From Lie Groups to Lie Algebras

We've seen every Lie group G gives a Lie algebra $\mathfrak{g} = T_1 G$. Also every homomorphism of Lie groups gives a "homomorphism" of Lie algebras.

Def. — A <u>Lie algebra</u> is a vector space L with a <u>bracket</u> $[\cdot,\cdot] : L \times L \to L$ that is bilinear and antisymmetric and obeys the Jacobi identity.

Def. — Given Lie algebras L and M, a <u>Lie algebra homomorphism</u> $\varphi : L \to M$ is a linear map that preserves the bracket:

$$\varphi([v,w]) = [\varphi(v), \varphi(w)] \quad \forall v, w \in L$$

Given a Lie group homomorphism $f: G \to H$, its differential at the identity is a linear map

$$\begin{array}{ccc} (df)_1 : T_1 G & \to & T_1 H \\ \| & & \| \\ \mathfrak{g} & & \mathfrak{h} \end{array}$$

which we call $df: \mathfrak{g} \to \mathfrak{h}$ for short.

Theorem - If $f: G \to H$ is a Lie group homomorphism then $df: \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof - Calculate! Since the bracket is defined in terms of conjugation & a group homomorphism preserves that:

$$f(ghg^{-1}) = f(g)\, f(h)\, f(g)^{-1}$$

it's got to work! $\boxed{}$

Theorem - If $f: G \to H$ and $g: H \to K$ are Lie group homomorphisms then

$$d(g \circ f) = dg \circ dh$$

If $1_G : G \to G$ is the identity homomorphism then

$$d\, 1_G = 1_{\mathfrak{g}} .$$

Proof - These are properties of the differential. $\blacksquare$

This theorems says there's a functor

$$\underline{R} : \text{Lie Group} \to \text{Lie Alg}$$

where the category LieGroup has Lie groups as objects & Lie group homomorphisms as morphisms, similarly for Lie Alg, and

$$\underline{R}(G) = \mathfrak{g}$$

$$\underline{R}\,(f : G \to H) = df : \mathfrak{g} \to \mathfrak{h}$$

So, any commuting diagram of Lie groups:

$$\begin{tikzcd}
GL(n,\mathbb{R}) \arrow[rr, hook] & & GL(n,\mathbb{C}) \arrow[rr, hook] & & GL(n,\mathbb{H}) & \\
& O(n) \arrow[ul, hook] \arrow[rr, hook] & & U(n) \arrow[ul, hook] \arrow[rr, hook] & & Sp(n) \arrow[ul, hook] \\
SL(n,\mathbb{R}) \arrow[uu, hook] \arrow[rr, hook] & & SL(n,\mathbb{C}) \arrow[uu, hook] \arrow[rr, hook] & & SL(n,\mathbb{H}) \arrow[uu, hook] & \\
& SO(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & SU(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & Sp(n) \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}$$

gives a commuting diagram of Lie algebras:

$$\begin{tikzcd}
\mathfrak{gl}(n,\mathbb{R}) \arrow[rr, hook] & & \mathfrak{gl}(n,\mathbb{C}) \arrow[rr, hook] & & \mathfrak{gl}(n,\mathbb{H}) & \\
& \mathfrak{o}(n) \arrow[ul, hook] \arrow[rr, hook] & & \mathfrak{u}(n) \arrow[ul, hook] \arrow[rr, hook] & & \mathfrak{sp}(n) \arrow[ul, hook] \\
\mathfrak{sl}(n,\mathbb{R}) \arrow[uu, hook] \arrow[rr] & & \mathfrak{sl}(n,\mathbb{C}) \arrow[uu, hook] \arrow[rr, hook] & & \mathfrak{sl}(n,\mathbb{H}) \arrow[uu, hook] & \\
& \mathfrak{so}(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr] & & \mathfrak{su}(n) \arrow[ul, hook] \arrow[uu, hook] \arrow[rr, hook] & & \mathfrak{sp}(n) \arrow[ul, hook] \arrow[uu, hook]
\end{tikzcd}$$

(The differential of an inclusion of Lie groups is an inclusion of Lie algebras, so all these arrows are $\hookrightarrow$'s.)