Matrix Lie Algebras

Theorem - The Lie algebra of $GL(n,\mathbb{C})$ is

$$\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$$

with Lie bracket $[x,y] = xy - yx$.

Proof - $GL(n,\mathbb{C})$ is an open subset of $M_n(\mathbb{C})$ so

$$\mathfrak{gl}(n,\mathbb{C}) = T_1 GL(n,\mathbb{C}) = T_1 M_n(\mathbb{C}) = M_n(\mathbb{C})$$

and we already know the Lie bracket in any matrix Lie algebra is $[x,y] = xy - yx$. ▨

Theorem - The Lie algebra of $GL(n,\mathbb{R})$ is

$$\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$$

and the Lie algebra of $GL(n,\mathbb{H})$ is

$$\mathfrak{gl}(n,\mathbb{H}) = M_n(\mathbb{H})$$

with Lie bracket $[x,y] = xy - yx$.

Proof – Similar; note we can think of quaternions as 2×2 complex matrices

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

with ordinary matrix multiplication, giving an algebra homomorphism

$$\mathbb{H} \hookrightarrow M_2(\mathbb{C})$$

and thus

$$M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})$$

and a Lie group homomorphism

$$GL(n,\mathbb{H}) \hookrightarrow GL(2n,\mathbb{C})$$

making $M_n(n,\mathbb{H})$ into a matrix Lie group.

$$GL(n,\mathbb{R}) \hookrightarrow GL(n,\mathbb{C})$$

is more obvious. $\boxed{}$

Theorem – The Lie algebra of

$$SL(n;\mathbb{C}) = \{ g \in M_n(\mathbb{C}) : \det g = 1 \}$$

is

$$\mathfrak{sl}(n,\mathbb{C}) = \{ a \in M_n(\mathbb{C}) : \operatorname{tr} a = 0 \}$$

We can use these facts:

Lemma 1 – Given a matrix Lie group $G \subseteq GL(n,\mathbb{C})$ and $a \in M_n(\mathbb{C})$,

$$a \in \mathfrak{g} \iff \text{for all } t \in \mathbb{R} \ \exp(ta) \in G.$$

Proof: $\Rightarrow$: We showed this in the section on the exponential map.

$\Leftarrow$: Given that $\exp(ta) \in G$ for all $t \in \mathbb{R}$, $C(t) = \exp(ta)$ defines a smooth curve $C : \mathbb{R} \to G$, so $C'(0) \in T_1 G = \mathfrak{g}$, but $C'(0) = a$. ▨

Lemma 2 - For any $a \in M_n(\mathbb{C})$,

$$\det(\exp(a)) = e^{tr(a)}$$

where the trace of a is

$$tr(a) = \sum_{i=1}^{n} a_{ii}.$$

Proof - Later!

Proof of Theorem - Given $a \in M_n(\mathbb{C})$,

$$
\begin{aligned}
& a \in \mathfrak{sl}(n, \mathbb{C}) \\
\text{Lemma 1} \quad & \Updownarrow \\
& \exp(ta) \in SL(n, \mathbb{C}) \quad \forall t \in \mathbb{R} \\
& \Updownarrow \\
& \det(\exp(ta)) = 1 \quad \forall t \in \mathbb{R} \\
\text{Lemma 2} \quad & \Updownarrow \\
& e^{tr(ta)} = 1 \quad \forall t \in \mathbb{R} \\
& \Updownarrow \\
& e^{t\, tr(a)} = 1 \quad \forall t \in \mathbb{R} \\
& \Updownarrow \\
& tr(a) = 0. \quad \blacksquare
\end{aligned}
$$

Proof of Lemma 2: First show it for diagonal matrices $a \in M_n(\mathbb{C})$:

$$a = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}.$$

$$\det(\exp(a)) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{tr(a)}$$

Then show it for <u>diagonalizable</u> a:

$a \in M_n(\mathbb{C})$ such that there's a basis v_i of $\mathbb{C}^n$ with

$$a v_i = \lambda_i v_i \qquad \exists \lambda_i \in \mathbb{C}$$

In this case

$$g a g^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $g \in GL(n, \mathbb{C})$ is the "change of basis" matrix with $g v_i = e_i$, e_i the standard basis of $\mathbb{C}^n$:

$$g a g^{-1} e_i = g a v_i = g \lambda_i v_i = \lambda_i e_i .$$

Note that tr, det & exp get along with change of basis:

$$\mathrm{tr}(gag^{-1}) = \mathrm{tr}(g^{-1}ga) = \mathrm{tr}(a)$$

cyclic property : $\mathrm{tr}(ab) = \mathrm{tr}(ba)$

$$\det(gag^{-1}) = \det(g)\det(a)\det(g)^{-1} = \det(a)$$

$$\exp(gag^{-1}) = \sum_{n=0}^{\infty} \frac{(gag^{-1})^n}{n!} = g\sum_{n=0}^{\infty}\frac{a^n}{n!}g^{-1} = g\exp(a)g^{-1}.$$

So: if gag^{-1} is diagonal,

$$\begin{aligned}\det(\exp(a)) &= \det(g\exp(a)g^{-1}) \\ &= \det(\exp(gag^{-1})) \\ &= e^{\mathrm{tr}(gag^{-1})} \\ &= e^{\mathrm{tr}(a)}\end{aligned}$$

Finally, diagonalizable matrices are dense in $M_n(\mathbb{C})$ since any $a \in M_n(\mathbb{C})$ for which the polynomial $\det(a-\lambda 1)$ has no repeated roots is diagonalizable. Since $\det(\exp(a))$ & $e^{tr(a)}$ are continuous functions of $a \in M_n(\mathbb{C})$ and they agree on a dense set, they're equal. ▨

Similarly:

Theorem – The Lie algebra of $SL(n, \mathbb{R})$ is

$$\mathfrak{sl}(n, \mathbb{R}) = \{a \in M_n(\mathbb{R}) : tr\, a = 0\}$$

Proof: it's a corollary. ▨

We can define a complex-valued det and tr on $M_n(\mathbb{H})$ using

$$M_n(\mathbb{H}) \hookrightarrow M_{2n}(\mathbb{C})$$

and then:

Theorem - The Lie algebra of

$$SL(n,\mathbb{H}) = \{ a \in M_n(\mathbb{H}) : \det a = 1 \}$$

is

$$\mathfrak{sl}(n,\mathbb{H}) = \{ a \in M_n(\mathbb{H}) : \operatorname{tr} a = 0 \}$$

Proof: again it's a corollary of the complex case. $\blacksquare$

Theorem – The Lie algebra of

$$U(n) = \{ g \in M_n(\mathbb{C}) : gg^* = g^*g = 1 \}$$

is

$$\mathfrak{u}(n) = \{ a \in M_n(\mathbb{C}) : a + a^* = 0 \}$$

A matrix with $a^* = -a$ is called <u>skew-adjoint</u>.

Proof – Note that if $t \in \mathbb{R}$,

$$\exp(ta^*) = \sum_{n=0}^{\infty} \frac{(ta^*)^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{(ta)^n}{n!} \right)^* = \exp(ta)^*$$

so

$$a \in \mathfrak{u}(n)$$
$$\Updownarrow$$
$$\exp(ta) \in U(n) \qquad \forall t \in \mathbb{R}$$
$$\Updownarrow$$
$$\exp(ta)\exp(ta)^* = \exp(ta)^*\exp(ta) = 1 \qquad \forall t \in \mathbb{R}$$
$$\Downarrow \ d/dt,\ t=0$$
$$a + a^* = 0$$

while $a = -a^* \Rightarrow \exp(ta^*) = \exp(-ta) = \exp(ta)^{-1}$

so $\exp(ta)\exp(ta)^* = \exp(ta)^*\exp(ta) = 1 \quad \forall t \in \mathbb{R}$. $\blacksquare$

Similarly,

Theorem - The Lie algebra of

$$O(n) = \{g \in M_n(\mathbb{R}) : gg^* = g^*g = 1\}$$

is

$$\mathfrak{o}(n) = \{a \in M_n(\mathbb{R}) : a + a^* = 0\}.$$

Theorem - The Lie algebra of

$$Sp(n) = \{g \in M_n(\mathbb{H}) : gg^* = g^*g = 1\}$$

is

$$\mathfrak{sp}(n) = \{a \in M_n(\mathbb{H}) : a + a^* = 0\}$$

Theorem - The Lie algebra of

$$SU(n) = SL(n, \mathbb{C}) \cap U(n)$$

is

$$\mathfrak{su}(n) = \mathfrak{sl}(n, \mathbb{C}) \cap \mathfrak{u}(n)$$

$$= \{a \in M_n(\mathbb{C}) : \mathrm{tr}(a) = 0,\ a + a^* = 0\}$$

Theorem - The Lie algebra of

$$SO(n) = SL(n,\mathbb{R}) \cap O(n)$$

is

$$\mathfrak{so}(n) = \mathfrak{sl}(n,\mathbb{R}) \cap \mathfrak{o}(n)$$

$$= \mathfrak{o}(n)$$

$$= \{ a \in M_n(\mathbb{R}) : a + a^* = 0 \}$$

because $\mathfrak{o}(n) \subseteq \mathfrak{sl}(n,\mathbb{R})$.

Proof - For a <u>real</u> matrix $a \in M_n(\mathbb{R})$

we have

$$a + a^* = 0$$
$$\Downarrow$$
$$tr(a + a^*) = 0$$
$$\Downarrow$$
$$\sum_{i=1}^{n} a_{ii} + a_{ii} = 0$$
$$\Downarrow$$
$$tr(a) = 0$$

since complex conjugation doesn't affect a_{ii}. $\blacksquare$