# The Lie Bracket

Suppose $G \subseteq GL(n, \mathbb{C})$ is a matrix Lie group and $\mathfrak{g} \subseteq M_n(\mathbb{C})$ is its Lie algebra. If $v, w \in \mathfrak{g}$ then $\exp(v), \exp(w) \in G$ so $\exp(v)\exp(w) \in G$. If $v, w$ commute then we've seen

$$\exp(v)\exp(w) = \exp(v+w)$$

What if they don't?

$$\exp(v)\exp(w) = \exp(?)$$

The question must have an answer if $v, w \in \mathfrak{g}$ are close enough to $0$, since then $\exp(v)\exp(w)$ is close to $1 \in G$, and we've shown exp maps onto some open neighborhood of $1 \in G$!

We can work it out! Suppose $v, w \in \mathfrak{g}$ are small (close to 0):

$$\exp(v)\exp(w) = \left(1 + v + \frac{v^2}{2!} + \cdots\right)\left(1 + w + \frac{w^2}{2!} + \cdots\right)$$

$$= 1 + v + w + \frac{v^2}{2!} + \underline{vw} + \frac{w^2}{2!} + \cdots$$

$$\exp(v+w) = 1 + (v+w) + \frac{(v+w)^2}{2!} + \cdots$$

$$= 1 + v + w + \frac{v^2}{2!} + \underline{\frac{vw}{2!}} + \underline{\frac{wv}{2!}} + \frac{w^2}{2!} + \cdots$$

They're not equal, in general, since

$$vw \neq \frac{vw}{2} + \frac{wv}{2}$$

when $v, w$ don't commute! Instead

$$vw = \frac{vw + wv}{2} + \frac{vw - wv}{2}$$

So define the <u>bracket</u> $[v, w]$ by

$$[v, w] = vw - wv$$

$$\exp(v)\exp(w) = 1 + v + w + \frac{v^2}{2!} + vw + \frac{w^2}{2!} + \cdots$$

$$= 1 + v + w + \frac{v^2}{2!} + \frac{vw + wv}{2} + \frac{[v,w]}{2} + \frac{w^2}{2!} + \cdots$$

while

$$\exp\left(v + w + \frac{[v,w]}{2}\right) =$$

$$1 + v + w + \frac{[v,w]}{2} + \frac{v^2}{2!} + \frac{vw}{2!} + \frac{wv}{2!} + \frac{w^2}{2!} + \cdots$$

where in both case we're leaving out cubic & higher-order terms. So:

$$\exp(v)\exp(w) = \exp\left(v + w + \frac{[v,w]}{2}\right) + \cdots$$

In fact:

Baker-Campbell-Hausdorff Formula: for any $v, w \in \mathfrak{g}$ we have

$$\exp(v)\exp(w) = \exp\left(v + w + \tfrac{1}{2}[v,w] + \cdots\right)$$

where the series, defined using brackets, converges when $v, w$ are small enough.

A bit more precisely:

$$\exp(v)\exp(w) =$$

$$\exp\left(v + w + \frac{1}{2}[v,w] + \frac{1}{12}[v,[v,w]] + \frac{1}{12}[w,[w,v]] + \cdots\right)$$

with convergence if $\|v\| + \|w\| < \frac{\ln 2}{2}$.

But the details matter less than this:

FOR A MATRIX LIE GROUP, THE BRACKET $[\cdot,\cdot]$ ON $\mathfrak{g}$ HAS ENOUGH INFORMATION TO LET YOU RECONSTRUCT MULTIPLICATION IN $G$ NEAR THE IDENTITY!

But we haven't shown this yet:

Theorem – If $G$ is a matrix Lie group then

$$v, w \in \mathfrak{g} \Rightarrow [v, w] \in \mathfrak{g}$$

To prove it we'll use this:

Lemma – If $G$ is a matrix Lie group then

$$g \in G, \ w \in \mathfrak{g} \Rightarrow g w g^{-1} \in \mathfrak{g}$$

where $gwg^{-1}$ is defined using matrix multiplication.

Proof – There is a smooth map given by conjugation

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto g h g^{-1} \end{aligned}$$

so for any $g \in G$,

$$\begin{aligned} AD(g) : G &\to G \\ h &\mapsto g h g^{-1} \end{aligned}$$

is smooth. ("AD" means "adjoint" for some reason.)

Differentiating $AD(g): G \to G$ we get

$$(d\,AD(g))_1 : \begin{array}{ccc} T_1 G & \longrightarrow & T_1 G \\ \shortparallel & & \shortparallel \\ \mathfrak{g} & & \mathfrak{g} \end{array}$$

We call this linear map

$$Ad(g): \mathfrak{g} \to \mathfrak{g}$$

and we claim

$$Ad(g)(w) = gwg^{-1} \quad \forall w \in \mathfrak{g}$$

This implies $gwg^{-1} \in \mathfrak{g}$, as desired. To check this remember if $f: X \to Y$ & $C: \mathbb{R} \to X$ are smooth with $C(0) = x$ then

$$(df)_x (C'(0)) = \frac{d}{dt} f(C(t))\Big|_{t=0}$$

Thus

$$(d\,AD(g))_1 \left(\frac{d}{dt} \exp(tw)\Big|_{t=0}\right) = \frac{d}{dt} AD(g)(\exp(tw))\Big|_{t=0}$$

$$= \frac{d}{dt}\, g \exp(tw) g^{-1}\Big|_{t=0}$$

$$= gwg^{-1}. \quad \blacksquare$$