# The exponential map

Let $G$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra: the tangent space $T_1 G$.

Theorem – For any $v \in \mathfrak{g}$ there exists a unique Lie group homomorphism $C : \mathbb{R} \to G$ with $C'(0) = v$. We write $C(t) = \exp(tv)$.

Proof – We'll only prove this when $G$ is a <u>matrix Lie group</u>: a closed subgroup of $GL(n, \mathbb{C})$ (e.g. a closed subgroup of $GL(n, \mathbb{R})$). We've seen that in this case $G$ is a submanifold of $GL(n, \mathbb{C})$, which in turn is an open subset of $M_n(\mathbb{C})$, so

$$\mathfrak{g} = T_1 G \subseteq T_1 GL(n, \mathbb{C}) \cong M_n(\mathbb{C}).$$

This lets us regard any $v \in \mathfrak{g}$ as an $n \times n$ matrix, which we do henceforth.

Define <u>exponentiation</u>

$$\exp : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

by

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \qquad \forall A \in M_n(\mathbb{C})$$

This converges in the usual topology on $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, since if

$$\|A\| = \sup_{0 \neq x \in \mathbb{C}^n} \frac{\|Ax\|}{\|x\|}$$

is the <u>operator norm</u> then $\|AB\| \leq \|A\| \, \|B\|$ so we get absolute convergence:

$$\sum_{n=0}^{\infty} \left\| \frac{A^n}{n!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} \leq \exp \|A\| < \infty$$

hence convergence.

Lemma 1 - If $A \in M_n(\mathbb{C})$ then

$$\frac{d}{dt}\exp(tA) = A\exp(tA) = \exp(tA)A$$

Proof - Thanks to absolute convergence the following manipulations can be justified just as in complex analysis:

$$\frac{d}{dt}\exp(tA) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{(tA)^n}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{d}{dt}\frac{(tA)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n t^{n-1}\frac{A^n}{n!}$$

$$= \sum_{m=0}^{\infty} t^m \frac{A^{m+1}}{m!} \qquad \text{reindexing: } m = n-1$$

$$= A\sum_{m=0}^{\infty}\frac{(tA)^m}{m!}$$

$$= A\exp(tA)$$

but we could also pull $A$ out to the right and get $\exp(tA)A$. $\blacksquare$

Lemma 2 – If $A, B \in M_n(\mathbb{C})$ commute then $\exp(A+B) = \exp(A)\exp(B)$.

Proof – If $A$ & $B$ commute we can show the binomial formula

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$
$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k}$$

so

$$\exp(A+B) = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!(n-k)!} A^k B^{n-k}$$

Reindexing:
let $n-k=\ell$
so $k+\ell=n$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{A^k}{k!} \frac{A^\ell}{\ell!}$$
$$= \sum_{k=0}^{\infty} \frac{A^k}{k!} \sum_{\ell=0}^{\infty} \frac{A^\ell}{\ell!}$$
$$= \exp(A)\exp(B). \qquad \blacksquare$$

Now given $v \in \mathfrak{g} \subseteq M_n(\mathbb{C})$ let

$$C(t) = \exp(tv) \in M_n(\mathbb{C})$$

We need to show

1) $C(0) = 1 \in M_n(\mathbb{C})$

2) $C(s+t) = C(s)\,C(t) \quad \forall s,t \in \mathbb{R}$

3) $C'(0) = v$

4) $C(t) \in G \quad \forall t \in \mathbb{R}$

These will imply there exists a Lie group homomorphism $C: \mathbb{R} \to G$ (since $C$ is smooth).

1) $C(0) = \sum_{n\geq 0}^{\infty} \frac{(0v)^n}{n!} = 1$ since $\frac{0^0}{0!} = 1$

2) Lemma 2 shows

$$\exp((s+t)v) = \exp(sv)\exp(tv)$$

3) Lemma 1 shows

$$\frac{d}{dt}\exp(tv) = v\exp(tv)$$

4) The hard part is showing

$$v \in \mathfrak{g} \Rightarrow \exp(tv) \in G$$

Sketch: if $v \in \mathfrak{g}$ there is a unique (smooth) vector field $\tilde{v}$ on $G$ such that

$$\tilde{v}(g) = vg$$

matrix multiplication of $v \in \mathfrak{g} \subseteq M_n(\mathbb{C})$ & $g \in G \subseteq M_n(\mathbb{C})$

(For this you need to check $vg \in M_n(\mathbb{C})$ is actually in $T_g G \subseteq T_g M_n(\mathbb{C}) \cong M_n(\mathbb{C})$.)

By local existence & uniqueness of solutions of first-order ODE with smooth coefficients, there's a unique smooth map $D: (-\varepsilon, \varepsilon) \to G$ with

$$\frac{d}{dt} D(t) = \tilde{v}(D(t))$$

This obeys

$$\frac{d}{dt} D(t) = v \, D(t)$$

matrix multiplication

by definition of $\tilde{v}$, and we've already seen

$$\frac{d}{dt} C(t) = v \, C(t)$$

so by uniqueness we have

$$C(t) = D(t) \in G \qquad \forall t \in (-\varepsilon, \varepsilon)$$

Then using

$$C(t_1 + \cdots + t_n) = C(t_1) \cdots C(t_n)$$

we get $C(t) \in G$ for all $t \in \mathbb{R}$. $\blacksquare$

Note: the result is true for all Lie groups, and the existence/uniquess of solutions for ODE is key, but for matrix Lie groups we get a nice formula for $C(t) = \sum_{n\geq 0}^{\infty} \frac{(tv)^n}{n!}$.

Defining $\exp(v) = \exp(1v)$ for $v \in \mathfrak{g}$ we get

$$\exp : \mathfrak{g} \to G$$

Theorem - For any Lie group $G$, $\exp : \mathfrak{g} \to G$ is a diffeomorphism from some open neighborhood of $0 \in \mathfrak{g}$ to some open neighborhood of $1 \in G$.


Proof - A smooth map $f : X \to Y$ between manifolds has a differential

$$(df)_x : T_x X \to T_{f(x)} Y \qquad \forall x \in X$$

and the Inverse Function Theorem says $f$ is a diffeomorphism in a neighborhood of $x$ iff the linear map $(df)_x$ is a bijection.

Since $\frac{d}{dt}\exp(tv) = v$ by our previous theorem, the map

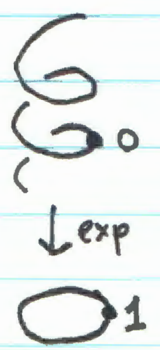
$$\begin{array}{ccc} (d\exp)_0 : T_0\mathfrak{g} & \longrightarrow & T_1 G \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathfrak{g} & & \mathfrak{g} \end{array}$$

is the identity on $\mathfrak{g}$:

$$(d\exp)_0(v) = \frac{d}{dt}\exp(tv) = v$$

hence bijective, so $\exp : \mathfrak{g} \to G$ is a diffeomorphism in a neighborhood of $0 \in \mathfrak{g}$. $\blacksquare$

Note if $G = U(1)$, $\mathfrak{g} = \mathfrak{u}(1) \cong \mathbb{R}$, then $\exp : \mathfrak{g} \to G$ is not a diffeomorphism, just a diffeomorphism near $0 \in \mathfrak{g}$:




$$\begin{array}{c} \mathfrak{u}(1) = \{ i\theta : \theta \in \mathbb{R} \} \\ \big\downarrow \exp \\ U(1) = \{ e^{i\theta} : \theta \in \mathbb{R} \} \end{array}$$