_x \times \psi_y : U_x \times V_y \to \mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\sim} \mathbb{R}^{n+m}\}_{(x,y) \in X \times Y})$$

Examples:

1) $(\mathbb{R}^n, +)$ is a Lie group

2) For any finite-dimensional real or complex vector space V, $(V, +)$ is a Lie group.

3) The <u>general</u> <u>linear</u> <u>group</u>

$$GL(n, \mathbb{R}) = \{n \times n \text{ real matrices w. } \det \neq 0\}$$

is a Lie group with matrix multiplication as the group operation. It's clearly a group, so why is it a manifold & why are multiplication & taking inverses smooth maps?

Note $GL(n,\mathbb{R})$ is an open subset of

$$M_n(\mathbb{R}) = \{n \times n \text{ real matrices}\}$$

since $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a polynomial function, hence continuous, even smooth.

$M_n(\mathbb{R})$ is an n^2-dim vector space hence a manifold. So we need:

Lemma – An open subset O of a d-dimensional manifold X is a d-dimensional manifold.

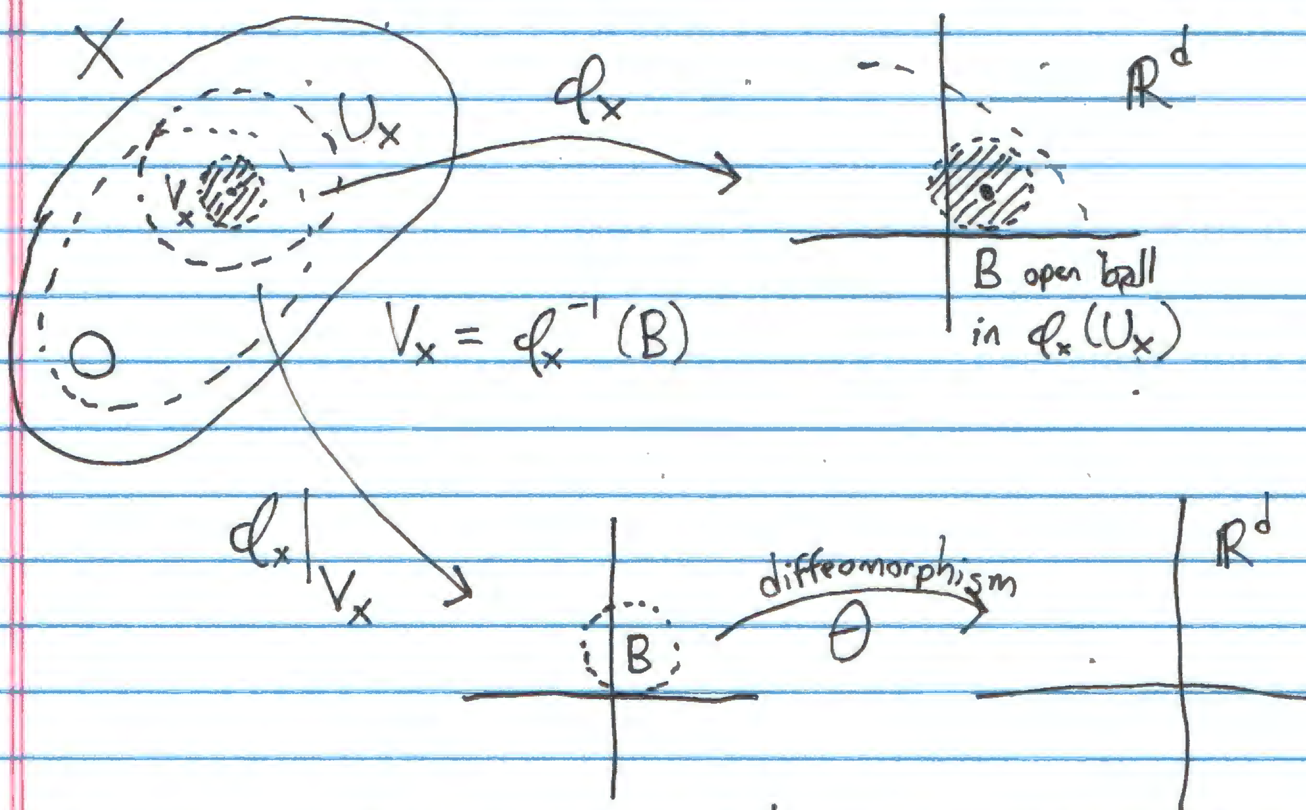
Thus $GL(n,\mathbb{R})$ is an n-dimensional manifold, and the maps

$$\cdot : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$$
$$(g,h) \longmapsto gh \qquad (gh)_{ik} = \sum_j g_{ij} h_{jk}$$

$$^{-1} : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$$
$$g \mapsto g^{-1} \qquad (g^{-1})_{ij} = \frac{(-1)^{i+j}(\text{minor})_{ji}}{\det(g)}$$

are smooth!

Proof of Lemma – If $(X, \{\varphi_x\}_{x \in X})$ is a manifold and $O \subseteq X$ is open



$$\Psi_x = \theta \circ \varphi_x|_{V_x}$$

$(O, \{\Psi_x\}_{x \in O})$ becomes a manifold & the inclusion $O \hookrightarrow X$ becomes a smooth map.

4) Similarly, the <u>general linear group</u>

$$GL(n,\mathbb{C}) = \{n\times n \text{ complex matrices with } \det \neq 0\}$$

is a Lie group, since it's an open subset of the $2n^2$-dimensional real vector space

$$M_n(\mathbb{C}) = \{n\times n \text{ complex matrices}\}$$

& the group operations are smooth.

Thm. – A closed subgroup of a Lie group is a Lie group. That is, if G is a Lie group & $H \subseteq G$ is a subgroup that is also a closed subset in the sense of topology, H is a submanifold of G, and the group operations restrict to smooth maps

$$\cdot : H\times H \to H \qquad {}^{-1} : H \to H$$

so it becomes a Lie group.

Cor. – A closed subgroup of $GL(n,\mathbb{C})$ is a Lie group. This is called a <u>matrix Lie group</u>.