Lie Groups in Quantum Physics

In physics we study "systems" that obey simple mathematical laws, like:

- an electron
- a proton
- a hydrogen atom
- a water molecule
- an ice crystal

In quantum physics each system is described using a <u>Hilbert space</u>: a complete complex inner product space. A <u>complex inner product space</u> is a complex vector space H with an <u>inner product</u>: a map $\langle -,- \rangle : H \times H \to \mathbb{C}$ with

$$\langle \psi, a\varphi + b\varphi' \rangle = a\langle \psi, \varphi \rangle + b\langle \psi, \varphi' \rangle$$

$$\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle} \qquad \forall \psi, \varphi, \varphi' \in H, \; a, b \in \mathbb{C}$$

$$\langle \psi, \psi \rangle > 0 \text{ unless } \psi = 0$$

A complex inner product space is <u>complete</u> if when we define a norm by

$$\|\psi\|^2 = \langle \psi, \psi \rangle$$

and then a metric by

$$d(\psi, \varphi) = \|\psi - \varphi\|$$

this metric is <u>complete</u>: every Cauchy sequence converges. Luckily every finite-dimensional complex (or real, or quaternionic) inner product space is complete!

Every finite-dimensional Hilbert space is isomorphic to $\mathbb{C}^n$ with its standard inner product

$$\langle \psi, \varphi \rangle = \sum_{i=1}^{n} \overline{\psi_i} \varphi_i \qquad \psi, \varphi \in \mathbb{C}^n$$

A "state" of a physical system is a way it can be. In quantum physics a state is described by a unit vector in the Hilbert space of that system : $\psi \in H$ with $\|\psi\| = 1$.

Axiom 1: If a system is in the state $\psi \in H$ and you check to see if it's in the state $\varphi \in H$, the answer is "yes" with probability

$$|\langle \varphi, \psi \rangle|^2 = |\langle \psi, \varphi \rangle|^2$$

Note: the Cauchy-Schwartz inequality

$$|\langle \psi, \varphi \rangle| \leq \|\psi\| \, \|\varphi\|$$

implies this probability is between 0 & 1, since ψ, φ are states so $\|\psi\| = \|\varphi\| = 1$.

If we do something to a system, or just wait, its states will become new states.

In quantum physics we describe this using a linear map

$$U : H \to H$$

that maps states to states:

$$\|\psi\| = 1 \implies \|U\psi\| = 1 \qquad \forall \psi \in H$$

$$\Downarrow \text{ linearity}$$

$$\|\psi\| = c \implies \|U\psi\| = c \qquad \forall \psi \in H, c \in \mathbb{R}$$

$$\Downarrow$$

$$\|U\psi\| = \|\psi\| \qquad \forall \psi \in H$$

$$\Downarrow \text{ polarization identity}$$

$$\langle U\psi, U\phi \rangle = \langle \psi, \phi \rangle \qquad \forall \psi, \phi \in H$$

$$\Downarrow \text{ definition of adjoint}$$

$$\langle \psi, U^* U \phi \rangle = \langle \psi, \phi \rangle \qquad \forall \psi, \phi \in H$$

$$\Downarrow$$

$$U^* U = 1_H$$

All these conditions are actually equivalent, and a linear operator obeying any one is called an "isometry."

Definition - An *isometry* from a Hilbert space H to a Hilbert space H' is a linear map

$$U : H \to H'$$

obeying any one of these equivalent conditions:

1) U maps unit vectors to unit vectors

2) U preserves the norm : $\|U\psi\| = \|\psi\|$

3) U preserves the inner product : $\langle U\psi, U\phi\rangle = \langle \psi, \phi\rangle$

4) $U^* U = 1_H$

A *unitary* operator is an isometry with an inverse, or equivalently a linear map $U : H \to H'$ with $U^* U = 1_H$, $UU^* = 1_{H'}$.

Thus, in quantum physics an *invertible* transformation of the states of a system is described by a unitary operator $U: H \to H$.

Given a Hilbert space H, the unitary operators $U: H \to H$ form a group:

$$U, V: H \to H \text{ unitary}$$

$$\Downarrow$$

$$\|UV\psi\| = \|V\psi\| = \|\psi\| \quad \forall \psi \in H$$

$$\Downarrow$$

$$UV \text{ unitary}$$

$$U^{-1} = U^* \quad \& \quad UU^* = U^*U = 1_H$$

$$\Downarrow$$

$$U^*U^{**} = U^{**}U^* = 1_H$$

We call this group $U(H)$. If H is finite-dim

$$U(H) \cong U(\mathbb{C}^n) = U(n) = \{U \in M_n(\mathbb{C}) : UU^* = U^*U = 1\}$$

so H is a Lie group.

In quantum physics we say a group G <u>acts as symmetries</u> of some system with Hilbert space if there's a group homomorphism

$$\rho : G \to U(H)$$

We'll focus on the case where:

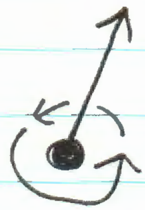
- G is a Lie group
- H is finite-dimensional so $U(H)$ is a Lie group
- ρ is a Lie group homomorphism

Definition — A <u>unitary representation</u> of a Lie group G on a finite-dimensional Hilbert space H is a Lie group homomorphism

$$\rho : G \to U(H)$$

Note $U(H) \subseteq GL(H)$ so a unitary representation gives, or "is", a representation.

Example: If we ignore where they are & where they are going, particles still have "spin".



Their spin is described by a state (unit vector) in a finite-dimensional Hilbert space. We can rotate the particle using the group $SO(3)$ or its double cover $\mathrm{Spin}(3) \cong SU(2)$.

For a <u>spin-1 particle</u> we do this using a unitary representation

$$\rho_1 : SO(3) \to U(3) = U(\mathbb{C}^3)$$

which is just the inclusion of $SO(3) =$ $\{g \in M_3(\mathbb{R}) : gg^* = g^*g = 1,\ \det(g) = 1\}$ in $U(3)$.

For a spin-$\frac{1}{2}$ particle we use the unitary representation

$$\rho_{\frac{1}{2}} : SU(2) \to U(2) = U(\mathbb{C}^2)$$

which is just the inclusion of $SU(2)$ in $U(2)$.

For a spin-0 particle we use the so-called "trivial" unitary representation

$$\rho_0 : SO(3) \to U(1) = U(\mathbb{C}^1)$$

defined by $\rho_0(g) = 1 \quad \forall g \in SO(3)$.

Definition - The trivial unitary rep of a Lie group G is the rep

$$\begin{aligned} \rho_0 : G &\to U(1) = U(\mathbb{C}) \\ g &\mapsto 1 \end{aligned}$$

sending every group element to the identity 1×1 matrix, $1 \in U(1)$.

Rotating a Spin-1 Particle

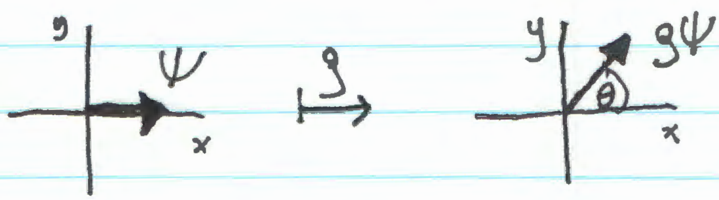
Take this state of the spin-1 particle:

$$\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{C}^3$$

and apply $\rho_1(g) : \mathbb{C}^3 \to \mathbb{C}^3$ where $g \in SO(3)$

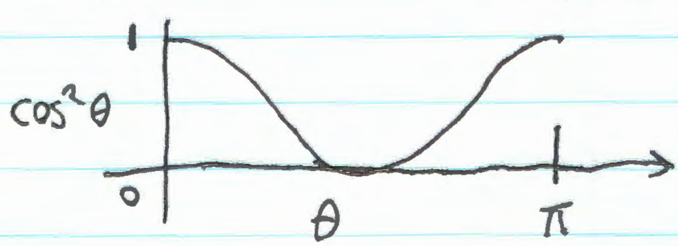
is rotation by θ in the xy plane:

$$\rho_1(g)\psi = g\psi = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$



If we check to see if the particle in this new state is in the original state ψ, the probability that we get the answer "yes" is

$$|\langle \psi, g\psi \rangle|^2 = \left| \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \right\rangle \right|^2 = \cos^2\theta$$



If this seems weird, it is!

If you rotate a spin-1 particle in the state $\psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ 180° in the xy plane & check to see if it's in the state ψ, the answer is yes with probability 100% even though now it's $g\psi = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$. In general for any Hilbert space H if $\|\psi\| = 1$ then $|\langle c\psi, \psi \rangle|^2 = 1$ when $c \in \mathbb{C}$ is a <u>phase</u>: $|c| = 1$. So really we should call a unit vector $\psi \in H$ a <u>state vector</u>, and call an equivalence class of state vectors where $c\psi \sim \psi$ when $|c| = 1$ a <u>state</u>. For $H = \mathbb{C}^n$, states form the projective space

$$\begin{aligned} \mathbb{C}P^{n-1} &\cong \{\text{lines through the origin in } \mathbb{C}^n\} \\ &\cong \{\text{unit vectors in } \mathbb{C}^n\} / c\psi \sim \psi \end{aligned}$$

In nature,

electrons
protons &
neutrons

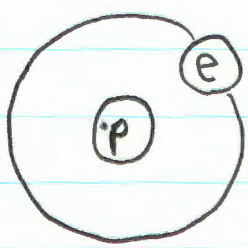
are all spin-$\frac{1}{2}$ particles. Sometimes two spin-$\frac{1}{2}$ particles will "stick together" and form a "bigger" particle:

electron $\otimes$ proton = hydrogen atom

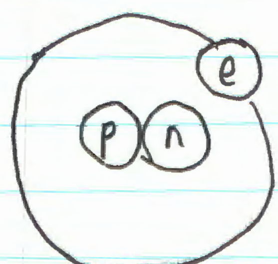
proton $\otimes$ neutron = deuteron

electron $\otimes$ deuteron = deuterium atom

"Deuterium" is a form of "heavy hydrogen":



hydrogen



deuterium

(Don't take these pictures too literally!)

When two spin-$\frac{1}{2}$ particles "stick together", they can form either a spin-0 or a spin-1 particle! Why?

We can think of all three as reps of $SU(2)$:

$$\begin{tikzcd} SU(2) \arrow[d, "\pi"'] \arrow[dr, "\alpha_0"] & \\ SO(3) \arrow[r, "\rho_0"'] & U(1) \end{tikzcd}$$

spin-0 rep of $SU(2)$

$$SU(2) \xrightarrow[\alpha_{1/2} := \rho_{1/2}]{} U(2)$$

spin-$\frac{1}{2}$ rep of $SU(2)$

$$\begin{tikzcd} SU(2) \arrow[d, "\pi"'] \arrow[dr, "\alpha_1"] & \\ SO(3) \arrow[r, "\rho_1"'] & U(3) \end{tikzcd}$$

spin-1 rep of $SU(2)$

Then we have this result:

Theorem- as representations of $SU(2)$ we have

$$\alpha_{1/2} \otimes \alpha_{1/2} \cong \underset{\uparrow \, 1d}{\alpha_0} \oplus \underset{\uparrow \, 3d}{\alpha_1}$$

This can be shown using something we've seen:

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \Lambda^2 \mathbb{C}^2 \oplus S^2 \mathbb{C}^2$$

$$\cong \langle \uparrow \otimes \downarrow - \downarrow \otimes \uparrow \rangle \oplus$$

$$\langle \uparrow \otimes \uparrow, \downarrow \otimes \downarrow, \uparrow \otimes \downarrow + \downarrow \otimes \uparrow \rangle$$

To prove it we'd need to check

$$\alpha_0 \cong \Lambda^2 \alpha_{1/2}$$

$$\alpha_1 \cong S^2 \alpha_{1/2}$$

which is just a computation, but a bit tiring since we don't have enough machinery now.

$$\alpha_{1/2} \otimes \alpha_{1/2} \cong \alpha_0 \oplus \alpha_1$$

has this meaning in physics: "when you stick together two spin-$\frac{1}{2}$ particles you can get a spin-0 particle or a spin-1 particle."

$\otimes$ = "stick together"

$\oplus$ = "or"

Protons & neutrons are spin-$\frac{1}{2}$ particles.

A "deuteron" is a proton & a neutron stuck together. It is a spin-1 particle:

$$\alpha_1 \subseteq \alpha_{1/2} \otimes \alpha_{1/2}$$

$$\langle \uparrow\otimes\uparrow, \downarrow\otimes\downarrow, \uparrow\otimes\downarrow + \downarrow\otimes\uparrow \rangle \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$$

A "dineutron" is 2 neutrons stuck together.

It falls apart so fast it's very hard to see!

But a dineutron is a spin-0 particle:

$$\alpha_0 \subseteq \alpha_{1/2} \otimes \alpha_{1/2}$$

$$\langle \uparrow\otimes\downarrow - \downarrow\otimes\uparrow \rangle \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$$

So is the even more short-lived "diproton".

Why is the deuteron spin-1, while the dineutron & diproton are spin-0? This question goes beyond what we can answer using just the math we have so far! But Lie group representation theory tells us the possible options, which is already amazing!