

From Lie Groups to Lie Algebras

We've seen every Lie group G gives a Lie algebra $\mathfrak{g} = T_1 G$. Also every homomorphism of Lie groups gives a "homomorphism" of Lie algebras.

Def. - A Lie algebra is a vector space L with a bracket $[\cdot, \cdot]: L \times L \rightarrow L$ that is bilinear and antisymmetric and obeys the Jacobi identity.

Def. - Given Lie algebras L and M , a Lie algebra homomorphism $\varphi: L \rightarrow M$ is a linear map that preserves the bracket:

$$\varphi([v, w]) = [\varphi(v), \varphi(w)] \quad \forall v, w \in L$$

Given a Lie group homomorphism $f: G \rightarrow H$,
its differential at the identity is a linear map

$$(df)_1 : T_1 G \rightarrow T_1 H$$

$\parallel \qquad \parallel$
 $\mathfrak{g} \qquad \mathfrak{h}$

which we call $df: \mathfrak{g} \rightarrow \mathfrak{h}$ for short.

Theorem - If $f: G \rightarrow H$ is a Lie group
homomorphism then $df: \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie
algebra homomorphism.

Proof - Calculate! Since the bracket is defined
in terms of conjugation & a group homomorphism
preserves that:

$$f(ghg^{-1}) = f(g)f(h)f(g)^{-1}$$

it's got to work! $\square$

Theorem - If $f: G \rightarrow H$ and $g: H \rightarrow K$ are Lie group homomorphisms then

$$d(g \circ f) = dg \circ df$$

If $1_G: G \rightarrow G$ is the identity homomorphism then

$$d1_G = 1_{\mathfrak{g}}$$

Proof - These are properties of the differential. 

This theorem says there's a functor

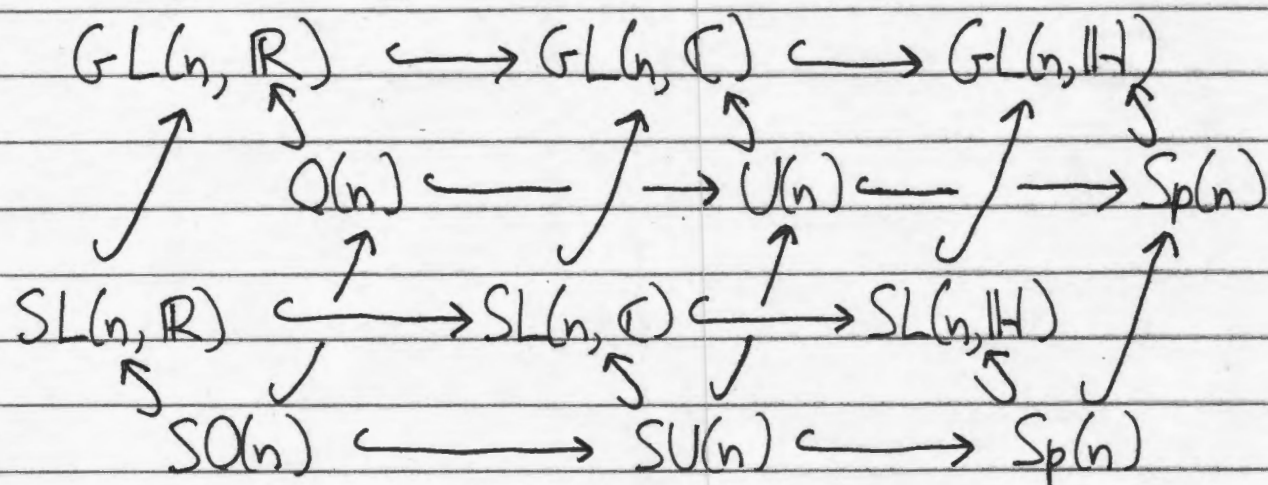
$$\underline{R}: \text{Lie Group} \rightarrow \text{Lie Alg}$$

where the category LieGroup has Lie groups as objects & Lie group homomorphisms as morphisms, similarly for Lie Alg, and

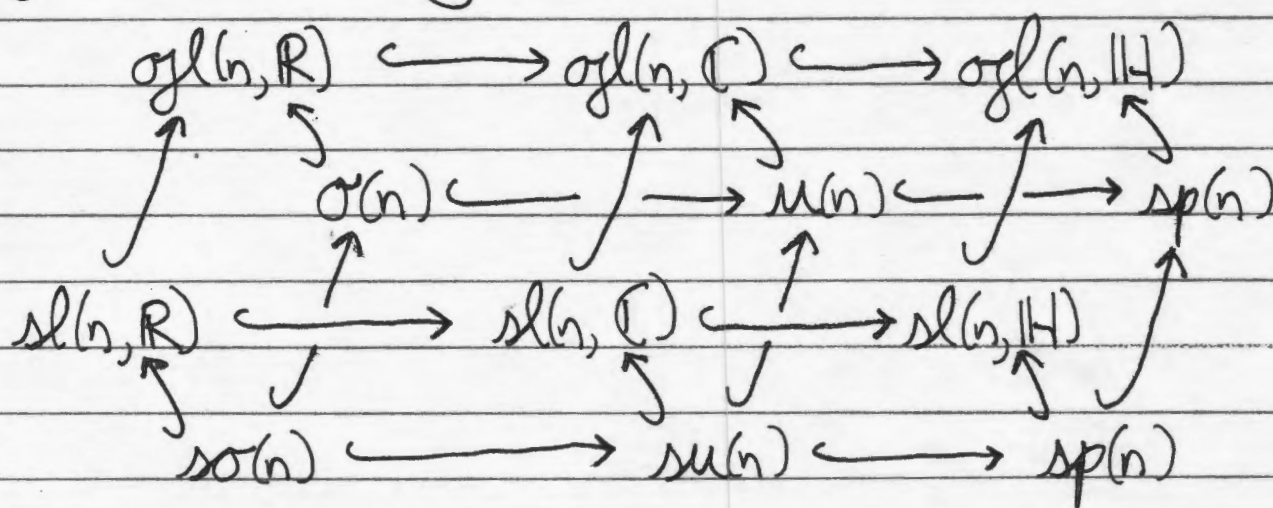
$$\underline{R}(G) = \mathfrak{g}$$

$$\underline{R}(f: G \rightarrow H) = df: \mathfrak{g} \rightarrow \mathfrak{h}$$

So, any commuting diagram of Lie groups:



gives a commuting diagram of Lie algebras:



(The differential of an inclusion of Lie groups is an inclusion of Lie algebras, so all these arrows are $\hookrightarrow$'s.)