Lie Algebras

Every Lie group G has a "Lie algebra" $\mathfrak{g}$, which is the tangent space of G at the point $1 \in G$.

The Lie algebra $\mathfrak{g}$ has a "bracket" operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which turns out to contain a lot of information about the Lie group G. In fact when G is (connected and) simply connected, it contains all the information about G!!!

This lets solve lots of problems in the theory of Lie groups using linear algebra! Let us start by briefly sketching 3 key results, and then go into details.

Thm. – Given a Lie group G, for any $v \in \mathfrak{g}$ there is a unique smooth map $C: \mathbb{R} \to G$ with

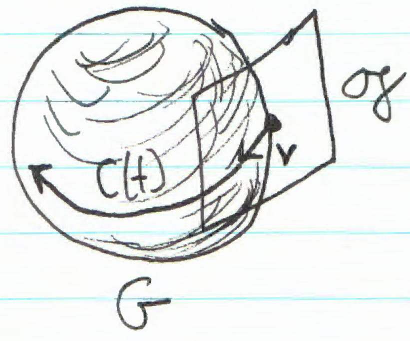
1) $C(0) = 1 \in G$

2) $C'(0) = v \in T_1 G = \mathfrak{g}$

3) $C(s+t) = C(s)\, C(t) \quad \forall s, t \in \mathbb{R}$

I.e., there is a unique Lie group homomorphism (1, 3) $C: \mathbb{R} \to G$ with $C'(0) = v$ (2).

The picture:



Thm. – The Lie algebra $\mathfrak{g}$ of a Lie group is a vector space with a "bracket" operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is:

1) linear in the second argument:

$$[u, av + bw] = a[u,v] + b[u,w] \quad \forall a,b \in \mathbb{R} \quad \forall u,v,w \in \mathfrak{g}$$

2) antisymmetric:

$$[u,v] = -[v,u] \quad \forall u,v \in \mathfrak{g}$$

(and thus linear in the first argument)

3) obeys the Jacobi identity, which says $[u,\cdot]$ obeys the product rule:

$$[u,[v,w]] = [[u,v],w] + [v,[u,w]] \quad \forall u,v,w \in \mathfrak{g}$$

(like $\frac{d}{dx} vw = \left(\frac{d}{dx} v\right) w + v\left(\frac{d}{dx} w\right)$).

I'll explain later how $\mathfrak{g} = T_1 G$ gets this bracket operation.

Thm. – Any finite-dimensional vector space L with a bracket $[\cdot,\cdot] : L \times L \to L$ obeying 1)–3) is isomorphic to the one $\mathfrak{g}$ coming from some Lie group G. Moreover there is a unique (connected and) simply connected G with $\mathfrak{g} \cong L$, up to isomorphism.

Since $\mathfrak{g}$ only "sees near the identity" of G, the Lie algebra of G is the same as that of the connected component containing $1 \in G$ (which is a Lie group too), and the Lie algebra of G is the same as that of any cover of G.

$$\begin{tikzcd}
\text{Lie Group} \arrow[d, bend left=60, "\underline{R}"] \\
\text{Lie Alg} \arrow[u, bend left=60, "\underline{\underline{L}}"]
\end{tikzcd}$$

The functor sending $G \in$ LieGroup to $\underline{R}(G) = \mathfrak{g}$ is a "right adjoint", and the functor sending $L \in$ LieAlg to $\underline{\underline{L}}(L)$, the simply connected Lie group with Lie algebra L, is a "left adjoint".

$\underline{R}(\underline{\underline{L}}(L)) \cong L$.